ON GENERALIZED NARKIEWICZ CONSTANTS OF FINITE ABELIAN GROUPS

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Abstract. For finite abelian groups G, we introduce some generalized zero-sum invariants $\mathsf{D}^N(G)$, $\eta^N(G)$, and $\mathsf{s}^N(G)$. For example, $\mathsf{D}^N(G)$ is the smallest integer t such that every sequence $S = g_1 \cdot \ldots \cdot g_t$ over $G \setminus \{0\}$ of length t has two zero-sum subsequences $T_1 = \prod_{i\in I} g_i$ and $T_2 = \prod_{j\in J} g_j$ such that $\prod_{k\in I\cap J} g_k$ is not zero-sum, where I, J are distinct subsets of $[1, t]$. These invariants have close connection with Narkiewicz constant and significant applications in Factorization Theory. We first systematically studied these three invariants.

1. Introduction

Let G be an additive finite abelian group, let $G^\bullet = G \setminus \{0\}$, and let $G_0 \subset G$ be a nonempty subset. We denote by $\mathcal{F}(G_0)$ the free abelian monoid with basis G_0. Elements of $\mathcal{F}(G_0)$ are called sequences over G_0. In other words, sequences over G_0 are finite unordered sequences with terms from G_0 and repetition allowed. Let

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g\in G_0} g^{\mathsf{v}_g(S)}$$

be a sequence over G_0, where ℓ is a positive integer and $g_1, \ldots, g_\ell \in G_0$. Then $|S| = \ell$ is the length of S and $\mathsf{v}_g(S)$ is the multiplicity of g in S. We say S is a *zero-sum sequence* if the sum of all the terms equals zero, i.e., $\sigma(S) = g_1 + \ldots + g_\ell = \sum_{g\in G_0} \mathsf{v}_g(S)g = 0$. Let T be another sequence over G_0. We say T is a subsequence of S (denoted by $T \mid S$) if T divides S in $\mathcal{F}(G_0)$, or in other words, $\mathsf{v}_g(T) \le \mathsf{v}_g(S)$ for all $g \in G_0$.

A typical zero-sum problem studies conditions which ensure that given sequences have nontrivial zero-sum subsequences with prescribed properties. Let Ω be a nonempty subset of zero-sum sequences with prescribed properties. In 2018, to give a unifying look at zero-sum invariants, Gao, Li, Peng, and Wang [GLPW18] introduced $\mathsf{s}_\Omega(G)$ (note that $\mathsf{d}_\Omega(G)$ is used in the original paper), which is the smallest integer t such that every sequence S of length t over G has a subsequence belonging to Ω. Therefore special sets Ω lead to the following classic zero-sum invariants (the reader may want to consult one of the surveys or monographs [GG06, GH06, G13]).

- $\mathsf{s}_\Omega(G) = \mathsf{D}(G)$ is the Davenport constant, if Ω is the set of all zero-sum sequences;

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- $\mathsf{s}_\Omega(G) = \mathsf{s}(G)$ is the Erdős-Ginzberg-Ziv constant, if Ω is the set of all zero-sum sequences of length $\exp(G)$, where $\exp(G)$ is the exponent of G;
- $\mathsf{s}_\Omega(G) = \eta(G)$ is the η-constant, if Ω is the set of all nontrivial zero-sum sequences of length not larger than $\exp(G)$.

For recent progress of these classic invariants, we refer to [BGH20, L20, N20, S20]. Furthermore, the invariant $\mathsf{s}_\Omega(G)$ has been studied for various other sets Ω (see [GHHLYZ21, GLPW18]). Recent years, these invariants are also generalized to non-abelian groups, please see [B07, CDS18, GL10, GL08, H15, HZ19, OZ20, OhZh20].

A natural generalization is to study conditions which ensure that given sequences S have two nontrivial zero-sum subsequences with prescribed properties and relations. In 2012, B. Girard [G12] initially studied the constant $\mathsf{disc}(G)$, which is the smallest integer t such that all sequences of length t over G have two nontrivial zero-sum subsequences having distinct lengths. The generalized Davenport constant $\mathsf{D}_2(G)$ is the smallest integer t such that every sequence S of length t over G has two disjoint nontrivial zero-sum subsequences (see [H92]). These two invariants have been studied by many researchers (see [GHLYZ20, GLZZ16, GZZ15, GH06, PS11]).

To continue, we need to introduce some relations like "disjoint" for subsequences. Let $S = g_1 \cdot \ldots \cdot g_\ell$ be a sequence over G_0 and let T_1, T_2 be two subsequences of S. If $Y \mid T_1$ and $Y \mid T_2$, we say Y is a *common divisor* of T_1 and T_2. Let Y be a common divisor of T_1 and T_2. We say Y is a *S-inner common divisor* of T_1 and T_2, if $T_1T_2 \mid SY$, or equivalently, there exist subsets $I, J \subset [1,\ell]$ such that $T_1 = \prod_{i\in I} g_i$, $T_2 = \prod_{i\in J} g_j$, and $Y = \prod_{k\in I\cap J} g_k$. In particular, $\gcd(T_1, T_2)$ is a S-inner common divisor of T_1 and T_2. Furthermore, we say

- T_1 and T_2 are (S-)innerly distinct if T_1 and T_2 have a (S-)inner common divisor Y such that either $Y \ne T_1$ or $Y \ne T_2$, or equivalently, there exist distinct subsets $I, J \subset [1,\ell]$ such that $T_1 = \prod_{i\in I} g_i$ and $T_2 = \prod_{i\in J} g_j$.
- T_1 and T_2 are (S-)innerly joint if T_1 and T_2 are (S-)innerly distinct and have a nontrivial (S-)inner common divisor.
- T_1 and T_2 are (S-)innerly disjoint if the trivial sequence is a (S-)inner common divisor of T_1 and T_2, or equivalently $T_1T_2 \mid S$.
- T_1 and T_2 are (S-)innerly non-zero-sum-joint if T_1 and T_2 are (S-)innerly distinct and have a non-zero-sum (S-)inner common divisor.

By our definition, subsequences T_1 and T_2 of S could be both S-innerly joint and disjoint. For example, let $S = g^{2\,\mathrm{ord}(g)}$ and $T_1 = T_2 = g^{\mathrm{ord}(g)}$, where g is a nonzero element. Then g^k is a S-inner common divisor of T_1 and T_2 for each $k \in [0, \mathrm{ord}(g)]$, which implies that T_1 and T_2 are S-innerly distinct, joint, disjoint, and non-zero-sum-joint.

Let $T_1 = 0^{s_1}W_1$ and $T_2 = 0^{s_2}W_2$ be two subsequences of $S = 0^sW$, where W_1, W_2, W are sequences over $G^\bullet$ and $s_1, s_2, s \in \mathbb{N}_0$. Then T_1, T_2 are S-innerly non-zero-sum-joint if and only if W_1, W_2 are W-innerly non-zero-sum-joint. Therefore we only need to consider sequences over $G^\bullet$ when studying the "innerly non-zero-sum-joint" property. Now we can define some generalized zero-sum invariants associated with the innerly non-zero-sum-joint property.

Definition 1.1. Let G be a finite abelian group with $|G| > 1$. We define

- $\mathsf{D}^N(G)$ to be the smallest integer ℓ such that every sequence S over $G^\bullet$ of length ℓ has two innerly non-zero-sum-joint zero-sum subsequences;
- $\eta^N(G)$ to be the smallest integer ℓ such that every sequence S over $G^\bullet$ of length ℓ has two innerly non-zero-sum-joint zero-sum subsequences of length not larger than $\exp(G)$;
- $\mathsf{s}^N(G)$ to be the smallest integer ℓ such that every sequence S over $G^\bullet$ of length ℓ has two innerly non-zero-sum-joint zero-sum subsequences of length $\exp(G)$.

Let S be a sequence over $G^\bullet$ of length $|S| \geq (|G|-1)\exp(G)+1$. Then there exists $g \in G^\bullet$ such that $\mathsf{v}_g(S) \geq \exp(G)+1$, whence $g^{\exp(G)}$ and $g^{\exp(G)}$ have a S-inner common divisor $g^{\exp(g)-1}$. It follows that all the three invariants above are finite.

Like the Davenport constant, these invariants have significant applications in Factorization Theory. In fact, we have $\mathsf{D}^N(G) = \mathsf{N}_1(G) + 1$ and $\eta^*(G) = \eta^N(G)$ (see Definitions 2.2, 2.3 and Lemma 2.4), where $\mathsf{N}_1(G)$ is the Narkiewicz constant and $\eta^*(G)$ is a Narkiewicz-sense constant introduced by Gao, Geroldinger, and Wang [GGW11] to study $\mathsf{N}_1(G)$. The Narkiewicz constant was first used by Narkiewicz in 1960 to study the asymptotic behavior of counting functions associated with non-unique factorizations (see [GH06, N04] for an overview and historical references). For recent progress of $\mathsf{N}_1(G)$ and $\eta^*(G)$, the readers may refer to [GGW11, GLP11, GPZ13]. Since these new invariants have a flavor of Narkiewicz constants, we can view them as generalized Narkiewicz constants.

In section 2, we collect necessary notation, build connection with Narkiewicz constants, and gather the required machinery. In section 3, we study the invariants $\mathsf{D}^N(G)$ and $\eta^N(G)$ for finite abelian groups G. The main theorems are Theorems 3.6 and 3.14. In Section 4, we introduce two more invariants to help study $\mathsf{s}^N(G)$ and the main theorems are Theorems 4.8 and 4.18.

2. Preliminaries

We denote by $\mathbb{N}$ the set of positive integers and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a, b \in \mathbb{R}$, we set $[a,b] = \{x \in \mathbb{Z} \colon a \leq x \leq b\}$. For $n, r \in \mathbb{N}$, let C_n denote a cyclic group with n elements and let C_n^r denote the direct sum of r copies of C_n. Let G be an abelian group and let $G_0 \subset G$ be a subset. We let $\langle G_0 \rangle \subset G$ be the subgroup generated by G_0, $G_0^\bullet = G_0 \setminus \{0\}$, and $-G_0 = \{-g \colon g \in G_0\}$. A family $(e_i)_{i \in I}$ of nonzero elements of G is said to be *independent* if

$$\sum_{i\in I} m_i e_i = 0 \quad \text{implies} \quad m_i e_i = 0 \quad \text{for all } i \in I, \qquad \text{where } m_i \in \mathbb{Z}\,.$$

If $I = [1, r]$ and $(e_1, \ldots, e_r)$ is independent, then we simply say that $e_1, \ldots, e_r$ are independent elements of G. The tuple $(e_i)_{i\in I}$ is called a *basis* if $(e_i)_{i\in I}$ is independent and $\langle \{e_i \colon i \in I\}\rangle = G$. If $1 < |G| < \infty$, then we have

$$G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}\,, \qquad \text{where } r \in \mathbb{N} \text{ and } 1 < n_1 \mid \ldots \mid n_r\,.$$

Then $r = \mathsf{r}(G)$ is the *rank* of G and $n_r = \exp(G)$ is the *exponent* of G. Set

$$\mathsf{D}^*(G) = 1 + \sum_{i=1}^{r} (n_i - 1).$$

Let $G_0 \subset G$ be a nonempty subset and let

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g\in G_0} g^{\mathsf{v}_g(S)}$$

be a sequence over G_0, where $\mathsf{v}_g(S) \in \mathbb{N}_0$ for all $g \in G_0$. We call

- $\mathsf{h}(S) = \max\{\mathsf{v}_g(S) \colon g \in G_0\}$ the *height* of S;
- $\operatorname{supp}(S) = \{g \in G_0 \colon \mathsf{v}_g(S) > 0\}$ the *support* of S;

and we say S is

- *short* if $1 \le |S| \le \exp(G)$;
- a *squarefree sequence* if $\mathsf{v}_g(S) \le 1$ for all $g \in G_0$.

Let $g \in G$ and let T be a subsequence of S. We set $g + S = (g + g_1) \cdot \ldots \cdot (g + g_\ell)$ and denote $T^{-1}S = \prod_{g\in G_0} g^{\mathsf{v}_g(S)-\mathsf{v}_g(T)}$. If $1 \le |T| < |S|$, we say T is a *proper subsequence* of S. We call S

- a *zero-sum free sequence* if there is no nontrivial zero-sum subsequence of S;
- a *minimal zero-sum sequence* if S is a nontrivial zero-sum sequence, but S has no proper zero-sum subsequence.

Let $s \ge 2$ and let $T_1, \ldots, T_s$ be subsequences of S. We say $T_1, \ldots, T_s$ are S-innerly disjoint if $T_1 \cdot \ldots \cdot T_s$ is a subsequence of S and we say S has no innerly non-zero-sum-joint

- zero-sum subsequences if for any two zero-sum subsequences T_1, T_2 of S, the subsequences T_1 and T_2 have only zero-sum S-inner common divisors;
- short zero-sum subsequences if for any two short zero-sum subsequences T_1, T_2 of S, the subsequences T_1 and T_2 have only zero-sum S-inner common divisors;
- zero-sum subsequences of length N if for any two zero-sum subsequences T_1, T_2 of S of length $|T_1| = |T_2| = N$, the subsequences T_1 and T_2 have only zero-sum S-inner common divisors, where $N \in \mathbb{N}$.

Let H be another abelian group and let $\varphi \colon G \to H$ be a group homomorphism. Then we can extend φ to a homomorphism $\varphi \colon \mathcal{F}(G_0) \to \mathcal{F}(\varphi(G_0))$, where $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_l)$ for every sequence $S = g_1 \cdot \ldots \cdot g_l$ over G_0.

We have the following easy lemma, which will be used often without further mention.

Lemma 2.1. *Let $\varphi\colon G \to H$ be a group homommorphism, let S be a sequence over G, and let T_1, T_2 be two subsequences of S.*

(1) If T_1 and T_2 have a S-inner common divisor Y, then $\varphi(Y)$ is a $\varphi(S)$-inner common divisor of $\varphi(T_1)$ and $\varphi(T_2)$.

(2) If $\varphi(T_1)$ and $\varphi(T_2)$ have a $\varphi(S)$-inner common divisor X, then there exists a subsequence T_3 of S such that $\varphi(T_3) = \varphi(T_2)$ and T_1, T_3 have a S-inner common divisor Y such that $\varphi(Y) = X$.

(3) Suppose $S = S_1S_2$. If T_1, T_2 are subsequences of S_1 such that T_1, T_2 have a S_1-inner common divisor Y_1 and T_3, T_4 are subsequences of S_2 such that T_3, T_4 have a S_2-inner common divisor Y_2, then T_1T_3, T_2T_4 have a S-inner common divisor Y_1Y_2.

(4) If $|S| \ge \mathsf{D}^N(H)$ and $S \in \mathcal{F}(G \setminus \ker(\varphi))$, then S has two subsequences T_1 and T_2 having a non-zero-sum S-inner common divisor Y such that $\varphi(T_1)$ and $\varphi(T_2)$ are zero-sum and have a non-zero-sum $\varphi(S)$-inner common divisor $\varphi(Y)$.

(5) If $|S| \ge \eta^N(H)$ and $S \in \mathcal{F}(G \setminus \ker(\varphi))$, then S has two subsequences T_1 and T_2 having a non-zero-sum S-inner common divisor Y such that $\varphi(T_1)$ and $\varphi(T_2)$ are short zero-sum subsequences and have a non-zero-sum $\varphi(S)$-inner common divisor $\varphi(Y)$.

(6) If $|S| \ge \mathsf{s}^N(H)$ and $S \in \mathcal{F}(G \setminus \ker(\varphi))$, then S has two subsequences T_1 and T_2 of length $\exp(H)$ having a non-zero-sum S-inner common divisor Y such that $\varphi(T_1)$ and $\varphi(T_2)$ are zero-sum and have a non-zero-sum $\varphi(S)$-inner common divisor $\varphi(Y)$.

(7) S has two innerly joint minimal zero-sum subsequences if and only if S has two innerly non-zero-sum-joint zero-sum subsequences.

(8) S has two innerly joint short minimal zero-sum subsequences if and only if S has two innerly non-zero-sum-joint short zero-sum subsequences.

Proof. 1. Suppose T_1 and T_2 have a S-inner common divisor Y. Then $Y^{-1}T_1T_2$ divides S and hence $\varphi(Y^{-1}T_1T_2)$ divdies $\varphi(S)$. It follows that $\varphi(T_1)$ and $\varphi(T_2)$ have a $\varphi(S)$-inner common divisor $\varphi(Y)$.

2. Suppose $\varphi(T_1)$ and $\varphi(T_2)$ have a $\varphi(S)$-inner common divisor X. Then T_1 has a subsequence Y_1 and T_2 has a subsequence Y_2 such that $\varphi(Y_1) = \varphi(Y_2) = X$, whence $X^{-1}\varphi(T_1)\varphi(T_2) = \varphi(Y_2^{-1}T_1T_2)$ divides $\varphi(S)$. Since $\varphi(Y_2^{-1}T_2)$ divides $\varphi(T_1^{-1}S)$, there exists a subsequence W of $T_1^{-1}S$ such that $\varphi(Y_2^{-1}T_2) = \varphi(W)$. Let $T_3 = Y_1W$. Then T_3 is a subsequence of $T_1(T_1^{-1}S) = S$ such that $\varphi(T_3) = \varphi(Y_1)\varphi(W) = \varphi(Y_2)\varphi(Y_2^{-1}T_2) = \varphi(T_2)$. Since $Y_1^{-1}T_3T_1 = WT_1$ divides S, we obtain that T_1, T_3 have a S-inner common divisor Y_1.

3. Since $Y_1^{-1}T_1T_2$ divides S_1 and $Y_2^{-1}T_3T_4$ divides S_2, we obtain that $(Y_1Y_2)^{-1}T_1T_3T_2T_4$ divides S and Y_1Y_2 divides both T_1T_3 and T_2T_4, whence T_1T_3, T_2T_4 have a S-inner common divisor Y_1Y_2.

The proofs of Items 4, 5, and 6 are similar. We only prove Item 4. Suppose $|S| \ge \mathsf{D}^N(H)$ and $S \in \mathcal{F}(G \setminus \ker(\varphi))$. Then $\varphi(S) \in \mathcal{F}(H^\bullet)$. By definition of $\mathsf{D}^N(H)$, there exist subsequences T_1, T_2 of S such that $\varphi(T_1)$ and $\varphi(T_2)$ are zero-sum and $\varphi(S)$-innerly non-zero-sum-joint. It follows by Item 2 that there exists a subsequence T_3 of S such that $\varphi(T_3)$ is zero-sum and T_1, T_3 have a S-inner common divisor Y such that $\varphi(Y)$ is not zero-sum. Now the assertion follows by Item 1.

The proofs of Items 7 and 8 are similar. We only prove Item 7. Suppose T_1 and T_2 are S-innerly distinct minimal zero-sum subsequences with a nontrivial S-inner common divisor Y. We assert that Y is not zero-sum. Assume to the contrary that Y is zero-sum, since $Y \mid T_1$ and T_1 is minimal, we have that $Y = T_1$ divides T_2, whence $T_1 = T_2 = Y$, a contradiction.

Suppose S has two innerly non-zero-sum-joint zero-sum subsequences T_1, T_2. Let $T_1 = W_1 \cdot \ldots \cdot W_r$, $T_2 = V_1 \cdot \ldots \cdot V_s$, and Y a non-zero-sum S-inner common divisor of T_1 and T_2, where $r, s \in \mathbb{N}$ and $W_1, \ldots, W_r, V_1, \ldots, V_s$ are minimal zero-sum subsequences. Then for each $i \in [1, r]$ and each $j \in [1, s]$, W_i and V_j have a S-inner common divisor $Y_{i,j}$ such that $Y = \prod_{i \in [1,r], j \in [1,s]} Y_{i,j}$. Since Y is not zero-sum, there exist $i_0 \in [1, r]$ and $j_0 \in [1, s]$ such that Y_{i_0,j_0} is not zero-sum, whence W_{i_0} and V_{j_0} have a nontrivial S-inner common divisor. □

Type monoids and Narkiewicz constants. Note that $G_0 \times \mathbb{N}$ is a subset of the abelian group $G \times \mathbb{Z}$. We call sequences over $G_0 \times \mathbb{N}$ are types over G_0. Let $\boldsymbol{\alpha} \colon \mathcal{F}(G_0 \times \mathbb{N}) \to \mathcal{F}(G_0)$ denote the unique homomorphism satisfying

$$\boldsymbol{\alpha}((g, n)) = g \quad \text{for all} \quad (g, n) \in G_0 \times \mathbb{N}$$

and let $\tau \colon \mathcal{F}(G_0) \to \mathcal{F}(G_0 \times \mathbb{N})$ be defined by

$$\tau(S) = \prod_{g \in G_0} \prod_{k=1}^{\mathsf{v}_g(S)} (g, k) \in \mathcal{F}(G_0 \times \mathbb{N}) \,.$$

For $S \in \mathcal{F}(G_0)$, we call $\tau(S)$ the *type associated with* S. We say that $\boldsymbol{T}$ is a *zero-sum type* (*short, zero-sum free* or a *minimal zero-sum type*) if the associated sequence has the relevant property. Types were introduced by F. Halter-Koch in [Hal92] and applied successfully in the analytic theory of so-called type-dependent factorization properties (see [GH06, Section 9.1], and [Ha92, Ha93] for some early papers).

For a given squarefree zero-sum type $\boldsymbol{T}$ (note that $\boldsymbol{\alpha}(\boldsymbol{T})$ may not be squarefree), we can always write $\boldsymbol{T}$ as follows

$$\boldsymbol{T} = \boldsymbol{V}_1 \cdot \ldots \cdot \boldsymbol{V}_r \,,$$

where $r \in \mathbb{N}_0$ and $\boldsymbol{V}_1, \ldots, \boldsymbol{V}_r$ are minimal zero-sum subtypes. We say $\boldsymbol{T}$ has *unique factorization* if the above factorization of $\boldsymbol{T}$ is unique, i.e., if $\boldsymbol{T}$ has another factorization

$$\boldsymbol{T} = \boldsymbol{U}_1 \cdot \ldots \cdot \boldsymbol{U}_s\,,$$

where $s \in \mathbb{N}_0$, $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_s$ are minimal zero-sum subtypes, then $r = s$ and there exists a permutation $\tau \in \mathcal{S}_r$ such that $\boldsymbol{V}_i = \boldsymbol{U}_{\tau(i)}$ for all $i \in [1, r]$.

Definition 2.2. Let G be a finite abelian group. The *Narkiewicz constant* $\mathsf{N}_1(G)$ of G is defined by

$$\mathsf{N}_1(G) = \sup\{|\boldsymbol{T}|\colon\ \boldsymbol{T} \text{ is a squarefree zero-sum type over } G^\bullet \text{ and } \boldsymbol{T} \text{ has unique factorization}\}\,.$$

Suppose $G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r$ and let $(e_1, \ldots, e_r)$ be a basis of G with $\operatorname{ord}(e_i) = n_i$ for all $i \in [1, r]$. Let

$$B = \prod_{i=1}^{r} e_i^{n_i}. \quad \text{Then} \quad \tau(B) = \prod_{i=1}^{r}\prod_{k=1}^{n_i}(e_i, k)$$

has unique factorization, and hence

$$\mathsf{N}_1(G) \ge n_1 + \ldots + n_r\,. \tag{2.1}$$

Let us recall the definition of $\eta^*(G)$ which was first introduced by Gao, Geroldinger and Wang [GGW11] to study $\mathsf{N}_1(G)$.

Definition 2.3. Let G be a finite abelian group and let $\eta^*(G)$ denote the smallest integer $\ell \in \mathbb{N}_0$ such that every squarefree type $\boldsymbol{T} \in \mathcal{F}(G^\bullet \times \mathbb{N})$ of length $|\boldsymbol{T}| \ge \ell$ has two distinct short minimal zero-sum subtypes $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ such that $\gcd(\boldsymbol{T}_1, \boldsymbol{T}_2)$ is not empty.

By the above definitions, we have the following lemma.

Lemma 2.4. *Let G be a finite abelian group. Then $\mathsf{N}_1(G) + 1 = \mathsf{D}^N(G)$ and $\eta^*(G) = \eta^N(G)$.*

Proof. We first show that $\mathsf{N}_1(G) + 1 = \mathsf{D}^N(G)$. Let $\boldsymbol{T}$ be a squarefree zero-sum type over $G^\bullet$ of length $\mathsf{N}_1(G)$ such that $\boldsymbol{T}$ has unique factorization. Assume that $\boldsymbol{\alpha}(\boldsymbol{T})$ has two innerly joint minimal zero-sum subsequences S_1 and S_2 with the $\boldsymbol{\alpha}(\boldsymbol{T})$-inner common divisor Y. Then there exist minimal zero-sum subtypes $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ such that $\boldsymbol{\alpha}(\boldsymbol{T}_1) = S_1$, $\boldsymbol{\alpha}(\boldsymbol{T}_2) = S_2$, and $\boldsymbol{\alpha}(\gcd(\boldsymbol{T}_1, \boldsymbol{T}_2)) = Y$. Therefore

$$\boldsymbol{T} = \boldsymbol{T}_1\boldsymbol{U}_1 \cdot \ldots \cdot \boldsymbol{U}_k = \boldsymbol{T}_2\boldsymbol{V}_1 \cdot \ldots \cdot \boldsymbol{V}_\ell\,,$$

where $k, \ell \in \mathbb{N}$ and $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_k, \boldsymbol{V}_1, \ldots, \boldsymbol{V}_\ell$ are minimal zero-sum types. Since $\boldsymbol{Y}$ is not trivial, we obtain $\boldsymbol{T}_2 \notin \{\boldsymbol{T}_1, \boldsymbol{U}_1, \ldots, \boldsymbol{U}_k\}$, a contradiction to the fact that $\boldsymbol{T}$ has unique factorization. Thus $\boldsymbol{\alpha}(\boldsymbol{T})$ has no innerly joint minimal zero-sum subsequences and hence has no innerly non-zero-sum-joint zero-sum subsequences, whence $\mathsf{D}^N(G) \ge \mathsf{N}_1(G) + 1$.

Let S be a sequence over $G^\bullet$ of length $\mathsf{D}^N(G) - 1$ such that S has no innerly non-zero-sum-joint zero-sum subsequences. We assert that S is zero-sum. Assume to the contrary that S is not zero-sum. Then $T := S(-\sigma(S))$ is zero-sum and $|T| = \mathsf{D}^N(G)$, whence there exist two zero-sum subsequences U_1, V_1 of T such that U_1, V_1 have a non-zero-sum T-inner common divisor. Let $U_2 = U_1^{-1}T$ and $V_2 = V_1^{-1}T$. Then for every $i \in [1,2]$ and every $j \in [1,2]$ we have U_i and V_j have a non-zero-sum T-inner common divisor. By symmetry, we may assume that $-\sigma(S)$ is a term of both U_2 and V_2, whence U_1 and V_1 are subsequences of S and have a non-zero-sum S-inner common divisor, a contradiction. Thus S is zero-sum and hence $\tau(S)$ is a squarefree zero-sum type over $G^\bullet$ of length $\mathsf{D}^N(G) - 1$. Assume that $\tau(S)$ does not have unique factorization. Then there are two minimal zero-sum subtypes $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ of $\tau(S)$ such that $\gcd(\boldsymbol{T}_1, \boldsymbol{T}_2)$ is nontrivial, whence $\boldsymbol{\alpha}(\boldsymbol{T}_1)$ and $\boldsymbol{\alpha}(\boldsymbol{T}_2)$ are minimal zero-sum subsequences of S and have a nontrivial S-inner common divisor $\boldsymbol{\alpha}(\gcd(\boldsymbol{T}_1, \boldsymbol{T}_2))$, a contradiction. Therefore $\tau(S)$ has unique factorization and hence $\mathsf{D}^N(G) - 1 \le \mathsf{N}_1(G)$. Therefore $\mathsf{D}^N(G) = \mathsf{N}_1(G) + 1$.

We next show that $\eta^*(G) = \eta^N(G)$. Let $\boldsymbol{T}$ be a squarefree type over $G^\bullet$ of length $\eta^N(G)$. Then $\boldsymbol{\alpha}(\boldsymbol{T})$ has two innerly joint short minimal zero-sum subsequences S_1, S_2 with the nontrivial $\boldsymbol{\alpha}(\boldsymbol{T})$-inner common divisor Y, whence there exist minimal zero-sum subtypes $\boldsymbol{T}_1, \boldsymbol{T}_2$ of $\boldsymbol{T}$ such that $\boldsymbol{\alpha}(\boldsymbol{T}_1) = S_1$, $\boldsymbol{\alpha}(\boldsymbol{T}_2) = S_2$, and $\boldsymbol{\alpha}(\gcd(\boldsymbol{T}_1, \boldsymbol{T}_2)) = Y$. Therefore $\eta^*(G) \le \eta^N(G)$. Let S be a sequence over $G^\bullet$ of length $\eta^*(G)$. Then $\tau(S)$ is a squarefree type over $G^\bullet$ of length $\eta^*(G)$, whence there exist two short minimal zero-sum subtypes $\boldsymbol{T}_1, \boldsymbol{T}_2$ of $\tau(S)$ such that $\gcd(\boldsymbol{T}_1, \boldsymbol{T}_2)$ is not empty. It follows that $\boldsymbol{\alpha}(\boldsymbol{T}_1)$ and $\boldsymbol{\alpha}(\boldsymbol{T}_2)$ are short minimal zero-sum subsequences of S and have a nontrivial S-inner common divisor $\boldsymbol{\alpha}(\gcd(\boldsymbol{T}_1, \boldsymbol{T}_2))$, whence $\eta^N(G) \le \eta^*(G)$. Therefore $\eta^N(G) = \eta^*(G)$. □

Property C and Property D. Let G be a finite abelian group. Gao [GG06] conjectured that $\mathsf{s}(G) = \eta(G) + \exp(G) - 1$. When considering the structure of extremal sequences that has no short zero-sum subsequence and that has no zero-sum subsequence of length $\exp(G)$, the following definitions are introduced.

Definition 2.5. Let $G = C_n^r$, where $n, r \in \mathbb{N}$. We say G has

- **Property C** with respect to c if every sequence $S \in \mathcal{F}(G)$ of length $|S| = \eta(G) - 1$ which has no short zero-sum subsequence has the form $S = T^{n-1}$ for some sequence $T \in \mathcal{F}(G)$ of length c.
- **Property D** with respect to c if every sequence $S \in \mathcal{F}(G)$ of length $|S| = \mathsf{s}(G) - 1$ which has no zero-sum subsequence of length n has the form $S = T^{n-1}$ for some sequence $T \in \mathcal{F}(G)$ of length c.

If $G = C_n^r$ has Property D with respect to c, then $\mathsf{s}(G) = \eta(G) + n - 1 = c(\exp(G) - 1) + 1$ and G has Property C with respect to $c - 1$ (See [GT03, Corollary 1.2]). For groups of rank 2, Property C was first considered by van Emde Boas and Property D by Gao (see [E69, G00, Ga00]). It is conjectured that every group $G = C_n^r$ has Property D, where

$r \in \mathbb{N}$ and $n \geq 2$ (see [GG06, Conjecture 7.2]). Among others, we collect some known results.

Lemma 2.6. *Let $a, b \in \mathbb{N}_0$ and $r \in \mathbb{N}$.*

(1) $C_{2^a}^r$ has Property D with respect to 2^r.
(2) $C_{3^a}^4$ has Property D with respect to 20.
(3) $C_{3^a 5^b}^3$ has Property D with respect to 9.
(4) C_3^r has Property D.
(5) C_n and $C_n \oplus C_n$ have Property C, where $n \geq 2$.
(6) $C_{2^a 3^b 5^c 7^d}^2$ has Property D with respect to 4, where $c, d \in \mathbb{N}_0$.

Proof. For Items 1, 2, and 3, see [FGZ11, Lemma 2.4] and Item 4 follows from [H73, Hilfssatz 3] and [EEGKR07, Lemma 2.3.3]. For Item 5, we refer to [GH06, Theorem 5.1.10] and [R10, Section 11.3]. For Item 6, see [Ga00, Theorems 1.4 and 1.5] and [ST02, Theorem 3.1]. □

Lemma 2.7. *If n is odd, then there exists a sequence $T \in \mathcal{F}(C_n^3)$ of length $|T| = 9$ such that T^{n-1} has no zero-sum subsequence of length n. In particular, we have $\eta(C_n^3) \geq 8n-7$ and $\mathsf{s}(C_n^3) \geq 9n - 8$.*

Proof. See [EEGKR07, Theorem 1.2]. □

Lemma 2.8. *If n is odd, then there exists a sequence $T \in \mathcal{F}(C_n^4)$ of length $|T| = 20$ such that T^{n-1} has no zero-sum subsequence of length n. In particular, we have $\eta(C_n^4) \geq 19n - 18$ and $\mathsf{s}(C_n^4) \geq 20n - 19$.*

Proof. See [EEGKR07, Theorem 1.3]. □

Lemma 2.9. *Let $n, m, r \in \mathbb{N}$.*

(1) $\eta(C_n \oplus C_m) = 2n + m - 2$ and $\mathsf{s}(C_n \oplus C_m) = 2n + 2m - 3$ with $1 \leq n \mid m$.
(2) $\eta(C_{2^n 3}^3) = 21 \cdot 2^n - 6$ and $\mathsf{s}(C_{2^n 3}^3) = 24 \cdot 2^n - 7$.
(3) $\eta(C_2^3 \oplus C_{2n}) = 2n + 6$ for $n \geq 2$ and $\mathsf{s}(C_2^3 \oplus C_{2n}) = 4n + 5$ for $n \geq 36$.
(4) Let $G = C_2 \oplus C_{2m} \oplus C_{2mn}$. If C_m^2 has Property D or $n = 1$, then

$$\mathsf{s}(G) = 4m + 4mn - 1.$$

(5) $\eta(C_n^r) \geq (2^r - 1)(n - 1) + 1$.

Proof. For Item 1, see [GH06, Theorem 5.8.3] and for Item 2, see [GHST07, Theorem 1.8]. Item 3 follows from [FZ16, Theorem 1.2] and Item 4 follows from [GS19, Theorem 3.2]. Item 5 follows from [H73]. □

Lemma 2.10. *Let $\alpha, \beta \in \mathbb{N}_0$. Then*

$$\eta(C_{3^\alpha 5^\beta}^3) = 8 \cdot 3^\alpha 5^\beta - 7 \quad \text{and} \quad \eta(C_{3^\alpha}^4) = 19 \cdot 3^\alpha - 18\,.$$

Proof. See [GHST07, Theorem 1.7, Theorem 1.8] and [FGZ11, Theorem B]. □

Lemma 2.11. [GH06, Proposition 5.7.11] *Let G be a finite abelian group, and let H be a subgroup of G with $\exp(G) = \exp(H)\exp(G/H)$. Then*

(1) $\eta(G) \le \exp(G/H)(\eta(H) - 1) + \eta(G/H)$.

(2) $\mathsf{s}(G) \le \exp(G/H)(\mathsf{s}(H) - 1) + \mathsf{s}(G/H)$.

Lemma 2.12. *Let $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}_0$ and $\alpha \ge \beta$. Then*

$$\eta(C^3_{2^\alpha 3^\beta}) = 7 \cdot 2^\alpha 3^\beta - 6 \quad \text{and} \quad \mathsf{s}(C^3_{2^\alpha 3^\beta}) = 8 \cdot 2^\alpha 3^\beta - 7\,.$$

Proof. Let $G = C^3_{2^\alpha 3^\beta}$. Let $G = C^3_{2^\alpha 3^\beta}$. By Lemma 2.9.5, we have $\eta(G) \ge 7 \cdot 2^\alpha 3^\beta - 6$. Since $\mathsf{s}(G) \ge \eta(G) + \exp(G) - 1$, it suffices to show that $\mathsf{s}(G) \le 8 \cdot \exp(G) - 7$.

We proceed by induction on α. If $\alpha = 1$, then $\beta = 1$ or 0 and Lemma 2.9.2 implies that $\mathsf{s}(G) \le 8 \cdot \exp(G) - 7$. Suppose $\alpha \ge 2$ and suppose the assertion $\mathsf{s}(C^3_{2^t 3^\beta}) \le 8 \cdot 2^t 3^\beta - 7$ holds for all (t, β) with $\beta \le t < \alpha$. Let H be a subgroup of G such that $G/H \cong C_6^3$ if $\beta \ge 1$ and otherwise $G/H \cong C_2^3$. By induction hypothesis, we have $\mathsf{s}(H) \le 8 \cdot \exp(H) - 7$ and $\mathsf{s}(G/H) \le 8 \cdot \exp(G/H) - 7$. It follows by applying Lemma 2.11.2 that

$$\mathsf{s}(G) \le \exp(G/H)(\mathsf{s}(H) - 1) + \mathsf{s}(G/H) \le 8 \cdot \exp(G/H)\exp(H) - 7 = 8\exp(G) - 7\,.$$ □

3. On $\eta^N(G)$ and $\mathsf{D}^N(G)$

3.1. **On $\eta^N(G)$.** In this subsection, our main theorem is Theorem 3.6. We need the following lemmas.

Lemma 3.1. *Let $G = C_n^r$ be a finite abelian group, where $n, r \in \mathbb{N}$. If $r \in [1, 2]$ or $n = 2$, then*

$$\eta^N(G) = (2^r - 1) \cdot n + 1\,.$$

Proof. The assertion follows from [GGW11, Corollary 3.11] and [GPZ13, Theorem 2.6]. □

Lemma 3.2. *Let $G = C_n \oplus C_{nm}$ with $n, m \in \mathbb{N}_{\ge 2}$. Then $\eta^N(G) \ge 2n + nm$.*

Proof. Let (e_1, e_2) be a basis of G and let

$$S = e_1^n (e_1 + e_2)^{n-1} e_2^{nm}.$$

It is easy to see that S has no innerly joint short minimal zero-sum subsequences. Therefore $\eta^N(G) \ge |S| + 1 = 2n + nm$. □

Lemma 3.3. *Let $G = C_n^r$ with $n, r \in \mathbb{N}$ and $n \ge 2$.*

(1) $\eta^N(G) \ge (2^r - 1)n + 1$.

(2) If G has Property C, then

$$\eta^N(G) \ge \frac{n(\eta(G) - 1)}{n - 1} + 1\,.$$

(3) If $n=3$, then
$$\eta^N(G)=\frac{3(\eta(G)-1)}{2}+1\,.$$

Proof. 1. Let $(e_1,\ldots,e_r)$ be a basis of G and let
$$S=\prod_{\emptyset\neq I\subset[1,r]}\left(\sum_{i\in I}e_i\right)^n\,.$$
Since every short zero-sum subsequence of S has the form $(\sum_{i\in I}e_i)^n$, where $\emptyset\neq I\subset[1,r]$, we have that S has no innerly non-zero-sum-joint short zero-sum subsequences, whence $\eta^N(G)\geq|S|+1=(2^r-1)\cdot n+1$.

2. Let T be a sequence over G of length $\eta(G)-1$ which has no short zero-sum subsequence. It follows from G has Property C that T has the form
$$T=U^{n-1}\,,$$
where U is a squarefree sequence over $G^\bullet$. Let $S=U^n$. Then $S\in\mathcal{F}(G^\bullet)$ has no innerly non-zero-sum-joint short zero-sum subsequences. Thus, $\eta^N(G)\geq|S|+1=\frac{n(\eta(G)-1)}{n-1}+1$.

3. Since $G=C_3^r$ has Property D by Lemma 2.6.4, we have $\mathsf{s}(G)=\eta(G)+2$ and G has Property C. Then it suffices to show $\eta^N(G)\leq\frac{3(\eta(G)-1)}{2}+1$. Let S be a sequence over $G^\bullet$ of length $|S|=\frac{3(\eta(G)-1)}{2}+1$. We need to show that S has two innerly joint short minimal zero-sum subsequences. Assume to the contrary that S has no innerly joint short minimal zero-sum subsequences.

Suppose $S=B_1\cdot\ldots\cdot B_lB_{l+1}\cdot\ldots\cdot B_sS'$, where $s,l\in\mathbb{N}_0$, $|B_i|=3$ for $i\in[1,l]$, $|B_i|=2$ for $i\in[l+1,s]$, $B_1,\ldots,B_s$ are short minimal zero-sum subsequences and S' has no short zero-sum subsequence. For every $i\in[1,s]$, we choose an element $g_i\in\operatorname{supp}(B_i)$. Since S has no innerly joint short minimal zero-sum subsequences, we obtain that $(g_1\cdot\ldots\cdot g_l)^{-1}S$ has no zero-sum subsequence of length 3 and that $(g_1\cdot\ldots\cdot g_s)^{-1}S$ has no short zero-sum subsequence. It follows that $|(g_1\cdot\ldots\cdot g_l)^{-1}S|\leq\mathsf{s}(G)-1=\eta(G)+1$ and $|(g_1\cdot\ldots\cdot g_s)^{-1}S|\leq\eta(G)-1$. Therefore $l\geq\frac{\eta(G)-3}{2}$ and $s\geq\frac{\eta(G)+1}{2}$. It follows that
$$\frac{3(\eta(G)-1)}{2}+1=|S|=3l+2(s-l)+|S'|=l+2s+|S'|\geq\frac{3(\eta(G)-1)}{2}+1\,,$$
which implies that $l=\frac{\eta(G)-3}{2}$, $s=\frac{\eta(G)+1}{2}$, and $|S'|=0$. We infer that $(g_1\cdot\ldots\cdot g_{\frac{\eta(G)+1}{2}})^{-1}S$ has length $\eta(G)-1$ and has no short zero-sum subsequence, a contradiction to the fact that G has Property C. $\square$

Proposition 3.4. *Let $G=C_{mn}^r$, where $r\in\mathbb{N}$ and $m,n\in\mathbb{N}_{\geq2}$. If there exists $c\in\mathbb{N}$ such that $\eta^N(C_m^r)\leq cm+1$ and $\eta(C_n^r)\leq c(n-1)+1$, then $\eta^N(G)\leq cmn+1$.*

Proof. Let S be a sequence over $G^\bullet$ of length $cmn+1$. We need to show that S has two innerly non-zero-sum-joint short zero-sum subsequences. Assume to the contrary that S has no innerly non-zero-sum-joint short zero-sum subsequences.

Let $\varphi:G\to G$ be the multiplication by n. Then $\ker(\varphi)\cong C_n^r$ and $\varphi(G)=nG\cong C_m^r$. Suppose $S=S_0S_1$, where S_0 is a subsequence over $\ker(\varphi)^\bullet$ and S_1 is a subsequence over

$G\setminus\ker(\varphi)$. Suppose $S_0 = T_1\cdot\ldots\cdot T_tT_0$, where $t\in\mathbb{N}_0$, $T_1,\ldots,T_t$ are S_0-innerly disjoint short minimal zero-sum subsequences over $\ker(\varphi)^\bullet$, and T_0 has no short zero-sum subsequence over $\ker(\varphi)^\bullet$. Choose a term h_i of T_i for each $i\in[1,t]$. If $W := (h_1\cdot\ldots\cdot h_t)^{-1}S_0$ has a short minimal zero-sum subsequence T over $\ker(\varphi)^\bullet$, then there exists $i\in[1,t]$ such that T_i and T are S_0-innerly joint, a contradiction. Therefore W has no short zero-sum subsequence over $\ker(\varphi)^\bullet$. Note that $m|W|\ge 2|W|\ge |S_0|$.

Let $k\in\mathbb{N}_0$ be maximal such that there are S_1-innerly disjoint subsequences $V_1,\ldots,V_k$ satisfying the following properties.

- For every $i\in[1,k]$, we have $\varphi(V_i)$ is a short zero-sum subsequence over $\varphi(G)$;
- $W\cdot\sigma(V_1)\cdot\ldots\cdot\sigma(V_k)$ has no short zero-sum subsequence over $\ker(\varphi)$.

Let $U = (V_1\cdot\ldots\cdot V_k)^{-1}S_1$. Then $|W|+k\le \eta(C_n^r)-1$ and

$$|U|\ge |S_1|-km\ge cmn+1-|S_0|-(\eta(C_n^r)-1-|W|)m\ge cm+1\ge \eta^N(C_m^r)\,,$$

whence U has two subsequences U_1,U_2 such that $\varphi(U_1),\varphi(U_2)$ are $\varphi(U)$-innerly joint short minimal zero-sum subsequences and U_1,U_2 have a U-inner common divisor Y such that $\sigma(Y)\notin\ker(\varphi)$. By the maximality of k, there exist subsequences W_1,W_2 of W and subsets $I_1,I_2\subset[1,k]$ such that $\sigma(U_1)W_1\prod_{i\in I_1}\sigma(V_i)$ and $\sigma(U_2)W_2\prod_{i\in I_2}\sigma(V_i)$ are short zero-sum subsequence over $\ker(\varphi)$, whence $X_1 := U_1W_1\prod_{i\in I_1}V_i$ and $X_2 := U_2W_2\prod_{i\in I_2}V_i$ are short zero-sum subsequence over $G^\bullet$. Let Y_0 be a W-inner common divisor of W_1 and W_2. Then X_1 and X_2 have a S-inner common divisor $YY_0\prod_{i\in I_1\cap I_2}V_i$, which is not zero-sum, a contradiction. □

Corollary 3.5. *Let $G=C_{2m}^r$ with $m\ge 2$. If $\eta^N(C_m^r)=(2^r-1)\cdot m+1$, then $\eta^N(G)=(2^r-1)\cdot 2m+1$.*

Proof. By Lemma 3.3.1, it suffices to prove $\eta^N(G)\le (2^r-1)\cdot 2m+1$. Note that $\eta(C_2^r)=2^r=(2^r-1)2-(2^r-2)$. The assertion follows by applying Proposition 3.4 for $c=2^r-1$. □

Now we can prove our main theorem in this subsection.

Theorem 3.6. *Let $r,\alpha\in\mathbb{N}$, $\beta\in\mathbb{N}_0$, and let $n,m\in\mathbb{N}$.*

(1) If $n,m\ge 2$ and $\gcd(n,m)=1$, then $\eta^N(C_n\oplus C_{nm})=2n+nm$
(2) $\eta^N(C_{2^\alpha}^r)=(2^r-1)\cdot 2^\alpha+1$.
(3) $\eta^N(C_{2^\alpha 3^\beta}^3)=7\cdot 2^\alpha 3^\beta+1$ with $\alpha>\beta$.
(4) If n is odd, then $\eta^N(C_n^3)\ge 8n+1$ and $\eta^N(C_{3^\alpha 5^\beta}^3)=8\cdot 3^\alpha 5^\beta+1$.
(5) If n is odd, then $\eta^N(C_n^4)\ge 19n+1$ and $\eta^N(C_{3^\alpha}^4)=19\cdot 3^\alpha+1$.

Proof. 1. Let $G=C_n\oplus C_{nm}$. By Lemma 3.2, it suffices to prove that $\eta^N(G)\le 2n+nm$. Let S be a sequence over $G^\bullet$ of length $2n+nm$. We need to show that S has two innerly non-zero-sum-joint short zero-sum subsequences. Assume to the contrary that S has no innerly non-zero-sum-joint short zero-sum subsequences.

Let $\varphi : G \to G$ be the multiplication by m. Note that $\gcd(n, m) = 1$. We have $\ker(\varphi) \cong C_m$ and $\varphi(G) = mG \cong C_n^2$. Let $S = S_0 S_1$, where S_0 is a subsequence over $\ker(\varphi)^\bullet$ and S_1 is a subsequence over $G \setminus \ker(\varphi)$. Suppose $S_0 = T_1 \cdot \ldots \cdot T_t T_0$, where $t \in \mathbb{N}_0$, $T_1, \ldots, T_t$ are S_0-innerly disjoint short minimal zero-sum subsequences over $\ker(\varphi)^\bullet$, and T_0 has no short zero-sum subsequence over $\ker(\varphi)^\bullet$. Choose a term h_i of T_i for each $i \in [1, t]$. If $W := (h_1 \cdot \ldots \cdot h_t)^{-1} S_0$ has a short minimal zero-sum subsequence T over $\ker(\varphi)^\bullet$, then there exists $i \in [1, t]$ such that T_i and T are S_0-innerly joint, a contradiction. Therefore W has no short zero-sum subsequence over $\ker(\varphi)^\bullet$.

Let $k \in \mathbb{N}_0$ be maximal such that there are S_1-innerly disjoint subsequences $V_1, \ldots, V_k$ satisfying the following properties.

- For every $i \in [1, k]$, we have $\varphi(V_i)$ is a short zero-sum subsequence over $\varphi(G)$;
- $W \cdot \sigma(V_1) \cdot \ldots \cdot \sigma(V_k)$ has no short zero-sum subsequence over $\ker(\varphi)$.

Let $U = (V_1 \cdot \ldots \cdot V_k)^{-1} S_1$. Then $|W| + k \le \eta(C_m) - 1 = m - 1$ and

$$|U| \ge |S_1| - kn \ge 2n + nm - |S_0| - (m - 1 - |W|)n \ge 3n. \tag{3.1}$$

If $\varphi(U)$ has two innerly joint short minimal zero-sum subsequences over $\varphi(G)^\bullet$, whence U has two subsequences U_1, U_2 such that $\varphi(U_1), \varphi(U_2)$ are $\varphi(U)$-innerly joint short minimal zero-sum subsequences and U_1, U_2 have a U-inner common divisor Y such that $\sigma(Y) \notin \ker(\varphi)$. By the maximality of k, there exist subsequences W_1, W_2 of W and subsets $I_1, I_2 \subset [1, k]$ such that $\sigma(U_1) W_1 \prod_{i \in I_1} \sigma(V_i)$ and $\sigma(U_2) W_2 \prod_{i \in I_2} \sigma(V_i)$ are short zero-sum subsequence over $\ker(\varphi)$, whence $X_1 := U_1 W_1 \prod_{i \in I_1} V_i$ and $X_2 := U_2 W_2 \prod_{i \in I_2} V_i$ are short zero-sum subsequence over $G^\bullet$. Let Y_0 be a W-inner common divisor of W_1 and W_2. Then X_1 and X_2 have a S-inner common divisor $Y Y_0 \prod_{i \in I_1 \cap I_2} V_i$, which is not zero-sum, a contradiction.

If $\varphi(U)$ has no innerly joint short minimal zero-sum subsequences over $\varphi(G)^\bullet$, we have $|\varphi(U)| = |U| \le \eta^N(C_n \oplus C_n) - 1 = 3n$. Combining inequality (3.1) and $n|W| \ge 2|W| \ge |S_0|$, we obtain that $n = 2$, $|U| = 6$, $|W| = t$, $k = m-1-t$, and $|T_1| = \ldots = |T_t| = |V_1| = \ldots = |V_k| = 2$. Since $W \cdot \sigma(V_1) \cdot \ldots \cdot \sigma(V_{m-1-t})$ has no short zero-sum subsequence over $\ker(\varphi)$ and $|W \cdot \sigma(V_1) \cdot \ldots \cdot \sigma(V_{m-1-t})| = m - 1 = \eta(C_m) - 1$, it follows from Lemma 2.6.5 that $W \cdot \sigma(V_1) \cdot \ldots \cdot \sigma(V_{m-1-t}) = g^{m-1}$, where g is a generator of $\ker(\varphi) \cong C_m$. Thus $|T_1| = \ldots = |T_t| = 2$ implies that $T_1 = \ldots = T_t = g(-g)$. If $t \ge 2$, then it is easy to see that $g(-g)$ and $g(-g)$ are two S-innerly joint short minimal zero-sum subsequences, a contradiction. If $t = 1$, then it is easy to see that T_1 and $(-g)V_1$ are two S-innerly joint short minimal zero-sum subsequences, a contradiction. Therefore $t = 0$ and $k = m - 1$.

Let $U = g_1 \cdot \ldots \cdot g_6$, where $g_1, \ldots, g_6 \in G \setminus \ker(\varphi)$. Since $\varphi(U)$ has no innerly joint short minimal zero-sum subsequences over $\varphi(G) \cong C_2^2$ and $|\varphi(U)| = \eta^N(C_2 \oplus C_2) - 1 = 6$, after renumbering if necessary, we may assume that $\varphi(g_1 g_2) = e_1^2$, $\varphi(g_3 g_4) = e_2^2$, and $\varphi(g_5 g_6) = e_3^2$, where $\{e_1, e_2, e_3\} = \varphi(G) \setminus \{0\}$. Let $L_1 = g_1 g_2$, $L_2 = g_3 g_4$, $L_3 = g_5 g_6$, $M_1 = g_1 g_3 g_5$, and $M_2 = g_2 g_4 g_6$. For every $L \in \{L_1, L_2, L_3, M_1, M_2\}$, we have $\sigma(L) \in$

$\ker(\varphi) = \{0, g, \ldots, (m-1)g\}$. We claim that $\sigma(L) \in \{0, g\}$. Assume to the contrary that $\sigma(L) = xg$ with $2 \le x \le m-1$. Then $LV_1 \cdot \ldots \cdot V_{m-x}$ and $LV_1 \cdot \ldots \cdot V_{m-x-1}V_{m-x+1}$ are short zero-sum subsequence of S over $G^\bullet$ with a non-zero-sum S-inner common divisor $LV_1 \cdot \ldots \cdot V_{m-x-1}$, a contradiction. Therefore $\sigma(L) \in \{0, g\}$.

If there exist l_1, l_2 with $\{l_1, l_2\} \subset \{1, 2, 3\}$ such that $\sigma(L_{l_1}) = \sigma(L_{l_2}) = g$, then $L_{l_1}V_1 \cdot \ldots \cdot V_{m-1}$ and $L_{l_2}V_1 \cdot \ldots \cdot V_{m-1}$ are short zero-sum subsequence of S over $G^\bullet$ with a non-zero-sum S-inner common divisor $V_1 \cdot \ldots \cdot V_{m-1}$, a contradiction. Thus, after renumbering if necessary, we may assume that $\sigma(L_1) = \sigma(L_2) = 0$. If there exists $i \in [1, 2]$ such that $\sigma(M_i) = 0$, then $L_1 = g_1g_2$ and M_i are short zero-sum subsequence of S over $G^\bullet$ with a non-zero-sum S-inner common divisor g_i, a contradiction. Then $\sigma(M_1) = \sigma(M_2) = g$, which implies that $\sigma(L_3) = g_5 + g_6 = \sigma(L_1) + \sigma(L_2) + g_5 + g_6 = \sigma(M_1) + \sigma(M_2) = 2g$, a contradiction.

2. If $\alpha = 1$, then the assertion follows from Lemma 3.1. Suppose $\alpha \ge 2$. We proceed by induction on α and the assertion follows by applying Corollary 3.5.

3. Let $G = C_{2^\alpha 3^\beta}^3$ with $\alpha > \beta$. By Lemma 3.3.1, we have $\eta^N(G) \ge 7 \cdot 2^\alpha 3^\beta + 1$. It suffices to show the upper bound. Since Lemma 3.1 implies that $\eta^N(C_2^3) = 7 \cdot 2 + 1$ and Lemma 2.12 implies that $\eta(C_{2^{\alpha-1}3^\beta}^3) = 7 \cdot 2^{\alpha-1}3^\beta - 6$, it follows by Proposition 3.4 that $\eta^N(G) \le 7 \cdot 2^\alpha 3^\beta + 1$.

4. Let $G = C_n^3$. By Lemma 2.7, there exists a sequence $T \in \mathcal{F}(G)$ of length $|T| = 9$ such that T^{n-1} has no zero-sum subsequence of length n. Let $g \mid T$ and set $S' = (0^{-1}(-g+T))^n$. It is easy to see that $S' \in \mathcal{F}(G^\bullet)$ has no innerly non-zero-sum-joint short zero-sum subsequences, whence $\eta^N(G) \ge |S'| + 1 = 8n + 1$.

Suppose $n = 3^\alpha 5^\beta$. It suffices to show $\eta^N(G) \le 8n + 1$. The result follows from Lemma 3.3.3, Proposition 3.4 and Lemma 2.10.

5. Let $G = C_n^4$. By Lemma 2.8, there exists a sequence $T \in \mathcal{F}(G)$ of length $|T| = 20$ such that T^{n-1} has no zero-sum subsequence of length n. Let $g \mid T$ and set $S' = (0^{-1}(-g+T))^n$. It is easy to see that $S' \in \mathcal{F}(G^\bullet)$ has no innerly non-zero-sum-joint short zero-sum subsequences, whence $\eta^N(G) \ge |S'| + 1 = 19n + 1$.

Suppose $n = 3^\alpha$. It suffices to show $\eta^N(G) \le 19n+1$. The result follows from Lemma 3.3.3, Proposition 3.4 and Lemma 2.10. □

3.2. **On $\mathsf{D}^N(G)$ and $\mathsf{N}_1(G)$.** In this subsection, our main theorem is Theorem 3.14. Note that $\mathsf{D}^N(G) = \mathsf{N}_1(G) + 1$. We first collect some known results of $\mathsf{N}_1(G)$ and $\mathsf{D}(G)$.

Lemma 3.7. *Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ be a finite abelian group, where $1 \le n_1 \mid \ldots \mid n_r$. Then $\mathsf{N}_1(G) = n_1 + \ldots + n_r$ provided that G has one of the following forms.*

(1) $G = C_n$ with $n \ge 2$.

(2) $G = C_{n_1} \oplus C_{n_2}$ with $1 < n_1 \mid n_2$.

(3) $G = C_2^r$ with $r \in \mathbb{N}$.

(4) $G = C_3^r$ *with* $r \in \mathbb{N}$.
(5) $G = C_2^r \oplus C_4^t \oplus C_{2^m}$ *with* $0 \le t \le 1$ *and* $m \ge 1$.
(6) $G = C_2^r \oplus C_4^t \oplus C_{2^m l}$ *with* $0 \le t \le 1, l \ge 4,$ *and* $2^m \ge r + 3t + 1$.
(7) $G = C_3^r \oplus C_9^t \oplus C_{3^m}$ *with* $0 \le t \le 1$ *and* $m \ge 1$.
(8) $G = C_3^r \oplus C_9^t \oplus C_{3^m l}$ *with* $0 \le t \le 1, l \ge 4,$ *and* $3^m \ge 2r + 8t + 1$.
(9) $G = C_5^2 \oplus C_{25m}$ *with either* $m = 1$ *or* $m \ge 4$.

Proof. Item 2 follows from [GPZ13, Theorem 2.3]. For Items 1, 3, and 4, see [GH06, Theorem 6.2.8] and for Items 5-9, see [G97, Theorem 1]. □

Lemma 3.8. *Let* p *be a prime and* $m, n \in \mathbb{N}$. *Then* $\mathsf{D}(G) = \mathsf{D}^*(G)$ *holds for one of the following groups.*

(1) $G = C_n \oplus C_{mn}$.
(2) G *is a finite abelian* p*-group.*
(3) $G = C_2 \oplus C_{2m} \oplus C_{2mn}$.
(4) $G = C_2^3 \oplus C_{2m}$ *with* m *odd.*

Proof. For Items 1 and 2, see [GH06, Theorems 5.8.3 and 5.5.9]. Item 3 follows from [GS19, Theorem 2.7] and Item 4 follows from [E69, Page 1]. □

We need the following lemmas.

Lemma 3.9. *Let* G *be a finite abelian group with* $|G| > 1$ *and let* $T = U_1 \cdot \ldots \cdot U_r$ *be a sequence over* $G^\bullet$, *where* $r \in \mathbb{N}$ *and* $U_1, \ldots, U_r$ *are minimal zero-sum sequences. If* T *has no innerly joint minimal zero-sum subsequences, then* $\prod_{i=1}^r |U_i| \le |G|$.

Proof. See [GGW11, Lemma 3.9]. □

Lemma 3.10. *Let* G *be a finite abelian* p*-group with* p *an odd prime and let* $B = B_1 \cdot \ldots \cdot B_r$ *be a sequence over* $G^\bullet$, *where* $r \in \mathbb{N}$ *and* $B_1, \ldots, B_r$ *are minimal zero-sum subsequences. Suppose* B *has no innerly joint minimal zero-sum subsequences. If exactly* t *of* $|B_1|, \ldots, |B_r|$ *are odd, then* $|B| \le \mathsf{D}(G) + t - 1$.

Proof. See [G97, Proposition 1]. □

Definition 3.11. Let $S = g_1 \cdot \ldots \cdot g_l \in \mathcal{F}(G)$ be a sequence of length $|S| = l \in \mathbb{N}_0$ and let $g \in G$.

(1) For every $k \in \mathbb{N}_0$, let
$$\mathbf{N}_g^k(S) = \left| \left\{ I \subset [1, l] \,:\, \sum_{i \in I} g_i = g \text{ and } |I| = k \right\} \right|$$
denote the number of distinct subsequences T in S having sum $\sigma(T) = g$ and length $|T| = k$.

(2) We define

$$\mathbf{N}_g(S)=\sum_{k\ge 0}\mathbf{N}_g^k(S),\quad \mathbf{N}_g^+(S)=\sum_{k\ge 0}\mathbf{N}_g^{2k}(S),\quad \mathbf{N}_g^-(S)=\sum_{k\ge 0}\mathbf{N}_g^{2k+1}(S)\,.$$

Thus $\mathbf{N}_g(S)$ denotes the number of distinct subsequences T in S with $\sigma(T)=g$, $\mathbf{N}_g^+(S)$ denotes the number of such subsequences in S of even length, and $\mathbf{N}_g^-(S)$ denotes the number of such subsequences in S of odd length.

Lemma 3.12. *Let G be a finite abelian p-group and let S be a sequence over G of length $\mathsf{D}(G)-2$. Suppose that $\mathbf{N}_0^+(S)\not\equiv\mathbf{N}_0^-(S) \pmod p$. Then there exist a subgroup H of G and an element $x\in G\setminus H$ such that $G\setminus\Sigma(S)\subset x+H$.*

Proof. See [G97, Lemma 10]. □

Lemma 3.13. *Let $G=C_p^r$, where $p\ge 3$ is prime and $r\ge 2$, and let $S=S_1\cdot\ldots\cdot S_l$ be a sequence over $G^\bullet$, where $l\in\mathbb{N}$ and $S_1,\ldots,S_l$ are minimal zero-sum subsequences. Suppose S has no innerly joint minimal zero-sum subsequences and that $|S|=rp+t$ with $t\ge 1$. Then at least $t+r+2$ of $|S_1|,\ldots,|S_l|$ are odd.*

Proof. Suppose that exactly k of $|S_1|,\ldots,|S_l|$ are odd. Note that $\mathsf{D}(G)=r(p-1)+1$ by Lemma 3.8.2. Then $k\ge t+r$ follows from Lemma 3.10. We need to show that $k\ge t+r+2$. Assume to the contrary that $k\le t+r+1$. Then $k=t+r$ by $k\equiv rp+t\equiv r+t \pmod 2$. After renumbering if necessary, we may assume that $|S_1|,\ldots,|S_{r+t}|$ are odd and $|S_{r+t+1}|,\ldots,|S_l|$ are even.

Let $a_i\in\operatorname{supp}(S_i)$ for every $i\in[1,r+t]$, choose a term x of $a_1^{-1}S_1$, and set

$$T=a_1^{-1}x^{-1}S_1a_2^{-1}S_2\cdot\ldots\cdot a_{r+t}^{-1}S_{r+t}S_{r+t+1}\cdot\ldots\cdot S_l\,.$$

Then $\mathbf{N}_0^+(T)=2^{l-r-t}$, $\mathbf{N}_0^-(T)=0$, $|T|=r(p-1)-1=\mathsf{D}(G)-2$, and

$$\{-a_1,-a_1-a_2,\ldots,-a_1-a_{r+t},-x,-x-a_2,\ldots,-x-a_{r+t}\}\cap\Sigma(T)=\emptyset.$$

It follows from Lemma 3.12 that there exist a subgroup H of G and an element $g\in G\setminus H$ such that

$$\{-a_1,-a_1-a_2,\ldots,-a_1-a_{r+t},-x,-x-a_2,\ldots,-x-a_{r+t}\}\subset g+H\,.$$

This implies that $x-a_1=(-a_1)-(-x)\in H$. Since x was chosen arbitrarily, we obtain S_1 is over $a_1+H=-g+H$. In view of $\sigma(S_1)=0$, we obtain $|S_1|g\in H$ and hence $p\mid |S_1|$. Similarly, we can show

$$p\mid |S_2|,\ldots,p\mid |S_{r+t}|\,,$$

which implies that $|S|\ge |S_1|+\ldots+|S_{r+t}|\ge p(r+t)>rp+t$, a contradiction. Thus $k\ge t+r+2$ and we are done. □

Now we are ready to state our main theorem of this subsection.

Theorem 3.14. *Let $G=C_{n_1}\oplus\ldots\oplus C_{n_r}$ be a finite abelian group, where $1\le n_1\mid\ldots\mid n_r$. Then $\mathsf{N}_1(G)=n_1+\ldots+n_r$ holds for any one of the following groups.*

(1) $G = C_2^r \oplus C_{2m}$ *with* $m \in \mathbb{N}$ *and* $r \in [2,3]$.
(2) $G = C_5^r$ *with* $r \in [1,9]$.
(3) $G = C_7^r$ *with* $r \in [1,6]$.
(4) $G = C_{11}^r$ *with* $r \in [1,4]$.
(5) $G = C_p^r$ *with* $r \in [1,3]$ *and* $p \in \{13, 17, 19, 23\}$.

Proof. 1. We only prove the case that $r = 2$, since the proof is similar for $r = 3$. If $m = 1$, the assertion follows from Lemma 3.7.3. Suppose $m \ge 2$. By (2.1), it suffices to show that $\mathsf{N}_1(G) \le 2m + 4$. Let S be a zero-sum sequence over $G^\bullet$ of length $|S| \ge 2m + 5$. We need to show that S has two innerly joint minimal zero-sum subsequences. Assume to the contrary that S has no innerly joint minimal zero-sum subsequences.

Let $S = U_1 \cdot \ldots \cdot U_t$, where $U_1, \ldots, U_t$ are minimal zero-sum subsequences. Choose $g_i \in \operatorname{supp}(U_i)$ for every $i \in [1,t]$. Then $T := (g_1 \cdot \ldots \cdot g_t)^{-1} S$ has no zero-sum subsequence. If $t \le 3$, then by Lemma 3.8.3 $|T| \ge 2m + 2 = \mathsf{D}(G)$, a contradiction. If $t \ge 4$, then Lemma 3.9 implies that $|G| \ge \prod_{i=1}^t |U_i| \ge 2^3(2m-1) > |G|$, a contradiction.

2. By (2.1), it suffices to show that $\mathsf{N}_1(G) \le 5r$, where $r \in [1,9]$. Let S be a zero-sum sequence over $G^\bullet$ of length $|S| = 5r + k$ with $k \ge 1$. We need to show that S has two innerly joint minimal zero-sum subsequences. Assume to the contrary that S has no innerly joint minimal zero-sum subsequences.

Let $S = U_1 \cdot \ldots \cdot U_t U_{t+1} \cdot \ldots \cdot U_l$, where $U_1, \ldots, U_l$ are minimal zero-sum subsequences such that $|U_i|$ is odd for every $i \in [1,t]$ and $|U_i|$ is even for every $i \in [t+1, l]$. By Lemma 3.13, we have $t \ge r + k + 2 \ge r + 3$.

If $r < 3 + k$, then $5r + k = |S| \ge 3(r + k + 2) > 5r + k$, a contradiction. If $r \ge 3 + k \ge 4$, then Lemma 3.9 implies that

$$|G| \ge \prod_{i=1}^{l} |U_i| \ge 3^{r+2}(2r-5) > 5^r = |G|,$$

a contradiction.

3. By (2.1), it suffices to show that $\mathsf{N}_1(G) \le 7r$, where $r \in [1,6]$. Let S be a zero-sum sequence over $G^\bullet$ of length $|S| = 7r + k$ with $k \ge 1$. We need to show that S has two innerly joint minimal zero-sum subsequences. Assume to the contrary that S has no innerly joint minimal zero-sum subsequences.

Let $S = U_1 \cdot \ldots \cdot U_t U_{t+1} \cdot \ldots \cdot U_l$, where $U_1, \ldots, U_l$ are minimal zero-sum subsequences such that $|U_i|$ is odd for every $i \in [1,t]$ and $|U_i|$ is even for every $i \in [t+1, l]$. By Lemma 3.13, we have $t \ge r + k + 2 \ge r + 3$.

If $r = 1$, then Lemma 3.9 implies that

$$|G| \ge \prod_{i=1}^{l} |U_i| \ge 3^4 > 7 = |G|,$$

a contradiction. If $r \geq 2$, then Lemma 3.9 implies that

$$|G| \geq \prod_{i=1}^{l} |U_i| \geq 3^{r+2}(4r-5) > 7^r = |G|\,,$$

a contradiction.

4. By (2.1), it suffices to show that $\mathsf{N}_1(G) \leq 11r$, where $r \in [1,4]$. Let S be a zero-sum sequence over $G^\bullet$ of length $|S| = 11r + k$ with $k \geq 1$. We need to show that S has two innerly joint minimal zero-sum subsequences. Assume to the contrary that S has no innerly joint minimal zero-sum subsequences.

Let $S = U_1 \cdot \ldots \cdot U_t U_{t+1} \cdot \ldots \cdot U_l$, where $U_1, \ldots, U_l$ are minimal zero-sum subsequences such that $|U_i|$ is odd for every $i \in [1,t]$ and $|U_i|$ is even for every $i \in [t+1,l]$. By Lemma 3.13, we have $t \geq r+k+2 \geq r+3$.

Then Lemma 3.9 implies that

$$|G| \geq \prod_{i=1}^{l} |U_i| \geq 3^{r+2}(8r-5) > 11^r = |G|\,,$$

a contradiction.

5. By (2.1), it suffices to show that $\mathsf{N}_1(G) \leq pr$, where $r \in [1,3]$ and $p \in \{13,17,19,23\}$. Let S be a zero-sum sequence over $G^\bullet$ of length $|S| = pr+k$ with $k \geq 1$. We need to show that S has two innerly joint minimal zero-sum subsequences. Assume to the contrary that S has no innerly joint minimal zero-sum subsequences.

Let $S = U_1 \cdot \ldots \cdot U_t U_{t+1} \cdot \ldots \cdot U_l$, where $U_1, \ldots, U_l$ are minimal zero-sum subsequences such that $|U_i|$ is odd for every $i \in [1,t]$ and $|U_i|$ is even for every $i \in [t+1,l]$. By Lemma 3.13, we have $t \geq r+k+2 \geq r+3$.

Then Lemma 3.9 implies that

$$|G| \geq \prod_{i=1}^{l} |U_i| \geq 3^{r+2}((p-3)r-5) > p^r = |G|\,,$$

a contradiction. □

4. On $\mathsf{s}^N(G)$

In this section we shall investigate $\mathsf{s}^N(G)$. We first introduce two more invariants which are lower and upper bounds of $\mathsf{s}^N(G)$. We define

- $\mathsf{s}^*(G)$ to be the smallest integer t such that every sequence S of length t over $G^\bullet$ has two innerly joint zero-sum subsequences of length $\exp(G)$.
- $\mathsf{s}^{**}(G)$ to be the smallest integer t such that every sequence S of length t over G with $\mathsf{v}_0(S) \leq \exp(G)$ has two innerly non-zero-sum-joint zero-sum subsequences of length $\exp(G)$.

It follows from the definitions of $\mathsf{s}^*(G)$, $\mathsf{s}^{**}(G)$ and $\mathsf{s}^N(G)$ that

$$\mathsf{s}^{**}(G) \geq \mathsf{s}^N(G) \geq \mathsf{s}^*(G)\,.$$

We will frequently use the following easy observation without further mention.

Lemma 4.1. *Let G be a finite abelian group and let S be a sequence over G with $\mathsf{v}_0(S) \le k\exp(G)$ such that S has no innerly non-zero-sum-joint zero-sum subsequences of length $k\exp(G)$, where $k \in \mathbb{N}$.*

(1) Let T be a zero-sum subsequence of S of length $|T| = k\exp(G)$. Then $\mathsf{v}_g(T) = \mathsf{v}_g(S)$ for every $g \in \operatorname{supp}(T) \setminus \{0\}$. In particular, $\mathsf{h}(S) \le k\exp(G)$.

(2) Suppose $S = T_1 \cdot \ldots \cdot T_r T_0$, where $r \in \mathbb{N}_0$, $T_1, \ldots, T_r$ are zero-sum subsequences of S of length $|T_1| = \ldots = |T_r| = k\exp(G)$, and T_0 has no zero-sum subsequence of length $k\exp(G)$. Then $\operatorname{supp}(T_i) \cap \operatorname{supp}(T_j) = \emptyset$ or $\{0\}$ for any distinct $i, j \in [0, r]$.

Proof. 1. Assume to the contrary that there exists $g \in \operatorname{supp}(T) \setminus \{0\}$ such that $\mathsf{v}_g(T) < \mathsf{v}_g(S)$. Then Tg divides S and $g^{-1}T$ is not zero-sum, whence T and T have a non-zero-sum S-inner common divisor $g^{-1}T$, a contradiction.

The "in particular" part follows from the fact that for every $g \in G^\bullet$, we have $g^{k\exp(G)}$ is a zero-sum sequence of length $k\exp(G)$.

2. Assume to the contrary that there exist distinct $i, j \in [0, r]$ such that $g \in \operatorname{supp}(T_i) \cap \operatorname{supp}(T_j)$ and $g \in G^\bullet$. By symmetry we may assume that $i \ge 1$ and hence $\mathsf{v}_g(T_i) < \mathsf{v}_g(S)$, a contradiction to Item 1. □

Lemma 4.2. *$\mathsf{s}^*(C_2^r) = \mathsf{s}^N(C_2^r) = 2^{r+1} - 1$ and $\mathsf{s}^{**}(C_2^r) = 2^{r+1} + 1$, where $r \in \mathbb{N}$.*

Proof. Let $G = C_2^r$. We first show that $\mathsf{s}^*(G) = \mathsf{s}^N(G) = 2^{r+1} - 1$. Let

$$S = \prod_{g \in G^\bullet} g^2 \,,$$

and it is easy to see that $S \in \mathcal{F}(G^\bullet)$ has no innerly joint zero-sum subsequences of length 2. It follows that $\mathsf{s}^*(G) \ge |S| + 1 = 2(2^r - 1) + 1 = 2^{r+1} - 1$.

We only need to show $\mathsf{s}^N(G) \le 2^{r+1} - 1$. Let U be a sequence over $G^\bullet$ of length $2^{r+1} - 1$. Then $\mathsf{h}(U) \ge |U|/(|G| - 1) > 2$, whence there exists $g \in G^\bullet$ such that $\mathsf{v}_g(U) \ge 3$. By Lemma 4.1.1 we obtain that U has two innerly non-zero-sum-joint zero-sum subsequences of length 2 and we are done.

We next show that $\mathsf{s}^{**}(G) = 2^{r+1} + 1$. Let

$$S' = \prod_{g \in G} g^2 \,.$$

Then $\mathsf{v}_0(S') = 2$ and S' has no innerly non-zero-sum-joint zero-sum subsequences of length 2, whence $\mathsf{s}^{**}(G) \ge |S'| + 1 = 2^{r+1} + 1$. Let U' be a sequence over G of length $2^{r+1} + 1$ with $\mathsf{v}_0(U') \le 2$. Since $(|U'| - \mathsf{v}_0(U'))/(|G| - 1) > 2$, there exists $g \in G^\bullet$ such that $\mathsf{v}_g(U') \ge 3$. By Lemma 4.1.1 we obtain that U' has two innerly non-zero-sum-joint zero-sum subsequences of length 2, whence $\mathsf{s}^{**}(G) \le 2^{r+1} + 1$ and we are done. □

Remark: Let G be a finite abelian group such that $\exp(G) \ge 3$. Then $S = \prod_{g \in G} g^{\exp(G)}$ has a zero-sum subsequence T of length $\exp(G)$ such that $|\operatorname{supp}(T)| \ge 2$, whence S

has two innerly non-zero-sum-joint zero-sum subsequences of length $\exp(G)$ and hence $\mathsf{s}^{**}(G) \le |G| \exp(G)$.

We distinguish three subsections depending on the rank of the groups.

4.1. **Cyclic groups.** Let G be a finite abelian group and let $A_1, A_2, \ldots, A_h$ be nonempty subsets of G, where $h \ge 2$. We define

$$A_1 + \ldots + A_h = \{a_1 + \ldots + a_h \colon a_i \in A_i \text{ for } i \in [1, h]\} .$$

The following lemma is the famous Cauchy-Davenport Theorem.

Lemma 4.3. *Let G be a cyclic group of prime order p and let $A_1, \ldots, A_h$ be nonempty subsets of G, where $h \ge 2$. Then*

$$|A_1 + \ldots + A_h| \ge \min\{p, \sum_{i=1}^{h} |A_i| - h + 1\} .$$

Proof. See [N96, Theorem 2.3]. □

Lemma 4.4. *Let $G = C_n$, where $n \ge 3$. Then $\mathsf{s}^{**}(G) \ge \mathsf{s}^{*}(G) \ge 2n + 1$.*

Proof. Let $g \in G$ such that $\operatorname{ord}(g) = n$ and let $S = g^n (2g)^n$. Then S has only two zero-sum subsequences g^n and $(2g)^n$ of length n. Since g^n and $(2g)^n$ are not S-innerly joint, we have $\mathsf{s}^{*}(G) \ge |S| + 1 = 2n + 1$. □

Lemma 4.5. *If p is a prime, then $\mathsf{s}^{**}(C_p) = 2p + 1$.*

Proof. By Lemmas 4.2 and 4.4, it suffices to prove that $\mathsf{s}^{**}(C_p) \le 2p + 1$. Let S be a sequence over C_p of length $|S| = 2p + 1$ such that $\mathsf{v}_0(S) \le p$. We need to show that S has two innerly non-zero-sum-joint zero-sum subsequences of length p. Assume to the contrary that S has no innerly non-zero-sum-joint zero-sum subsequences of length p. It follows from Lemma 4.1.1 that $\mathsf{v}_g(S) \le p$ for every $g \in C_p$.

Note that there exists $g \in C_p^{\bullet}$ such that $\mathsf{v}_g(S) \ge 2$. Set $T = (g^2)^{-1} S$. It follows from $\mathsf{h}(T) \le p$ that there exist T-innerly disjoint squarefree subsequences $A_1, \ldots, A_{p-1}$ of length $|A_1| = \ldots = |A_{p-1}| = 2$ such that $A_1 \cdot \ldots \cdot A_{p-1}$ divides T. Applying Lemma 4.3, we obtain that $|A_1 + \ldots + A_{p-1}| \ge \min\{p, |A_1| + \ldots + |A_{p-1}| - (p - 2)\} = p$, whence $A_1 + \ldots + A_{p-1} = C_p$. Let W be a subsequence of $A_1 \cdot \ldots \cdot A_{p-1}$ of length $p - 1$ such that $\sigma(W) = -g$. Then gW is a zero-sum subsequence of S of length p with $\mathsf{v}_g(gW) < \mathsf{v}_g(S)$, a contradiction to Lemma 4.1.1. □

Lemma 4.6. *Let G be a cyclic group of order $m \ge 2$ and let $n \ge m$. Let S be a nontrivial sequence over G such that $\mathsf{v}_0(S) \le mn - 3$. Suppose that $\mathsf{s}^{**}(C_m) = 2m + 1$ and that S has no innerly non-zero-sum-joint zero-sum subsequences of length mn. Then $|S| \le mn + 2m - 3$ and there exist $t \in \mathbb{N}_0$ and S-innerly disjoint subsequences $S_1, \ldots, S_t$ such that $|S_i| = n$ for each $i \in [1, t]$, $|(S_1 \cdot \ldots \cdot S_t)^{-1} S| \le n$, and the sequence $\sigma(S_1) \cdot \ldots \cdot \sigma(S_t)$ has no zero-sum subsequence of length m.*

Proof. Assume to the contrary that $|S| \ge mn + 2m - 2$. Then $|(0^{\mathsf{v}_0(S)})^{-1}S| \ge 2m+1$. It follows from $\mathsf{s}^{**}(C_m) = 2m+1$ that S has two innerly non-zero-sum-joint zero-sum subsequences W_0 and W_1 of length $|W_0| = |W_1| = m$. Let Y be a non-zero-sum S-inner common divisor of W_0 and W_1 and let $W = (W_0W_1)^{-1}YS$. Then, $|W| \ge |S| - (2m-1) \ge (n-2)m + 2m - 1$. Now applying $\mathsf{s}(C_m) = 2m-1$ repeatedly to W, one can find $n-1$ W-innerly disjoint zero-sum subsequences $W_2, \ldots, W_n$ of length $|W_2| = \ldots = |W_n| = m$. Set $T_1 = W_0 \prod_{i=2}^n W_i$ and $T_2 = W_1 \prod_{i=2}^n W_i$. Now T_1 and T_2 are two zero-sum subsequences of S of length $|T_1| = |T_2| = mn$, and T_1 and T_2 have a non-zero-sum S-inner common divisor $Y \prod_{i=2}^n W_i$, a contradiction.

Therefore $|S| \le mn + 2m - 3$. If $|S| = tn + r \le mn$, where $t \in [0, m-1]$ and $r \in [1, n]$, then choose S-innerly disjoint subsequences $S_1, \ldots, S_t$ of length $|S_1| = \ldots = |S_t| = n$, whence $\sigma(S_1) \cdot \ldots \cdot \sigma(S_t)$ has no zero-sum subsequence of length m.

Suppose that $mn + 1 \le |S| \le mn + n$. Let $S_1, \ldots, S_m$ be m S-innerly disjoint subsequences of length $|S_1| = \ldots = |S_m| = n$ such that $\operatorname{supp}((S_1 \cdot \ldots \cdot S_m)^{-1}S) \not\subset \{0\}$. Set $S' = (S_1 \cdot \ldots \cdot S_m)^{-1}S$. Then $|S'| \le n$. If $\sigma(S_1) + \ldots + \sigma(S_m) \ne 0$, then we are done. Otherwise choose a term $x \ne 0$ of S' and a term y of S_m. Let $S'_m = y^{-1}xS_m$. If $\sigma(S_1) + \ldots + \sigma(S_{m-1}) + \sigma(S'_m) \ne 0$, then we are done. Otherwise $x = y$ and there are two zero-sum subsequences $S_1 \cdot \ldots \cdot S_m$ and $S_1 \cdot \ldots \cdot S_{m-1}S'_m$ with a non-zero-sum S-inner common divisor $x^{-1}S_1 \cdot \ldots \cdot S_m$, a contradiction.

Suppose $mn + n + 1 \le |S| \le mn + 2n - 3$. Then $n \ge 4$. Set $\mathsf{v}_0(S) = t_1 n + r_1$, where $t_1 \in [0, m-1]$ and $r_1 \in [0, n-1]$ and $\ell = |S| - \mathsf{v}_0(S) \ge mn + n + 1 - (mn - 3) = n + 4$. Let $U = a_1 \cdot \ldots \cdot a_x$ be the maximal subsequence of $(0^{\mathsf{v}_0(S)})^{-1}S$ such that U^2 divides $(0^{\mathsf{v}_0(S)})^{-1}S$, then $2x = |U^2| = |(0^{\mathsf{v}_0(S)})^{-1}S| - |\operatorname{supp}((0^{\mathsf{v}_0(S)})^{-1}S)| \ge l - (m-1)$. We assert that $x \ge m+1-t_1$. If $t_1 = m-1$, then $2x - 2(m+1-t_1) \ge \ell - (m-1) - 4 \ge n + 4 - m - 3 > 0$. If $t_1 \le m-2$, then

$$\begin{aligned}
2x - 2(m+1-t_1) &\ge \ell - (m-1) - 2(m+1-t_1) \\
&\ge mn + n + 1 - (t_1 n + r_1) - (m-1) - 2(m+1-t_1) \\
&= (m+1-t_1)(n-2) - m - r_1 + 2 \\
&\ge 3(n-2) - m - (n-1) + 2 \\
&\ge n - 3 > 0\,,
\end{aligned}$$

whence $x \ge m + 1 - t_1$. By this assertion, we can choose S-innerly disjoint subsequences $S_1, \ldots, S_{m+1}$ of length $|S_1| = \ldots = |S_{m+1}| = m$ such that $S_1 = \ldots = S_{t_1} = 0^n$, a_1 divides S_{t_1+1}, $a_i a_{i+1}$ divides S_{t_1+i+1} for $i \in [1, m-t_1]$, and a_{m+1-t_1} divides $(S_1 \cdot \ldots \cdot S_{m+1})^{-1}S$. If $\sigma(S_1) \cdot \ldots \cdot \sigma(S_{m+1})$ has no zero-sum subsequence of length m, then we are done. If there exists $i \in [1, t_1]$, say $i = 1$, such that $\sigma(S_2) \cdot \ldots \cdot \sigma(S_{m+1})$ is zero-sum, then $V_1 := S_2 \cdot \ldots \cdot S_{m+1}$ is a zero-sum subsequence of length mn and $\mathsf{v}_{a_{m+1-t_1}}(V_1) < \mathsf{v}_{a_{m+1-t_1}}(S)$, a contradiction to Lemma 4.1.1. If there exists $i \in [t_1+1, m]$ such that $\sigma(S_1) \cdot \ldots \cdot$

$\sigma(S_{i-1})\sigma(S_{i+1}) \cdot \ldots \cdot \sigma(S_{m+1})$ is zero-sum, then $V_i := S_1 \cdot \ldots \cdot S_{i-1}S_{i+1} \cdot \ldots \cdot S_{m+1}$ is a zero-sum subsequence of length mn and $\mathsf{v}_{a_{i-t_1}}(V_i) < \mathsf{v}_{a_{i-t_1}}(S)$, a contradiction to Lemma 4.1.1. If $\sigma(S_1) \cdot \ldots \cdot \sigma(S_m)$ is zero-sum, then $V' := S_1 \cdot \ldots \cdot S_m$ is a zero-sum subsequence of length mn and $\mathsf{v}_{a_{m-t_1}}(V') < \mathsf{v}_{a_{m-t_1}}(S)$, a contradiction to Lemma 4.1.1. □

Lemma 4.7. *If* $\mathsf{s}^{**}(C_m) = 2m + 1$ *and* $\mathsf{s}^{**}(C_n) = 2n + 1$, *then* $\mathsf{s}^{**}(C_{mn}) = 2mn + 1$, *where* $m, n \ge 2$.

Proof. By symmetry, we may suppose $n \ge m \ge 2$. Let $G = C_{mn}$. By Lemma 4.4 it suffices to prove the upper bound. Let S be a sequence of length $|S| = 2mn + 1$ such that $\mathsf{v}_0(S) \le mn$. We need to show that S has two innerly non-zero-sum-joint zero-sum subsequences of length mn. Assume to the contrary that S has no innerly non-zero-sum-joint zero-sum subsequences of length mn. Let $S' = (0^{\mathsf{v}_0(S)})^{-1}S$. Then $|S'| \ge mn + 1 = \eta^N(C_{mn})$ by Lemma 3.1.1, whence S' has two innerly non-zero-sum-joint short minimal zero-sum subsequences W_1 and W_2. Let Y be a non-zero-sum S'-inner common divisor of W_1 and W_2. If $\mathsf{v}_0(S) \ge mn - 2$, then $W_3 := 0^{mn-|W_1|}W_1$ and $W_4 := 0^{mn-|W_2|}W_2$ are zero-sum subsequences of S of length mn and have a non-zero-sum S-inner common divisor 0^sY, where $s \in [0, \min\{mn - |W_1|, mn - |W_2|\}]$, a contradiction.

Therefore $\mathsf{v}_0(S) \le mn - 3$. Let $\varphi\colon G \to G$ denote the multiplication by m. Then $\ker(\varphi) \cong C_m$ and $\varphi(G) = mG \cong C_n$. Set $S = g_1 \cdot \ldots \cdot g_l$, where $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G$, such that $\varphi(g_i) = 0$ for all $i \in [1, t]$ and $\varphi(g_i) \ne 0$ for all $i \in [t + 1, l]$, where $t \in [0, l]$. Since the sequence $g_1 \cdot \ldots \cdot g_t$ has no innerly non-zero-sum-joint zero-sum subsequences of length mn, it follows by Lemma 4.6 that $t \le mn+2m-3$ and there exist $g_1 \cdot \ldots \cdot g_t$-innerly disjoint subsequences $S_1, \ldots, S_{u_0}$ of length $|S_1| = \ldots = |S_{u_0}| = n$ such that

- the sequence $\sigma(S_1) \cdot \ldots \cdot \sigma(S_{u_0})$ has no zero-sum subsequence of length m;
- $|(S_1 \cdot \ldots \cdot S_{u_0})^{-1} g_1 \cdot \ldots \cdot g_t| \le n$.

Let u_1 be the maixmal integer such that we can find $(S_1 \cdot \ldots \cdot S_{u_0})^{-1}S$-innerly disjoint subsequences $S_{u_0+1}, \ldots, S_{u_0+u_1}$ of length $|S_{u_0+1}| = \ldots = |S_{u_0+u_1}| = n$ satisfying the following properties.

- $\varphi(S_i)$ is zero-sum for every $i \in [1, u_0 + u_1]$;
- $\sigma(S_1) \cdot \ldots \cdot \sigma(S_{u_0})\sigma(S_{u_0+1}) \cdot \ldots \cdot \sigma(S_{u_0+u_1})$ has no zero-sum subsequence of length m.

Let $S'' = (S_1 \cdot \ldots \cdot S_{u_0+u_1})^{-1}S$. Therefore $u_0 + u_1 \le \mathsf{s}(C_m) - 1 = 2m - 2$ and hence $|S''| = |S| - n(u_0 + u_1) \ge 2n + 1 = \mathsf{s}^{**}(C_n)$. We infer that there exist two subsequences T_1 and T_2 of S'' with $|T_1| = |T_2| = n$ such that $\varphi(T_1), \varphi(T_2)$ are zero-sum and have a non-zero-sum $\varphi(S'')$-inner common divisor $\varphi(Y)$, where Y is a S''-inner common divisor of T_1 and T_2, whence $\sigma(Y) \notin \ker(\varphi)$. By the maximality of u_1, there exist $I, J \subset [1, u_0 + u_1]$ with $|I| = |J| = m - 1$ such that $T_1 \prod_{i \in I} S_i$ and $T_2 \prod_{j \in J} S_j$ are zero-sum subsequences of length mn and have a non-zero-sum S-inner common divisor $Y\prod_{k\in I\cap J} S_k$, a contradiction. □

By Lemmas 4.4, 4.5, and 4.7, we obtain our first main result of this section.

Theorem 4.8. *Let $G=C_n$ with $n\ge 3$. Then $\mathsf{s}^{**}(G)=\mathsf{s}^N(G)=\mathsf{s}^*(G)=2n+1$.*

4.2. **Abelian groups of rank 2.** We first consider $\mathsf{s}^*(G)$ for abelian groups G of rank 2.

Lemma 4.9. *If $G=C_n\oplus C_m$ with $1<n\mid m$, then*

$$\mathsf{s}^*(G)=\begin{cases}4m+1, & \text{if } n=m\ge 3;\\ 2n+2m-1, & \text{others.}\end{cases}$$

Proof. Let $G=C_n\oplus C_m$ with $1<n\mid m$. If $n=m=2$, then the assertion follows from Lemma 4.2.

Suppose $n=m\ge 3$. Let (e_1,e_2) be a basis of G and let

$$S=(2e_1\cdot e_1\cdot e_2\cdot(e_1+e_2))^m\,.$$

It is easy to see that S has no innerly joint zero-sum subsequences of length m. Then $\mathsf{s}^*(G)\ge |S|+1=4m+1$. To show $\mathsf{s}^*(G)\le 4m+1$. Let S be a sequence over $G^\bullet$ of length $|S|=4m+1$. We need to show that S has two innerly joint zero-sum subsequences of length m. Assume to the contrary that S has no innerly joint zero-sum subsequences of length m. Let t be maximal such that $S_1,\ldots,S_t$ are S-innerly disjoint zero-sum subsequences of length $|S_i|=m$ for $i\in[1,t]$, where $t\in\mathbb{N}_0$. Then $S_1\cdot\ldots\cdot S_t\mid S$. For every $i\in[1,t]$, we choose a term g_i of S_i. It follows that $(g_1\cdot\ldots\cdot g_t)^{-1}S$ has no zero-sum subsequence of length m, whence $|(g_1\cdot\ldots\cdot g_t)^{-1}S|\le \mathsf{s}(G)-1=4m-4$. We infer that $t=|S|-|(g_1\cdot\ldots\cdot g_t)^{-1}S|\ge 5$ and hence

$$4m+1=|S|\ge |S_1\cdot\ldots\cdot S_t|\ge 5m\,,$$

a contradiction.

Suppose $n<m$. Then $2n\le m$. Let (e_1,e_2) be a basis of G and let

$$T=(e_1+e_2)^{n-1}(e_1+2e_2)^{n-1}e_2^m(2e_2)^m\,.$$

Next we show that T has no innerly joint zero-sum subsequences of length m. Suppose that T' is a zero-sum subsequence of T of length $|T'|=m$. Then there exist $x,y\in[0,n-1]$ with $n\,|\,x+y$ and $z\in[0,m-x-y]$ such that

$$T'=(e_1+e_2)^x(e_1+2e_2)^y e_2^z(2e_2)^{m-x-y-z}\,,$$

whence $\sigma(T')=(x+y)e_1+(x+2y+z+2m-2x-2y-2z)e_2=(-x-z)e_2=0$ and hence m divides $x+z$. Since $x+z\in[x,m-y]$, we obtain either $x+z=x=0$ or $x+z=m-y=m$, whence $z\in\{0,m\}$. Note that $n\,|\,x+y$. Both cases imply that $x=y=0$. Therefore $T'\in\{e_2^m,(2e_2)^m\}$ and hence T has no innerly joint zero-sum subsequences of length m. So we have $\mathsf{s}^*(G) \ge |T|+1 = 2n+2m-1$. Next we need to prove that $\mathsf{s}^*(G) \le 2n+2m-1$. Let S' be a sequence over $G^\bullet$ of length $|S'| = 2n+2m-1$. We need to show that S' has two innerly joint zero-sum subsequences of length m. Assume to the contrary that S' has no innerly joint zero-sum subsequences of length m. Let t be maximal such that $S_1, \ldots, S_t$ are S'-innerly disjoint zero-sum subsequences with $|S_i| = m$ for $i \in [1,t]$, where $t \in \mathbb{N}_0$. Then $S_1 \cdot \ldots \cdot S_t \mid S'$. For every $i \in [1,t]$, we choose a term g_i of S_i. It follows that $(g_1 \cdot \ldots \cdot g_t)^{-1}S'$ has no zero-sum subsequence in S' of length m, whence $|(g_1 \cdot \ldots \cdot g_t)^{-1}S'| \le \mathsf{s}(G)-1 = 2n+2m-4$. We infer that $t = |S'| - |(g_1 \cdot \ldots \cdot g_t)^{-1}S'| \ge 3$ and hence

$$2n+2m-1 = |S'| \ge |S_1 \cdot \ldots \cdot S_t| \ge 3m\,.$$

It follows that $2n-1 \ge m$, a contradiction. □

Let $S = g_1 \cdot \ldots \cdot g_\ell$ be a sequence over G and let $k \in \mathbb{N}$. We define

$$\Sigma_k(S) = \{\sigma(T) \colon T \mid S \text{ with } |T| = k\}\,.$$

Note that we can view every subset A as a squarefree sequence over G and hence $\Sigma_k(A)$ is well-defined for every $k \in \mathbb{N}$. We need some preliminary results beginning with the following well known Dias da Silva-Hamidoune theorem.

Lemma 4.10. *Let p be a prime, and let $A \subset C_p$ with $|A| = k$. Let $1 \le h \le k$. Then*

$$|\Sigma_h(A)| \ge \min\{p, hk - h^2 + 1\}.$$

Proof. See [N96, Theorem 3.4]. □

Lemma 4.11. *Let G be a cyclic group of prime order p and let S be a sequence over $G^\bullet$. If $|S| \ge p$, then $\sum_{\le \mathrm{h}(S)}(S) = G$, where $\sum_{\le \mathrm{h}(S)}(S) = \bigcup_{r=1}^{\mathrm{h}(S)} \sum_r(S)$.*

Proof. Suppose $S = g_1^{r_1} \cdot \ldots \cdot g_t^{r_t}$, where $g_1, \ldots, g_t$ are different nonzero elements and $\mathsf{h}(S) = r_1 \ge r_2 \ge \ldots \ge r_t$. Then we can factor S into a product of $\mathsf{h}(S)$ nonempty subsets $A_1, \ldots, A_{\mathsf{h}(S)}$, that is to say

$$S = A_1 \cdot \ldots \cdot A_{\mathsf{h}(S)}.$$

Let $A_i' = A_i \cup \{0\}$ for $i \in [1, \mathsf{h}(S)-1]$. Then $\sum_{\le \mathrm{h}(S)}(S) \supset A_1' + \ldots + A_{\mathsf{h}(S)-1}' + A_{\mathsf{h}(S)}$. By Lemma 4.3,

$$|A_1' + \ldots + A_{\mathsf{h}(S)-1}' + A_{\mathsf{h}(S)}| \ge \min\{p, \sum_{i=1}^{\mathsf{h}(S)-1} |A_i'| + |A_{\mathsf{h}(S)}| - \mathsf{h}(S) + 1\} = p.$$

So we have $\sum_{\le \mathrm{h}(S)}(S) = G$. □

Lemma 4.12. *Let $G = C_p^2$ with p prime. Let S be a sequence over G of length $|S| = 4p-1$ such that $\mathsf{v}_0(S) \le p$. If there exist two distinct elements $e_1, e_2 \in G^\bullet$ such that $\mathsf{v}_{e_1}(S) = \mathsf{v}_{e_2}(S) = p-1$, then S has two innerly non-zero-sum-joint zero-sum subsequences of length p.*

Proof. Assume to the contrary that S has no innerly non-zero-sum-joint zero-sum subsequences of length p. It follows from Lemma 4.1.1 that $\mathsf{v}_g(S) \le p$ for every $g \in \operatorname{supp}(S) \setminus \{0\}$. Set $W = (e_1^{p-1}e_2^{p-1})^{-1}S = x_1 \cdot \ldots \cdot x_{2p+1}$, where $x_1, \ldots, x_{2p+1} \in \operatorname{supp}(S) \setminus \{e_1, e_2\}$.

Since $|S| = 4p-1 > \mathsf{s}(G) = 4p-3$, there exist zero-sum subsequences of length p of S. Let T be a zero-sum subsequence of S of length p. Then Lemma 4.1.1 implies $T \mid W$. After renumbering if necessary we may assume that there is a $t \in \mathbb{N}$ such that x_i is a term of some zero-sum subsequences of W of length p for each $i \in [1, p+t]$ and x_j is not a term of any zero-sum subsequence of W of length p for each $j \in [p+t+1, 2p+1]$.

Let $H = \langle e_2 - e_1 \rangle$ be the subgroup of $C_p \oplus C_p$ generated by $e_2 - e_1$. Then $H \cong C_p$.

Claim A. *$x_i \in H$ for every $i \in [1, p+t]$.*

Proof of Claim A. Let $W_0 = x_1 \cdot \ldots \cdot x_{p+t}$ and let T_1 be a zero-sum subsequence of W_0 of length $|T_1| = p$. Suppose that

$$T_1 = x_{i_1} \cdot \ldots \cdot x_{i_p}.$$

Let $U = x_{i_1}^{-1}T_1e_1^{p-1}e_2^{p-1}$. Then U has no zero-sum subsequence of length p. Assume that U has a zero-sum subsequence T_2 of length $|T_2| = 2p$. Then $T_2 = e_1^{\alpha}e_2^{\beta}T_0$, where $\alpha, \beta \in [1, p-1]$ and $T_0 \mid x_{i_1}^{-1}T_1$, whence $T_3 := e_1^{p-\alpha}e_2^{p-\beta}T_0^{-1}T_1$ is a zero-sum subsequence of S of length p. But $T_3 \nmid W$, a contradiction. Therefore U has no zero-sum subsequence of length p or $2p$, whence the sequence $-e_1 + U = (-e_1 + x_{i_1}^{-1}T_1)0^{p-1}(e_2 - e_1)^{p-1}$ has no zero-sum subsequence of length p or $2p$. It follows that $(x_{i_2} - e_1 + H) \cdot \ldots \cdot (x_{i_p} - e_1 + H)$ is a zero-sum free sequence over G/H, whence $x_{i_2} - e_1 + H = \ldots = x_{i_p} - e_1 + H$ by Lemma 2.6.5. In view of $\sigma(T_1) = 0$, we have

$$x_{i_1} + H = x_{i_2} + H = \ldots = x_{i_p} + H\,.$$

Let $T_4 = x_{j_1} \cdot \ldots \cdot x_{j_p}$ be another zero-sum subsequence of W_0 of length p such that $I := \{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_p\} \ne \emptyset$. Then $x_{j_1} + H = x_{j_2} + H = \ldots = x_{j_p} + H = x_{i_1} + H = x_{i_2} + H = \ldots = x_{i_p} + H$. Since $\prod_{k \in I} x_{i_k}$ is a S-inner common divisor of T_4 and T_1, we obtain that $0 = \sigma(\prod_{k \in I} x_{i_k}) \in |I|x_{i_1} + H$ and it follows from $0 < |I| < p$ that $x_{i_1} \in H$.

Assume to the contrary that there exists $i \in [1, p+t]$, say $i = 1$, such that $x_1 \notin H$. Let T be a zero-sum subsequence of W_0 of length p such that x_1 is a term of T. If there exists another zero-sum subsequence T' of W_0 of length p such that T' and T are W_0-innerly joint, then $x_1 \in H$ by the above argument, a contradiction. Thus W_0 and hence W have no innerly joint zero-sum subsequencesof length p. Choose two terms y_1, y_2 of $T^{-1}W$. Then $W_1 := (x_1y_1y_2)^{-1}W$ has no zero-sum subsequence of length p and hence $S_0 := (x_1y_1y_2)^{-1}S$ has no zero-sum subsequence of length p. Since $-e_1 + S_0 = 0^{p-1}(e_2 - e_1)^{p-1}(-e_1 + W_1)$, we obtain $(e_2 - e_1)^{p-1}(-e_1 + W_1)$ has no short zero-sum subsequence. It follows from Lemma 2.6.5 that there exist distinct $f_1, f_2 \in G^{\bullet}$ such that $(e_2 - e_1)^{p-1}(-e_1 + W_1) = (e_2 - e_1)^{p-1}(f_1 - e_1)^{p-1}(f_2 - e_1)^{p-1}$, whence $W_1 = f_1^{p-1}f_2^{p-1}$. Thus $x_1^{-1}T = f_1^{p-1}$ or f_2^{p-1}. By symmetry, we may assume $x_1^{-1}T = f_1^{p-1}$ and

$(y_1y_2T)^{-1}W = f_2^{p-1}$. Since y_1, y_2 are chosen arbitrarily, we infer that $T^{-1}W = f_2^{p+1}$, a contradiction to $\mathsf{h}(S) \le p$. $\square$(Claim A.)

By **Claim A** we set

$$x_1 \cdot \ldots \cdot x_{p+t} = h_1^{r_1} \cdot \ldots \cdot h_k^{r_k}\,,$$

where $h_1, \ldots, h_k \in H$ are distinct and $r_1 \ge r_2 \ge \ldots \ge r_k \ge 1$ with $r_1 + \ldots + r_k = p + t$. If $k = 2$, then by Lemma 4.1.1 we obtain that $(h_1h_2)^{-1}S$ has no zero-sum subsequence of length p, a contradiction to $|(h_1h_2)^{-1}S| = 4p - 3 = \mathsf{s}(G)$. Thus $k \ge 3$. If $r_1 + r_2 + r_3 \ge t + 1$. It follows from Lemma 4.1.1 that $(h_1h_2h_3)^{-1}S$ has no zero-sum subsequence of length p, whence $-e_1 + (h_1h_2h_3)^{-1}S = 0^{p-1}(-e_1 + e_2^{p-1})(-e_1 + (h_1h_2h_3)^{-1}W)$ has no zero-sum subsequence of length p. Therefore $(-e_1 + e_2^{p-1})(-e_1 + (h_1h_2h_3)^{-1}W)$ has no short zero-sum subsequence. By Lemma 2.6.5, there exist distinct $f_1, f_2 \in G^\bullet$ such that $(h_1h_2h_3)^{-1}W = f_1^{p-1}f_2^{p-1}$, whence $r_1 \ge r_2 \ge p - 1$. Thus $r_1 = r_2 = p$ and $r_3 = 1$. Let T_3 be a zero-sum subsequence of W of length p such that h_3 is a term of T_3. Then either $1 \le \mathsf{v}_{f_1}(T_3) \le p - 2$ or $1 \le \mathsf{v}_{f_2}(T_3) \le p - 2$, a contradiction to Lemma 4.1.1. Therefore $t \ge r_1 + r_2 + r_3$.

Let $l \in [1, k]$ be such that $r_1 = \ldots = r_l > r_{l+1} \ge \ldots \ge r_k$. Suppose $l \le t - r_1 + 1$. Let $R = -h_1 + (h_1^{r_1} \cdot \ldots \cdot h_k^{r_k})$ and let U be a subsequence of $(0^{r_1})^{-1}R$ of length $|U| = p$ such that $\mathsf{h}(U) \le r_1 - 1$. By Lemma 4.11, we obtain that $H = \sum_{\le \mathsf{h}(U)}(U) \subset \sum_{\le r_1-1}(U) \subset H$, whence $\sum_{\le r_1-1}(U) = H$. It follows that $\sum_{p-r_1+1}^{p-1}(U) = \sigma(U) - \sum_{\le r_1-1}(U) = H$, which implies U has a zero-sum subsequence U_0 of length $|U_0| \in [p-r_1+1, p-1]$. Thus $0^{p-|U_0|}U_0$ is a zero-sum subsequence of R and hence $h_1^{p-|U_0|}(h_1 + U_0)$ is a zero-sum subsequence of S of length p. But $1 \le p - |U_0| \le r_1 - 1$, a contradiction to Lemma 4.1.1. Therefore $l \ge t - r_1 + 2 \ge r_2 + r_3 + 2 \ge 4$, whence $r_1 = r_2 = r_3 \ge 2$, $l \ge r_2 + r_3 + 2 \ge 6$, and $r_1 \le \lfloor \frac{2p+1}{6} \rfloor \le (p-1)/2$.

There exist r_1 squarefree subsequences $B_1, \ldots, B_{r_1}$ such that $V := h_1^{r_1-2}h_2^{r_2}\cdot\ldots\cdot h_k^{r_k} = B_1 \cdot \ldots \cdot B_{r_1}$. Clearly, $|B_i| \ge l - 1 \ge 5$ for each $i \in [1, r_1]$. Since $r_1 \le (p-1)/2$, we can choose $A_i \mid B_i$ for each $i \in [1, r_1]$ such that $|A_i| \ge 3$ and $|A_1| + \ldots + |A_{r_1}| = 2r_1 + (p-1)/2$. Since $t \ge r_1 + r_2 + r_3 \ge 2r_1 + 2$, we obtain $|V| - |A_1 \cdot \ldots \cdot A_{r_1}| \ge (p-1)/2$, whence we can choose a subsequence V_1 of $(A_1 \cdot \ldots \cdot A_{r_1})^{-1}V$ of length $|V_1| = (p-1)/2$.

By Lemmas 4.10 and 4.3, we have that

$$|\Sigma_{|A_1|-2}(A_1) + \ldots + \Sigma_{|A_{r_1}|-2}(A_{r_1})| \ge \min\{p, \sum_{i=1}^{r_1}(2|A_i| - 4) + 1\} = p\,,$$

whence

$$\Sigma_{p-1}(V_1A_1 \cdot \ldots \cdot A_{r_1}) \supset \sigma(V_1) + \Sigma_{|A_1|+\ldots+|A_{r_1}|-2r_1}(A_1 \cdot \ldots \cdot A_{r_1}) = H\,.$$

Thus there exists a subsequence V' of V of length $p - 1$ such that $h_1 + \sigma(V') = 0$. But $\mathsf{v}_{h_1}(S) > r_1 - 1 \ge \mathsf{v}_{h_1}(h_1V')$, a contradiction to Lemma 4.1.1. This completes the proof. $\square$

Definition 4.13. Let $\mathsf{g}(G)$ denote the smallest integer t such that every squarefree sequence S of G of length $|S| \geq t$ contains a zero-sum subsequence of length $\exp(G)$.

Lemma 4.14. [GGS07, Theorem 5.1] *Let $G = C_p \oplus C_p$, where $p \geq 47$ is a prime. Then $\mathsf{g}(G) = 2p - 1$.*

Lemma 4.15. *Let $G = C_n^2$ with $n \in \mathbb{N}$.*

(1) $\mathsf{s}^N(G) \leq 8n - 7$.
(2) If $n = p$ is a prime such that $\mathsf{g}(G) = 2p - 1$, then $\mathsf{s}^N(G) \leq 6p - 5$. In particular, the assertion holds provided that $n = p \geq 47$.

Proof. 1. Let S be a sequence of length $|S| = 8n - 7$ over $G^\bullet$. We need to show that S has two innerly non-zero-sum-joint zero-sum subsequences of length n.

Let W be a subsequence of S with maxmial length such that W has no zero-sum subsequence of length n. Then by Lemma 2.9.1 we have $|W| \leq \mathsf{s}(G) - 1 = 4n - 4$, and hence $T = W^{-1}S$ has length at least $4n - 3$. Therefore T has a zero-sum subsequence T_0 of length n. Let g be a term of T_0. Then the maximality of $|W|$ implies that gW has a zero-sum subsequence T_1 of length n with $g|T_1$. It follows that T_0 and T_1 have a non-zero-sum S-inner common divisor g.

2. Suppose $n = p$ is a prime such that $\mathsf{g}(G) = 2p - 1$. Let S be a sequence of length $|S| = 6p - 5$ over $G^\bullet$. We need to show that S has two innerly non-zero-sum-joint zero-sum subsequences of length p.

Let W be a subsequence of S with maxmial length such that W has no zero-sum subsequence of length p. Then Lemma 2.9.1 implies that $|W| \leq \mathsf{s}(G) - 1 = 4p - 4$, and hence $T = W^{-1}S$ has length at least $2p - 1$.

If T is not squarefree, then there exists $g \in G^\bullet$ such that $g^2|T$. The maximality of $|W|$ implies that gW has a zero-sum subsequence V of length p with $g|V$. Since $\mathsf{v}_g(S) > \mathsf{v}_g(V) \geq 1$, it follows by Lemma 4.1.1 that S has two innerly non-zero-sum-joint zero-sum subsequences of length p.

If T is squarefree, then Lemma 4.14 implies that T has a zero-sum subsequence T_0 of length p. Let g be a term of T_0. Then the maximality of $|W|$ implies that gW has a zero-sum subsequence T_1 of length p with $g|T_1$. It follows that T_0 and T_1 have a non-zero-sum S-inner common divisor g.

For the "in particular" part, we finish the proof by applying Lemma 4.14. □

Lemma 4.16. *Let $G = C_n \oplus C_n$ with $n \geq 2$. Then every sequence of length $4n - 2$ over G has a zero-sum subsequence of length $2n$.*

Proof. The assertion follows from [GHPS14, Proposition 4.1]. □

Lemma 4.17. *Let $m, n, k \in \mathbb{N}$ with $n \ge 6$, let $G = C_m \oplus C_m$, and let S be a sequence over $G^\bullet$ of length $kn + n_0$, where $n_0 \in [1, n]$. Suppose that S has no innerly non-zero-sum-joint zero-sum subsequences of length mn.*

(1) There are S-innerly disjoint subsequences $S_1, \ldots, S_k$ of length $|S_i| = n$ for $i \in [1, k]$ such that $\sigma(S_1) \cdot \ldots \cdot \sigma(S_k)$ has no zero-sum subsequence of length m.

(2) $|S| \le mn + 3m - 3$.

Proof. 1. By Lemma 4.1.1, we have $\mathsf{h}(S) \le mn$. If $k \le m-1$, then the assertion is clear. If $k = m$, we can choose a subsequence S_0 of S of length $|S_0| = mn$ such that S_0 is not a zero-sum sequence and split S_0 into a product of m subsequences of length n, whence the assertion follows. Suppose $k \ge m+1$.

Let r be the maximal integer such that there exist S-innerly disjoint subsequences

$$T_1, W_1, T_2, W_2, \ldots, T_r, W_r$$

with $\sigma(T_i) = \sigma(W_i) \ne 0$ and $|T_i| = |W_i| = 2$ for all $i \in [1, r]$.

If $r \ge k$, then we can construct S-innerly disjoint subseqeuences $S_1, \ldots, S_k$ such that $T_1 \mid S_1$, $W_{i-1}T_i \mid S_i$ for every $i \in [2, k]$, and $|S_i| = n$ for every $i \in [1, k]$. We assert that the sequence $\sigma(S_1) \cdot \ldots \cdot \sigma(S_k)$ has no zero-sum subsequence of length m. Assume to the contrary that there is a subset $\{i_1, \ldots, i_m\} \subset [1, k]$ with $1 \le i_1 < \ldots < i_m \le k$ such that $\sum_{j=1}^m \sigma(S_{i_j}) = 0$. If $i_1 > 1$, then let $S'_{i_1} = W_{i_1-1}^{-1} S_{i_1} T_{i_1-1}$ and hence $\prod_{j=1}^m S_{i_j}$ and $S'_{i_1} \prod_{j=2}^m S_{i_j}$ are both zero-sum subsequences of S of length mn and have a non-zero-sum S-inner common divisor $W_{i_1-1}^{-1} \prod_{j=1}^m S_{i_j}$, a contradiction. Thus $i_1 = 1$. Since $k \ge m+1$, there exists $s \in [2, k]$ such that $s \notin \{i_1, \ldots, i_m\}$. Then we can choose $s \in [2, k]$ minimal such that $s \notin \{i_1, \ldots, i_m\}$, whence $s - 1 \in \{i_1, \ldots, i_m\}$. Let $S'_{s-1} = T_{s-1}^{-1} W_{s-1} S_{s-1}$. Then $\prod_{j=1}^m S_{i_j}$ and $S_{s-1}^{-1} S'_{s-1} \prod_{j=1}^m S_{i_j}$ are both zero-sum subsequences of S of length mn and have a non-zero-sum S-inner common divisor $T_{s-1}^{-1} \prod_{j=1}^m S_{i_j}$, a contradiction.

Thus $r \le k - 1$. Let

$$U = (T_1 W_1 \cdot \ldots \cdot T_r W_r)^{-1} S\,.$$

Assume that there is no element g of order two such that $\mathsf{v}_g(U) \ge 3$. Then the maximality of r implies that $\mathsf{h}(U) \le 3$. and that there are at most two elements y_1, y_2 such that $\mathsf{v}_{y_1}(U) \ge 2$ and $\mathsf{v}_{y_2}(U) \ge 2$. Let V be a squarefree subsequence of U with maximal length. Then

$$|V| \ge |U| - 4 = |S| - 4r - 4 \ge kn + 1 - 4(k-1) - 4 = (n-4)k + 1 \ge 2k + 1 \ge 2m + 3\,.$$

Since V has at least $\binom{|V|}{2} - \frac{|V|}{2}$ subsequences of length 2 which are not zero-sum and the sums of all these subsequences are pairwise distinct, we obtain that $\binom{|V|}{2} - \frac{|V|}{2} \le |G| - 1 = m^2 - 1$, whence

$$|V| - 1 = \sqrt{2\binom{|V|}{2} - |V| + 1} \le \sqrt{2m^2 - 1} \le \sqrt{2}m \le 2m\,,$$

a contradiction. Therefore there exists a order 2 element g with $\mathsf{v}_g(U) \ge 3$. Let $V = (g^{\mathsf{v}_g(U)})^{-1}U$. Then $\mathsf{h}(V) = 1$ and $\binom{|V|}{2} - \frac{|V|}{2} \le m^2 - 1$, whence

$$|V| - 1 = \sqrt{2\binom{|V|}{2} - |V| + 1} \le \sqrt{2m^2 - 1} \le \sqrt{2}m \le 2m\,.$$

It follows that

$$\mathsf{v}_g(U) + r = |U| - |V| + r = |S| - |V| - 3r \ge kn + 1 - 2m - 3(k-1) \ge 3k - 2m + 4 \ge k + 6\,.$$

Then we can construct S-innerly disjoint subsequences S_i for $i \in [1,k]$ with $T_1 \mid S_1$, $W_{i-1}T_i \mid S_i$ for every $i \in [2,r]$, $W_r g \mid S_{r+1}$, and $g \mid S_i$ for every $i \in [r+2,k]$. Assume to the contrary that the sequence $\sigma(S_1)\cdot\ldots\cdot\sigma(S_k)$ has a zero-sum subsequence of length m. Then there is a subset $\{i_1,\ldots,i_m\} \subset [1,k]$ such that $\sum_{j=1}^m \sigma(S_{i_j}) = 0$. If there exists $i \in [1,r]$ such that $i \notin \{i_1,\ldots,i_m\}$, then similarly as above we obtain two zero-sum subsequences of length mn which are S-innerly non-zero-sum-joint, a contradiction. Otherwise $[1,r] \subset \{i_1,\ldots,i_m\}$ and hence there exists $i \in [r+1,k]$ such that $i \notin \{i_1,\ldots,i_m\}$, whence $\mathsf{v}_g(\prod_{j=1}^m S_{i_j}) < \mathsf{v}_g(S)$, a contradiction to Lemma 4.1.1.

2. Assume to the contrary that $|S| \ge mn + 3m - 2$. By Lemma 4.15.1, we have $\mathsf{s}^N(G) \le 8m - 7 \le mn + 3m - 2$, whence S has two innerly non-zero-sum-joint zero-sum subsequences X_0 and X_0' of length $|X_0| = |X_0'| = m$. Let Y be the non-zero-sum S-inner common divisor and let $S' = (X_0X_0')^{-1}SY$. Then $|S'| \ge |S| - (2m-1) \ge (n-3)m + 4m - 1$. Now by Lemma 2.9.1 and applying $\mathsf{s}(G) = 4m - 3$ repeatedly to S', we can find $n-3$ S'-innerly disjoint zero-sum subsequences $X_1,\ldots,X_{n-3}$ of length $|X_1| = \ldots = |X_{n-3}| = m$. Let $S'' = (X_1\cdot\ldots\cdot X_{n-3})^{-1}S'$. Then $|S''| > 4m - 2$. It follows from Lemma 4.16 that S'' has a zero-sum subsequence X_{n-2} of length $|X_{n-2}| = 2m$. Let $Y_1 = X_0\prod_{i=1}^{n-2} X_i$ and $Y_2 = X_0'\prod_{i=1}^{n-2} X_i$. Then Y_1 and Y_2 are two zero-sum subsequences of length $|Y_1| = |Y_2| = mn$ with a non-zero-sum S-inner common divisor $Y\prod_{i=1}^{n-2} X_i$, a contradiction. □

We are now ready to prove our second theorem of this section.

Theorem 4.18. *Let $n \in \mathbb{N}$ and let $p \ge 47$ be a prime divisor of n such that $n \ge \frac{7p^4+p^3+2p^2}{2}$. Then $\mathsf{s}^N(C_n^2) = 4n+1$.*

Proof. Let $G = C_n^2$ and let $m = n/p$. By Lemma 4.9 it suffices to prove that $\mathsf{s}^N(G) \le 4n+1$. Let S be a sequence over $G^\bullet$ of length $|S| = l = 4n+1$. We need to show that S has two innerly non-zero-sum-joint zero-sum subsequences of length n.

Assume to the contrary that S has no innerly non-zero-sum-joint zero-sum subsequences of length n. Let $\varphi\colon G \to G$ denote the multiplication by m. Then $\ker(\varphi) \cong C_m^2$ and $\varphi(G) = mG \cong C_p^2$.

Let $S = X_0X_1$ such that X_0 is over $\ker(\varphi)$ and X_1 is over $G\setminus\ker(\varphi)$. Since X_0 has no innerly non-zero-sum-joint zero-sum subsequences of length pm, it follows from Lemma

4.17 that $|X_0| \le pm+3m-3$ and there exist X_0-innerly disjoint subsequences $S_1, \ldots, S_{u_0}$ such that $|S_i| = p$ for each $i \in [1, u_0]$, $\sigma(S_1) \cdot \ldots \cdot \sigma(S_{u_0})$ has no zero-sum sequence of length m, and the remaining subsequence $X_0' = (S_1 \cdot \ldots \cdot S_{u_0})^{-1} X_0$ is of length $|X_0'| \le p$.

We set

$$\varphi(X_1) = e_1^{r_1} \cdot \ldots \cdot e_k^{r_k}, \tag{4.1}$$

where $e_1, \ldots, e_k \in \varphi(G)$ are distinct and $r_1 \ge \ldots \ge r_k \ge 1$. We continue with the following assertion.

A1. $r_2 \ge (7p-6)(p-2)$.

Proof of **A1**. If $r_1 \le pm + 3m - 3$, then

$$\begin{aligned} r_2 &\ge \frac{|S| - |X_0| - (pm+3m-3)}{|\varphi(G) \setminus \{0, e_1\}|} \ge \frac{4pm + 1 - pm - 3m + 3 - pm - 3m + 3}{p^2 - 2} \\ &= \frac{2pm - 6m + 7}{p^2 - 2} \ge (7p-6)(p-2)\,. \end{aligned}$$

Suppose $r_1 \ge pm + 3m - 2 = (p-1)m + 4m - 2$. Let $X_{e_1} = h_1 \cdot \ldots \cdot h_{r_1}$ be the subsequence of X_1 such that $\varphi(X_{e_1}) = e_1^{r_1}$. Then there exist an element $h \in G$ with $\mathrm{ord}(h) = n$ and $s \in [1, n-1]$ with $p \nmid s$ such that $h_1 = sh$. Let m_0 be the maximal divisor of m such that $\gcd(m_0, s) = 1$. Then $\mathrm{ord}(h_1 + m_0 ph) = pm$ and $\varphi(h_1 + m_0 ph) = e_1$. Set $h_0 = h_1 + m_0 ph$. Then $(h_1 - h_0) \cdot \ldots \cdot (h_{r_1} - h_0)$ is a sequence over $\ker(\varphi)$. By using $\mathsf{s}(C_m^2) = 4m - 3$ repeatedly, we can find $p-1$ $(h_1 - h_0) \cdot \ldots \cdot (h_{r_1} - h_0)$-innerly disjoint zero-sum subsequences $E_1, \ldots, E_{p-1}$ of length $|E_1| = \ldots = |E_{p-1}| = m$. It follows from Lemma 4.16 that $(E_1 \cdot \ldots \cdot E_{p-1})^{-1}(h_1 - h_0) \cdot \ldots \cdot (h_{r_1} - h_0)$ has a zero-sum subsequence E_p of length $|E_p| = 2m$. Let $E = E_p \prod_{i=1}^{p-2} E_i$ and $E' = E_p \prod_{i=2}^{p-1} E_i$. Then $h_0 + E$ and $h_0 + E'$ are two zero-sum subsequences of X_{e_1} of length n and have a X_{e_1}-inner common divisor $h_0 + E_p \prod_{i=2}^{p-2} E_i$. But $\sigma(h_0 + E_p \prod_{i=2}^{p-2} E_i) = (p-1)mh_0 = -mh_0 \ne 0$, a contradiction. □(A1.)

Let W_1 be a subsequence of X_1 such that $\varphi(W_1) = e_1^{r_1} e_2^{r_2}$ and $W_2 = W_1^{-1} X_1$. Let $u_1 \in \mathbb{N}_0$ be maximal such that there exist W_2-innerly disjoint subsequences $S_{u_0+1}, \ldots, S_{u_0+u_1}$ with the following properties.

- $S_{u_0+1} \cdot \ldots \cdot S_{u_0+u_1} \mid W_2$;
- For every $\nu \in [1, u_1]$, $\varphi(S_{u_0+\nu})$ is a zero-sum sequence over $\varphi(G)$ of length p;
- The sequence $\sigma(S_1) \cdot \ldots \cdot \sigma(S_{u_0}) \sigma(S_{u_0+1}) \cdot \ldots \cdot \sigma(S_{u_0+u_1}) \in \mathcal{F}(\ker(\varphi))$ has no zero-sum subsequence of length m.

We set $W_2' = (S_{u_0+1} \cdot \ldots \cdot S_{u_0+u_1})^{-1} W_2$. If there exist two subsequence T_1 and T_2 of W_2' with $|T_1| = |T_2| = p$ such that $\varphi(T_1), \varphi(T_2)$ are zero-sum and have a non-zero-sum $\varphi(W_2')$-inner common divisor $\varphi(Y)$, where Y is a W_2'-inner common divisor of T_1 and T_2, whence $\sigma(Y) \notin \ker(\varphi)$. By the maximality of u_1, there exist $I, J \subset [1, u_0 + u_1]$ with $|I| = |J| = m - 1$ such that $T_1 \prod_{i \in I} S_i$ and $T_2 \prod_{j \in J} S_j$ are zero-sum subsequences of length pm and have a non-zero-sum S-inner common divisor $Y \prod_{k \in I \cap J} S_k$, a contradiction. Hence

$\varphi(W_2')$ has no innerly non-zero-sum-joint subsequences of length p. It follows from Lemma 4.15.2 that $|W_2'| = |\varphi(W_2')| \le 6p-6$.

Let $Q = X_0'W_2'$ and hence $|Q| \le p + 6p - 6 = 7p - 6$. Note that $r_2 \ge (7p-6)(p-2)$. Let $u_2 \in \mathbb{N}_0$ be maximal such that there exist W_1Q-innerly disjoint subsequences $S_{u_0+u_1+1}, \ldots, S_{u_0+u_1+u_2}$ with the following properties.

- $S_{u_0+u_1+1} \cdot \ldots \cdot S_{u_0+u_1+u_2} \mid W_1Q$;
- For every $\nu \in [1, u_2]$, $\varphi(S_{u_0+u_1+\nu})$ is a zero-sum subsequence of $e_1^{p-1}e_2^{p-1}\varphi(Q)$ over $\varphi(G)$ of length $|S_{u_0+u_1+\nu}| = p$. Note that $\mathsf{v}_{e_1}(\varphi(S_{u_0+u_1+\nu})) \le p-2$ and $\mathsf{v}_{e_2}(\varphi(S_{u_0+u_1+\nu})) \le p-2$;
- The sequence $\sigma(S_1)\cdot\ldots\cdot\sigma(S_{u_0+u_1+u_2}) \in \mathcal{F}(\ker(\varphi))$ has no zero-sum subsequence of length m.

Let $E = (S_{u_0+u_1+1} \cdot \ldots \cdot S_{u_0+u_1+u_2})^{-1}W_1Q$ and let $E_1 = \gcd(E, W_1)$, $E_2 = E_1^{-1}E$. If $|E_2| \ge 2p+1$, then $e_1^{p-1}e_2^{p-1}$ divides $\varphi(E_1)$ and $e_1^{p-1}e_2^{p-1}\varphi(E_2)$ has no innerly non-zero-sum-joint zero-sum subsequences of length p, a contradiction to Lemma 4.12. Thus $|E_2| \le 2p$.

Let $u_3 \in \mathbb{N}_0$ be maximal such that there are E_1-innerly disjoint subseqeunces $S_{u_0+u_1+u_2+1}, \ldots, S_{u_0+u_1+u_2+u_3}$ with the following properties.

- $S_{u_0+u_1+u_2+1} \cdot \ldots \cdot S_{u_0+u_1+u_2+u_3} \mid E_1$;
- For every $\nu \in [1, u_3]$, $\varphi(S_{u_0+u_1+u_2+u_3+\nu}) \in \{e_1^p, e_2^p\}$;
- The sequence $\sigma(S_1) \cdot \ldots \cdot \sigma(S_{u_0+u_1+u_2+u_3}) \in \mathcal{F}(\ker(\varphi))$ has no zero-sum subsequence of length m.

We set $F = (S_{u_0+u_1+u_2+1}\cdot\ldots\cdot S_{u_0+u_1+u_2+u_3})^{-1}E_1$ and observe that $\mathsf{h}(\varphi(F)) \le p$, whence $|FE_2| \le p+p+2p = 4p$. Therefore, $u_0+u_1+u_2+u_3 = \frac{|S|-|FE_2|}{p} \ge \lceil \frac{|S|-4p}{p} \rceil \ge 4m-3 = \mathsf{s}(\ker(\varphi))$, a contradiction.

4.3. **Abelian groups of higher rank.**

Lemma 4.19. *Let $G = C_n^r$ with $n, r \in \mathbb{N}$ and $n \ge 3$. Suppose that G has Property D. Then*

$$\mathsf{s}^*(G) = \frac{n(\mathsf{s}(G)-1)}{n-1} + 1.$$

Proof. Let T be a sequence over G of length $\mathsf{s}(G)-1$ which has no zero-sum subsequence of length n. It follows from G having Property D that T has the form

$$T = U^{n-1},$$

where U is squarefree. Let $g \in G$ such that $-g \notin \operatorname{supp}(U)$. Then $S := (g+U)^n \in \mathcal{F}(G^\bullet)$ has no innerly joint zero-sum subsequences of length n. Thus, $\mathsf{s}^*(G) \ge |S| + 1 = \frac{n(\mathsf{s}(G)-1)}{n-1} + 1$.

Next, we need to prove that $\mathsf{s}^*(G) \le \frac{n(\mathsf{s}(G)-1)}{n-1} + 1$. Let $S' \in \mathcal{F}(G^\bullet)$ be a sequence of length $\frac{n(\mathsf{s}(G)-1)}{n-1} + 1$. We need to show that S' has two innerly joint zero-sum subsequences of length n. Assume to the contrary that S' has no innerly joint zero-sum subsequences of length n.

Let t be maximal such that $S_1, \ldots, S_t$ are S'-innerly disjoint zero-sum subsequences of length n. Then

$$S_1 \cdot \ldots \cdot S_t \mid S'$$

and for every $i \in [1,t]$ we choose an element $g_i \mid S_i$. It follows that $(g_1 \cdot \ldots \cdot g_t)^{-1}S'$ has no zero-sum subsequence of length n. Then $|(g_1 \cdot \ldots \cdot g_t)^{-1}S'| \le \mathsf{s}(G) - 1$, which implies that $t \ge \frac{\mathsf{s}(G)-1}{n-1} + 1$. Therefore

$$\frac{n(\mathsf{s}(G)-1)}{n-1} + 1 = |S'| \ge |S_1 \cdot \ldots \cdot S_t| = n(\frac{\mathsf{s}(G)-1}{n-1} + 1)\,,$$

a contradiction. □

Theorem 4.20. *Let $n, m, r \in \mathbb{N}$.*

(1) If $G = C_2 \oplus C_{2m} \oplus C_{2mn}$, then

$$\mathsf{s}^*(G) = \begin{cases} 15, & \text{if } n = m = 1; \\ 8m+5, & \text{if } n = 1 \text{ and } m \ge 3; \\ 4mn + 4m + 1, & \text{if } n \ge 3 \text{ and } C_m^2 \text{ has Property D.} \end{cases}$$

(2) Let $G = C_n^r$ with $n \ge 3$. Then $\mathsf{s}^*(G) \ge 2^r \cdot n + 1$. If $r = 3$ and $n = 2^a 3^b$ with $a, b \in \mathbb{N}$ and $a \ge b$, then

$$\mathsf{s}^*(G) = 8n + 1\,.$$

(3) Let $G = C_n^3$ with $n \ge 3$ odd. Then $\mathsf{s}^*(G) \ge 9n+1$. If $n = 3^a 5^b \ge 3$ with $a, b \in \mathbb{N}_0$, then

$$\mathsf{s}^*(G) = 9n + 1.$$

(4) Let $G = C_n^4$ with $n \ge 3$ odd. Then $\mathsf{s}^*(G) \ge 20n + 1$. If $n = 3^a$ with $a \in \mathbb{N}$, then

$$\mathsf{s}^*(G) = 20n + 1.$$

(5) If $G = C_2^3 \oplus C_{2n}$ with $n \ge 36$, then

$$\mathsf{s}^*(G) = 4n + 7.$$

(6) If $G = C_{2^a}^r$ with $a \in \mathbb{N}$, then

$$\mathsf{s}^*(G) = \begin{cases} 2^{r+1} - 1, & \text{if } a = 1; \\ 2^{r+a} + 1, & \text{if } a \ge 2. \end{cases}$$

Proof. 1. Let $G = C_2 \oplus C_{2m} \oplus C_{2mn}$ and let (e_1, e_2, e_3) be a basis of G. If $n = m = 1$, then the assertion follows from Lemma 4.2.

Suppose $n = 1$ and $m \ge 3$. Then $\mathsf{s}(G) = 8m + 1$ by Lemma 2.9.4. Let

$$T = (e_1 + e_2)(e_1 + 2e_2)(e_1 + e_3)(e_1 + e_2 + e_3)e_2^{2m} e_3^{2m} (e_2 + e_3)^{2m} (2e_2)^{2m}\,.$$

It is easy to see that T has no innerly joint zero-sum subsequences of length $2m$. Therefore $\mathsf{s}^*(G) \ge |T| + 1 = 8m + 5$. Let S be a sequence over $G^\bullet$ of length $|S| = 8m + 5$. We need to show that S has two innerly joint zero-sum subsequences of length $2m$. Assume to the contrary that S has no innerly joint zero-sum subsequences of length $2m$. Let t be maximal such that $S_1, \ldots, S_t$ are S-innerly disjoint zero-sum subsequences of length $|S_i| = 2m$ for $i \in [1,t]$, where $t \in \mathbb{N}_0$. Then $S_1 \cdot \ldots \cdot S_t \mid S$. For every $i \in [1,t]$, we choose a term g_i of S_i. It follows that $(g_1 \cdot \ldots \cdot g_t)^{-1}S$ has no zero-sum subsequence of length $2m$, whence $|(g_1 \cdot \ldots \cdot g_t)^{-1}S| \le \mathsf{s}(G) - 1 = 8m$. Then $t = |S| - |(g_1 \cdot \ldots \cdot g_t)^{-1}S| \ge 5$ and hence

$$8m + 5 = |S| \ge |S_1 \cdot \ldots \cdot S_t| \ge 10m\,,$$

a contradiction to $m \ge 3$. Therefore $\mathsf{s}^*(G) = 8m + 5$.

Suppose $n \ge 3$ and C_m^2 has Property D. Then $\mathsf{s}(G) = 4mn + 4m - 1$ by Lemmas 2.9.4. Let

$$T' = (e_1 + e_3)(e_1 + 2e_3)(e_2 + e_3)^{2m-1}(e_2 + 2e_3)^{2m-1}e_3^{2mn}(2e_3)^{2mn}\,.$$

It is easy to see that T' has no innerly joint zero-sum subsequences of length $2mn$. Therefore $\mathsf{s}^*(G) \ge |T'| + 1 = 4mn + 4m + 1$. Let S' be a sequence over $G^\bullet$ of length $|S'| = 4mn+4m+1$. We need to show that S' has two innerly joint zero-sum subsequences of length $2mn$. Assume to the contrary that S' has no innerly joint zero-sum subsequences of length $2mn$. Let t be maximal such that $S_1, \ldots, S_t$ are S-innerly disjoint zero-sum subsequences of length $|S_i| = 2mn$ for $i \in [1,t]$, where $t \in \mathbb{N}_0$. Then $S_1 \cdot \ldots \cdot S_t \mid S'$. For every $i \in [1,t]$, we choose a term g_i of S_i. It follows that $(g_1 \cdot \ldots \cdot g_t)^{-1}S'$ has no zero-sum subsequence of length $2mn$, whence $|(g_1 \cdot \ldots \cdot g_t)^{-1}S'| \le \mathsf{s}(G) - 1 = 4mn + 4m - 2$. Then $t = |S'| - |(g_1 \cdot \ldots \cdot g_t)^{-1}S'| \ge 3$ and hence

$$4mn + 4m + 1 = |S'| \ge |S_1 \cdot \ldots \cdot S_t| \ge 6mn\,.$$

It follows that $2m \le 1$, a contradiction. Therefore $\mathsf{s}^*(G) = 4mn + 4m + 1$.

2. Let $(e_1, \ldots, e_r)$ be a basis of G and let

$$S = (2e_1)^n \prod_{\emptyset \ne I \subset [1,r]} \Big(\sum_{i \in I} e_i\Big)^n\,.$$

Suppose T is a zero-sum subsequence of S of length $|T| = n$. Then either $T = (2e_1)^n$ or $T = (\sum_{i\in I} e_i)^n$, where $\emptyset \ne I \subset [1,r]$. Therefore $S \in \mathcal{F}(G^\bullet)$ has no innerly joint zero-sum subsequences of length n, whence $\mathsf{s}^*(G) \ge |S| + 1 = 2^r n + 1$.

Suppose $r = 3$ and $n = 2^a 3^b$ with $a, b \in \mathbb{N}$ and $a \ge b$. Then $\mathsf{s}(G) = 8n - 7$ by Lemma 2.12. Let S be a sequence over $G^\bullet$ of length $|S| = 8n+1$. We need to show that S has two innerly joint zero-sum subsequences of length n. Assume to the contrary that S has no innerly joint zero-sum subsequences of length n. Let t be maximal such that $S_1, \ldots, S_t$ are S-innerly disjoint zero-sum subsequences with $|S_i| = n$ for $i \in [1,t]$, where $t \in \mathbb{N}_0$. Then $S_1 \cdot \ldots \cdot S_t \mid S$. For every $i \in [1,t]$, we choose a term g_i of S_i. It follows that $(g_1 \cdot \ldots \cdot g_t)^{-1}S$ has no zero-sum subsequence of length n, whence $|(g_1 \cdot \ldots \cdot g_t)^{-1}S| \le \mathsf{s}(G) - 1 = 8n - 8$. Then $t = |S| - |(g_1 \cdot \ldots \cdot g_t)^{-1}S| \ge 9$ and hence

$$8n + 1 = |S| \ge |S_1 \cdot \ldots \cdot S_t| \ge 9n\,,$$

a contradiction to $n \ge 3$. Therefore $\mathsf{s}^*(G) = 8n + 1$.

3. By Lemma 2.7, there exists a sequence $T \in \mathcal{F}(G)$ of length $|T| = 9$ such that T^{n-1} has no zero-sum subsequence in T^{n-1} of length n, there exists $g \in G$ such that $0 \nmid (g+T)$. Set $S = (g+T)^n$, then $S \in \mathcal{F}(G^\bullet)$ has no innerly joint zero-sum subsequences of length n. Therefore we have $\mathsf{s}^*(G) \ge |S| + 1 = 9n + 1$.

Suppose $n = 3^a 5^b$ with $a, b \in \mathbb{N}_0$. Then by Lemma 2.6.3 C_n^3 has Property D and hence by Lemma 4.19 that $\mathsf{s}^*(G) = 9n + 1$.

4. By Lemma 2.8, there exists a sequence $T \in \mathcal{F}(G)$ of length $|T| = 20$ such that T^{n-1} has no zero-sum subsequence in T^{n-1} of length n, there exists $g \in G$ such that $0 \nmid (g+T)$. Set $S = (g+T)^n$, then $S \in \mathcal{F}(G^\bullet)$ has no innerly joint zero-sum subsequences of length n. Therefore we have $\mathsf{s}^*(G) \ge |S| + 1 = 20n + 1$.

If $n = 3^a$ with $a \in \mathbb{N}$, then by Lemma 2.6.2 C_n^4 has Property D and hence by Lemma 4.19 $\mathsf{s}^*(G) = 20n + 1$.

5. Let (e_1, e_2, e_3, e_4) be a basis of G and let

$$T = (e_1+e_4)(e_1+2e_4)(e_2+e_4)(e_2+2e_4)(e_3+e_4)(e_3+2e_4)e_4^{2n}(2e_4)^{2n}\,.$$

It is easy to see that T has no innerly joint zero-sum subsequences of length $2n$. Therefore $\mathsf{s}^*(G) \ge |T| + 1 = 4n + 7$. Let S be a sequence over $G^\bullet$ of length $|S| = 4n + 7$. We need to show that S has two innerly joint zero-sum subsequences of length $2n$. Assume to the contrary that S has no innerly joint zero-sum subsequences of length $2n$. Let t be maximal such that $S_1, \ldots, S_t$ are S-innerly disjoint zero-sum subsequences of length $|S_i| = 2n$ for $i \in [1, t]$, where $t \in \mathbb{N}_0$. Then $S_1 \cdot \ldots \cdot S_t \mid S$. For every $i \in [1, t]$, we choose a term g_i of S_i. It follows that $(g_1 \cdot \ldots \cdot g_t)^{-1} S$ has no zero-sum subsequence of length $2n$, whence $|(g_1 \cdot \ldots \cdot g_t)^{-1} S| \le \mathsf{s}(G) - 1 = 4n+4$ by Lemma 2.9.3. Then $t = |S| - |(g_1 \cdot \ldots \cdot g_t)^{-1} S| \ge 3$ and hence

$$4n + 7 = |S| \ge |S_1 \cdot \ldots \cdot S_t| \ge 6n\,,$$

a contradiction to $n \ge 36$. Therefore $\mathsf{s}^*(G) = 4n + 7$.

6. If $a = 1$, then the assertion follows from Lemma 4.2. Suppose $a \ge 2$. Then by Lemma 2.6.1 $C_{2^a}^r$ has Property D and hence by Lemma 4.19 that $\mathsf{s}^*(G) = 2^{r+a} + 1$. □

Theorem 4.21. *Let $G = C_3^r$. Then*

$$\mathsf{s}^{**}(G) = \mathsf{s}^N(G) = \mathsf{s}^*(G) = \frac{3(\mathsf{s}(G)-1)}{2} + 1.$$

Proof. By Lemma 4.19 and $G = C_3^r$ has property D, we have $\mathsf{s}^*(G) = \frac{3(\mathsf{s}(G)-1)}{2} + 1$. It suffices to show that $\mathsf{s}^{**}(G) \le \frac{3(\mathsf{s}(G)-1)}{2} + 1$. Let $S \in \mathcal{F}(G)$ be a sequence of length $|S| = \frac{3(\mathsf{s}(G)-1)}{2} + 1$ such that $\mathsf{v}_0(S) \le 3$. We need to show that S has two innerly non-zero-sum-joint zero-sum subsequences of length 3. We distinguish two cases.

Case 1. $\mathsf{v}_0(S) \ge 1$.

Let $S' = (0^{\mathsf{v}_0(S)})^{-1} S$. Then $|S'| \ge |S| - 3 = \frac{3(\eta(G)-1)}{2} + 1 = \eta^*(G)$ by Lemma 3.1, whence there exist two S'-innerly non-zero-sum-joint short zero-sum subsequences T_1

and T_2. Let Y be the non-zero-sum S'-inner common divisor. Since $\min\{|T_1|,|T_2|\} \geq 2$, we have $\max\{3-|T_1|, 3-|T_2|\} \leq 1$, whence $0^{3-|T_1|}T_1$ and $0^{3-|T_2|}T_2$ are two zero-sum subsequences of S of length 3 and have a non-zero-sum S-inner common divisor 0^tY, where $t \leq \min\{3-|T_1|, 3-|T_2|\}$.

Case 2. $\mathsf{v}_0(S) = 0$.

Then every zero-sum subsequence of S of length 3 is minimal. Suppose $S = T_1 \cdot \ldots \cdot T_rT_0$, where $T_1, \ldots, T_r$ are zero-sum subsequences of S of length 3 and T_0 has no zero-sum subsequence of length 3. Then $r \leq \lfloor \frac{|S|}{3} \rfloor = \frac{\mathsf{s}(G)-1}{2}$. Choose a term $g_i \mid T_i$ for each $i \in [1,r]$ and let $S' = (g_1 \cdot \ldots \cdot g_r)^{-1}S$. Thus $|S'| \geq |S| - r \geq \mathsf{s}(G)$, whence S' has a zero-sum subsequence T of length 3. Since T_0 has no zero-sum subsequence of length 3, there exists $i \in [1,r]$ such that T_i and T have a non-zero-sum S-inner common divisor Y, where Y is a nontrivial subsequence of $g_i^{-1}T_i$. □

Proposition 4.22. *Suppose that $G = C_5^r$ has Property D. Then*

$$\mathsf{s}^N(G) = \mathsf{s}^*(G) = \frac{5(\mathsf{s}(G)-1)}{4} + 1.$$

Proof. By Lemma 4.19 and that $G = C_5^r$ has property D, we have $\mathsf{s}^*(G) = \frac{5(\mathsf{s}(G)-1)}{4} + 1$. Next we need to prove that $\mathsf{s}^N(G) \leq \frac{5(\mathsf{s}(G)-1)}{4} + 1$. Let S be a sequence over $G^\bullet$ of length $|S| = \frac{5(\mathsf{s}(G)-1)}{4} + 1$. We need to show that S has two innerly non-zero-sum-joint zero-sum subsequences of length 5. Assume to the contrary that S has no innerly non-zero-sum-joint zero-sum subsequences of length 5. By Lemma 4.1.1, if T is a zero-sum subsequence of S of length 5, then $\mathsf{v}_g(T) = \mathsf{v}_g(S)$ for every $g \in \mathrm{supp}(T)$.

Suppose $S = S_1 \cdot \ldots \cdot S_rS_0$, where $r \in \mathbb{N}$, $S_1, \ldots, S_r$ are S-innerly disjoint zero-sum subsequences of length $|S_1| = \ldots = |S_r| = 5$, and S_0 has no zero-sum subsequence of length 5. Thus $r \leq \lfloor |S|/5 \rfloor = \frac{\mathsf{s}(G)-1}{4}$.

Let $i \in [1,r]$. If S_i is not a minimal zero-sum sequence, then $S_i = T_iT_i'$, where T_i, T_i' are minimal zero-sum subsequences such that $|T_i| = 3$ and $|T_i'| = 2$, whence there exist an element $g_i \in \mathrm{supp}(T_i') \setminus \mathrm{supp}(T_i)$ such that $g_i^{-1}S_i$ has no zero-sum subsequence of length 2 and an element $h_i \in \mathrm{supp}(T_i) \setminus \mathrm{supp}(T_i')$ such that $h_i^{-1}S_i$ has no zero-sum subsequence of length 3. If S_i is a minimal zero-sum sequence, choose any two terms g_i, h_i of S_i.

We consider the sequence $S' := (g_1 \cdot \ldots \cdot g_r)^{-1}S$. If S' has no zero-sum subsequence of length 5, then $|S'| = \frac{5(\mathsf{s}(G)-1)}{4} + 1 - r \leq \mathsf{s}(G) - 1$, whence $r \geq \frac{\mathsf{s}(G)+1}{4}$, a contradiction. Thus we may assume that S' has a zero-sum subsequence T of length 5. Since S_0 has no zero-sum subsequence of length 5, there exists $i_0 \in [1,r]$ such that T and S_{i_0} have a nontrivial S-inner common divisor Y, where Y is a subsequence of $g_{i_0}^{-1}S_i$. By our assumption, Y is zero-sum. It follows that S_{i_0} is not a minimal zero-sum sequence, $Y = T_{i_0}$, $Y^{-1}T$ is a minimal zero-sum subsequence of length 2, and $Y^{-1}T$ divides $S_0 \prod_{j\in[1,r]\setminus\{i_0\}} g_j^{-1}S_j$. Assume that there exists $j \in [1,r] \setminus \{i_0\}$ such that $Y^{-1}T$ and $g_j^{-1}S_j$ are $S_0 \prod_{j\in[1,r]\setminus\{i_0\}} S_j$-innerly joint. Then the inner common divisor is not zero-sum and hence T and S_j have a non-zero-sum S-inner common divisor, a contradiction. Therefore $Y^{-1}T$ and $\prod_{j\in[1,r]\setminus\{i_0\}} g_j^{-1}S_j$ are not $S_0\prod_{j\in[1,r]\setminus\{i_0\}} S_j$-innerly joint, whence $Y^{-1}T$ divides S_0, i.e., S_0 has a zero-sum subsequence $U = Y^{-1}T$ of length 2.

Now we consider the sequence $S'' := (h_1 \cdot \ldots \cdot h_r)^{-1}S$ and similarly as above we can show S_0 has a zero-sum subsequence V of length 3. If U and V are S_0-innerly disjoint, then UV is a zero-sum subsequence of S_0 of length 5, a contradiction. If U and V have a S_0-inner common divisor Y_1, then $|Y_1| = 1$ and hence Y_1 is not zero-sum. It follows that $T = UT_{i_0}$ and VT'_{i_0} are zero-sum subsequences of S of length 5 and have a non-zero-sum S-inner common divisor Y_1, a contradiction. □

4.4. **Concluding remarks.** Let $G = C_n^2$. If $n \in [3, 10]$, then G has property D by Lemma 2.6.6, whence Lemma 4.19 implies that $\mathsf{s}^*(G) = 4n + 1$. It follows from Theorem 4.21 and Proposition 4.22 that $\mathsf{s}^N(C_n^2) = 4n + 1$ for $n = 3$ or 5. Let $m \geq \frac{7p^3+p^2+2p}{2}$ for some prime $p \geq 47$. Then Theorem 4.18 implies that $\mathsf{s}^N(C_{pm}^2) = 4pm + 1$. All these results support the following conjecture.

Conjecture 4.23. Let $G = C_n^2$, where $n \geq 3$. Then $\mathsf{s}^N(G) = 4n + 1$.

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