Mesh-Independence of an Augmented Lagrangian-SQP Method in Hilbert Spaces and Control Problems for the Burgers Equation

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PhD Thesis

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Abstract. In this research the augmented Lagrangian-SQP algorithm for optimal control of partial differential equations is analyzed. This algorithm has second-order convergence rate provided that a second-order sufficient optimality condition is satisfied. A mesh-independence principle for the augmented Lagrangian-SQP method in Hilbert spaces is proved. It asserts that asymptotically the infinite dimensional algorithm and finite dimensional discretizations have the same convergence property. More precisely, for sufficiently small mesh-size there is at most a difference of one iteration step between the number of steps required by the infinite dimensional method and its discretization to converge within a given tolerance $\varepsilon > 0$. The results are applied to distributed and boundary control problems for the Burgers equation. Finally, numerical examples are shown.

Zusammenfassung. In dieser Arbeit wird das augmentierte Lagrange-SQP Verfahren zur Kontrolle von partiellen Differentialgleichungen analysiert. Die Konvergenzrate dieses Verfahrens ist von zweiter Ordnung, falls eine hinreichende Optimalitätsbedingung zweiter Ordnung erfüllt ist. In der vorliegenden Arbeit wird Gitterunabhängigkeit für das augmentierte Lagrange-SQP Verfahren bewiesen. Damit haben sowohl der unendlich dimensionale Algorithmus wie auch seine endlich dimensionale Diskretisierung gleiches Konvergenzverhalten. Die Anzahl der Iterationen, die für das unendlich dimensionale Verfahren und seine Approximation notwendig sind, um innerhalb einer vorgegebenen Toleranz $\varepsilon > 0$ zu konvergieren, unterscheiden sich deshalb bei hinreichend kleiner Gitterweite höchstens um eine. Diese Resultate werden auf verteilte Kontroll- und Randkontrollprobleme für die Burgers Gleichung angewendet. Abschließend werden numerische Ergebnisse gezeigt.

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Contents

CHAPTER 0

Preface

The mathematical and more specifically the numerical treatment of optimal control problems for nonlinear partial differential equations arising in diverse areas of science receives an increasing amount of attention in the recent past. We mention optimal control problems in combustion (see [**Lio85**] for control of parabolic explosive systems), in phase field modeling [**CH91, HS94**], in super-conductivity [**GHS91, Tin75**] and in fluid dynamics (see [**Gun95**] and references therein). There is great interest for the numerical treatment of such problems.

In this work we study a general class of optimal control problems for nonlinear partial differential equations of the following type

$$(\mathbf{P}) \qquad \min J(y,u) \quad \text{subject to } e(y,u)=0,$$

where $J : \mathcal{Y} \times \mathcal{U} \to \mathbb{R}$ and $e : \mathcal{Y} \times \mathcal{U} \to Y^*$ are sufficiently smooth functions, $\mathcal{Y}$, $\mathcal{U}$ and Y are Hilbert spaces and Y^* is the dual of Y. The sets $\mathcal{Y}$ and $\mathcal{U}$ denote the state and the control space, respectively.

To solve (**P**) we use the augmented Lagrangian-SQP (sequential quadratic programming) technique as developed in [**IK96a**]. In this method the differential equation is treated as an equality constraint which is realized by a Lagrangian term together with a penalty functional. We present an algorithm which has second-order convergence rate and depends upon a second-order sufficient optimality condition. For this method we analyze internal approximations. Converge and rate of convergence results of the discretized algorithm are proved.

We prove different mesh-independence principles for this augmented Lagrangian-SQP method. The mesh-independence principle asserts that, when the augmented Lagrangian-SQP method is applied to an optimal control problem in Hilbert spaces as well as to some finite dimensional discretization of that problem, then the behavior of the discretized process is asymptotically the same as that for the original problem. Under additional convergence properties we are able to give upper bounds for the mesh-size such that the mesh-independence principle is valid for all smaller mesh-sizes.

As an application we investigate control problems for the Burgers equation, where the cost functional J is of tracking type. The Burgers equation was extensively developed by Burgers as a simplified fluid flow model which nonetheless exhibits some of the important aspects of turbulence [**Bur40**]. Later it was derived by Lighthill as a second-order approximation to the one dimensional unsteady Navier-Stokes equations [**Lig56**]. We consider both boundary and distributed control problems. Finally, numerical examples illustrate the theoretical results.

When compared to SQP methods, the augmented Lagrangian-SQP method has the advantage of a more global behavior. For certain examples we found it to be less sensitive with respect to the start-up values, and the region for second-order convergence rate was reached earlier. We shall point out that the penalty term of the augmented Lagrangian functional need not to be implemented but rather that it can be realized by an intermediate first-order Lagrangian update.

The work is organized as follows. In the first chapter we standardize our terminology and introduce abbreviations that we shall use in our work. Moreover, we shall briefly recall some basic facts and definitions, which are necessary in our work. The augmented Lagrangian-SQP method proposed by Ito and Kunisch in [**IK96a**] is studied in Chapter 2. We discuss the internal approximation of this method. Convergence results are presented. Since we have to solve a saddle-point problem at each level of the iteration, this result depends on the Babuška-Brezzi condition for the saddle-point problem. Under additional requirements for the approximation we prove that the rate of convergence of the discrete iterates to the infinite dimensional ones is the same as the discretization error of the used approximation scheme. In Chapter 3 both a strong and an asymptotic mesh-independence principle are proved for the augmented Lagrangian-SQP method. In the forth section we study distributed and boundary control problems for the stationary Burgers equation. The assumptions for the convergence of the infinite dimensional augmented Lagrangian-SQP method are shown. Moreover, we present some numerical examples, which illustrate the mesh-independence principle. For piecewise linear finite elements we give a sufficient condition such that the Babuška-Brezzi condition is satisfied. Chapter 5 deals with distributed and boundary control problems for the unsteady Burgers equation. The conditions for the convergence of the infinite dimensional augmented Lagrangian-SQP algorithm are also proved. Numerical examples are presented, which illustrate that the mesh-independence principle can be observed numerically.

CHAPTER 1

Preliminaries

It is appropriate to standardize our terminology and to introduce abbreviations that will occur throughout the book. Moreover, we shall briefly recall some basics facts and definitions.

1. Functional Analytic Preliminaries

1.1. Bounded Linear Operators. Let $(V, \|\cdot\|_V)$ be a normed linear space. The set

$$B(v;r) = \{w \in V : \|v - w\|_V < r\}$$

is called an *open ball* of radius r around the point v. The open set $U \subset X$ is called a *neighborhood* of $v \in U$ if $B(v;r) \subset U$ for some $r > 0$.

Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed linear spaces. We define the normed linear space

$$L(V,W) = \{A : V \to W \mid A \text{ is linear and continuous}\}$$

supplied with the norm

$$\|A\|_{L(V,W)} = \sup_{\|v\|_V = 1} \|Av\|_W \qquad \text{for all } A \in L(V,W)$$

and set $L(V) = L(V,V)$. It is well-known that $L(V,W)$ is a Banach space if W is complete [**Alt92**, p. 102].

Let V and W be real normed linear spaces and $A \in L(V,W)$. The set $A(V) \subseteq W$ will be called the *range* of A and will be denoted by $\operatorname{ran} A$. The set $\{v \in V : Av = 0\}$ is called the *kernel* of A and is denoted by $\ker A$.

1.2. Hilbert Spaces and their Duals. Let X be a Hilbert space equipped with the inner product $\langle \cdot\,, \cdot \rangle_X$ and the norm

$$\|x\|_X = \sqrt{\langle x, x\rangle_X} \qquad \text{for all } x \in X\,,$$

and let $M \subset X$ be closed. We denote by $M^\perp$ the set of vectors in X which are orthogonal to M, and $M^\perp$ is called the *orthogonal complement* of M.

Definition 1.1. Let V be a real normed linear space. The space $L(V,\mathbb{R})$ is called the (topological) dual space of V and is denoted by V^*. The elements of V^* are called continuous linear functionals. We write $\langle f, v\rangle_{V^*,V}$ for the duality pairing of $f \in V^*$ and $v \in V$.

Let us recall the *Riesz representation theorem*.

THEOREM 1.2. *Let $(X, \langle \cdot\,,\cdot\rangle_X)$ be a real Hilbert space with dual X^*. For each $f \in X^*$ there is a unique $x_f \in X$ such that $\langle f, x\rangle_{X^*,X} = \langle x_f, x\rangle_X$ for all $x \in X$ and $\|x_f\|_X = \|f\|_{X^*}$ hold.*

For the proof we refer the reader to [**RS80**, p. 43].

REMARK 1.3. By Theorem 1.2 the *Riesz isomorphism* $\mathfrak{J}_X : X^* \to X$ is a well-defined, linear and bounded operator, where $x_f = \mathfrak{J}_X f$ satisfies

$$\langle x_f, x\rangle_X = \langle f, x\rangle_{X^*,X} \qquad \text{for all } x \in X\,.$$

We point out that $\mathfrak{J}_X$ is an isometric isomorphism so that $\mathfrak{J}_X$ has norm one. Contents permitting X in the notation of $\mathfrak{J}_X$ is omitted.

The dual space X^* is also a real Hilbert space, and the natural inner product on X^* is given by

$$\langle f, g\rangle_{X^*} = \langle \mathfrak{J} f, \mathfrak{J} g\rangle_X \qquad \text{for all } f, g \in X^*\,. \tag{1.1}$$

For Hilbert spaces X_i, $i = 1, \ldots, m$, the product space $X = X_1 \times \ldots \times X_m$ is supplied with the natural Hilbert space topology. We identify the dual space of X with the product space $X_1^* \times \ldots \times X_m^*$.

DEFINITION 1.4. A sequence $\{v_n\}_{n\in\mathbb{N}}$ converges weakly to an element v of the real normed linear space V, $v_n \rightharpoonup v$, $n \to \infty$, if

$$\lim_{n\to\infty} \langle f, v_n\rangle_{V^*,V} = \langle f, v\rangle_{V^*,V} \qquad \text{for all } f \in V^*\,.$$

REMARK 1.5. By Theorem 1.2 a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to an element $\bar{x}$ of the real Hilbert space X if

$$\lim_{n\to\infty} \langle x_n, x\rangle_X = \langle \bar{x}, x\rangle_X \qquad \text{for all } x \in X\,.$$

1.3. Dual Operators. In this section we define dual operators of bounded linear operators on Hilbert spaces.

DEFINITION 1.6. Let X and Y be two real Hilbert spaces with duals X^* and Y^*, respectively. Then we associate with every $A \in L(X, Y^*)$ a dual (or conjugate) operator $A^* : Y \to X^*$ defined by

$$\langle A^*\lambda, x\rangle_{X^*,X} = \langle Ax, \lambda\rangle_{Y^*,Y} \qquad \text{for all } x \in X,\ \lambda \in Y\,. \tag{1.2}$$

REMARK 1.7. Let X and Y be two real Hilbert spaces and $A \in L(X, Y^*)$. Then the mapping $A^H = \mathfrak{J}_X\, A^*\, \mathfrak{J}_Y : Y^* \to X$ is called the *Hilbert-adjoint* of the operator A, [**TL58**, p. 242]. It follows that

$$\langle A^H g, x\rangle_X = \langle g, Ax\rangle_{Y^*} \quad \text{for all } x \in X,\ g \in Y^*\,.$$

The reader should be careful at the outset that the Hilbert-adjoint of a bounded linear operator is not equal to the dual operator although it is closely related to it. Since formula (1.2) is more suitable in the context of our optimal control problem posed in Chapter 2, we use the dual operator in place of the Hilbert-adjoint.

In the next lemma important properties of dual operators are summarized.

LEMMA 1.8. *Let X and Y be two real Hilbert spaces with duals X^* and Y^*, respectively, and let $A \in L(X, Y^*)$ be given. Its dual operator A^* has the following properties:*

a) $A^* \in L(Y, X^*)$ *and* $\|A^*\|_{L(Y,X^*)} = \|A\|_{L(X,Y^*)}$,
b) $\ker A^* = \ker (A\mathfrak{J}_X A^*)$,
$\overline{\operatorname{ran}(\mathfrak{J}_Y A)} = \ker (\mathfrak{J}_X A^*)^\perp$ *and*
$\overline{\operatorname{ran} A} = \overline{\operatorname{ran}(A\mathfrak{J}_X A^*)}$,
c) *if any of the two subspaces* $\operatorname{ran} A$ *and* $\operatorname{ran}(A\mathfrak{J}_X A^*)$ *is closed, than so the other.*

For the proof we refer the reader to [**DL88**, p. 322] and [**TL58**, p. 244].

LEMMA 1.9. *Let X and Y be two real Hilbert spaces with duals X^* and Y^*, respectively. If $A \in L(X, Y^*)$ is surjective, then A^* is injective.*

PROOF. Since A is surjective, we obtain $\operatorname{ran} A = Y^*$. By Lemma 1.8-b) this implies that $\ker (\mathfrak{J}_X A^*) = \{0\}$. Therefore, $\ker A^* = \{0\}$. $\square$

REMARK 1.10. If A^* is injective then $\overline{\operatorname{ran}(\mathfrak{J}_Y A)} = Y$ by Lemma 1.8-b). From this we derive at least $\overline{\operatorname{ran} A} = Y^*$. Therefore, $\operatorname{ran} A$ is at least dense in Y^*. If in addition $\operatorname{ran} A$ is closed, A is surjective.

LEMMA 1.11. *Let X and Y be two real Hilbert spaces with duals X^* and Y^*, respectively, and $A \in L(X, Y^*)$ be surjective. Then the linear operator $E = A\mathfrak{J}_X A^*$ is bounded and invertible, and $E^{-1} \in L(Y^*, Y)$ holds.*

PROOF. Lemma 1.9 implies that $\ker A^* = \{0\}$. From Lemma 1.8-b) we conclude that $\ker E = \{0\}$. Thus, E is injective. Since A is surjective, $\operatorname{ran} A$ is closed. Applying Lemma 1.8-c) it follows that the set $\operatorname{ran} E$ is also closed so that

$$Y^* = \operatorname{ran} A = \overline{\operatorname{ran} A} = \overline{\operatorname{ran} E} = \operatorname{ran} E\,.$$

Hence, E is surjective and therefore bijective. Since $A \in L(X, Y^*)$ holds, A^* is bounded by Lemma 1.8-a) and therefore E is continuous. Due to the inverse mapping theorem [**RS80**, p. 83] the operator E^{-1} exists and is linear and continuous. $\square$

2. Saddle-Point Problems

Using the augmented Lagrangian-SQP method posed in Chapter 2 we have to solve a saddle-point problem at each level of the iteration. Therefore, our concern will be existence and uniqueness of solutions of saddle-point problems. For more details we refer the books [**BF91, GR80**].

Let X and Y be real Hilbert spaces with dual spaces X^* and Y^*, respectively. We consider a continuous bilinear form $a : X \times X \to \mathbb{R}$ (not necessarily symmetric) satisfying

$$|a(\varphi, \phi)| \leq \|a\| \, \|\varphi\|_X \, \|\phi\|_X \,,$$

where

$$\|a\| = \sup \left\{ \frac{a(\varphi, \phi)}{\|\varphi\|_X \, \|\phi\|_X} \; : \; \varphi, \phi \in X \setminus \{0\} \right\} .$$

Thus, $a(\cdot\,,\cdot)$ defines a linear and continuous operator $A : X \to X^*$ by

$$\langle A\varphi, \phi \rangle_{X^*,X} = a(\varphi, \phi) \qquad \text{for all } \varphi, \phi \in X \,.$$

Let us choose another real-valued continuous bilinear form $b(\cdot\,,\cdot)$ on $X \times Y$ with

$$|b(\varphi, \psi)| \leq \|b\| \, \|\varphi\|_X \, \|\psi\|_Y \,,$$

where

$$\|b\| = \sup \left\{ \frac{b(\varphi, \psi)}{\|\varphi\|_X \, \|\psi\|_Y} \; : \; \varphi \in X \setminus \{0\}, \psi \in Y \setminus \{0\} \right\} .$$

Again we can define a bounded linear operator $B : X \to Y^*$ and its dual operator $B^* : Y \to X^*$ by

$$\langle B\varphi, \psi \rangle_{Y^*,Y} = \langle B^*\psi, \varphi \rangle_{X^*,X} = b(\varphi, \psi) \qquad \text{for all } \varphi \in X, \; \psi \in Y \,.$$

Let $f \in X^*$ and $g \in Y^*$ be given. Our aim is to find $(x, \lambda) \in X \times Y$ solving

$$\left\{ \begin{array}{rcll} a(x, \varphi) + b(\varphi, \lambda) & = & \langle f, \varphi \rangle_{X^*,X} & \text{for all } \varphi \in X \,, \\ b(x, \psi) & = & \langle g, \psi \rangle_{Y^*,Y} & \text{for all } \psi \in Y \,. \end{array} \right. \tag{1.3}$$

This can also be written as

$$\left\{ \begin{array}{rcll} Ax + B^*\lambda & = & f & \text{in } X^* \,, \\ Bx & = & g & \text{in } Y^* \,. \end{array} \right. \tag{1.4}$$

Remark 2.1. If the bilinear form $a(\cdot\,,\cdot)$ is symmetric, (1.3) are the optimality conditions of the saddle-point problem

$$\inf_{\varphi \in X} \sup_{\psi \in Y} \frac{1}{2} \, a(\varphi, \varphi) + b(\varphi, \psi) - \langle f, \varphi \rangle_{X^*,X} - \langle g, \psi \rangle_{Y^*,Y} \,.$$

Theorem 2.2. *Assume that there exists a constant $\kappa > 0$ such that*

$$\langle A\varphi, \varphi \rangle_{X^*,X} \geq \kappa \, \|\varphi\|_X^2 \qquad \textit{for all } \varphi \in \ker B \,.$$

Then (1.4) is well-defined, i.e. there is a unique pair (x, λ) solving (1.4), if and only if there exists a constant $\beta > 0$ such that

$$\inf_{\psi \in Y \setminus \{0\}} \; \sup_{\varphi \in X \setminus \{0\}} \frac{\langle B^*\psi, \varphi \rangle_{X^*,X}}{\|\psi\|_Y \, \|\varphi\|_X} \geq \beta \,. \tag{1.5}$$

For the proof we refer the reader to [**GR80**, p. 61].

REMARK 2.3. Condition (1.5) is called the *inf-sup-* or the *Babuška-Brezzi condition* [**Bab73, Bre74**].

LEMMA 2.4. *Let B be a bounded linear operator from the real Hilbert space X into the dual Y^* of the real Hilbert space Y. B is surjective if and only if the inf-sup condition (1.5) holds.*

PROOF. Applying the closed range theory [**Bre87**, p. 29] we derive that B is surjective if and only if there exists a constant $\beta > 0$ such that

$$\|\psi\|_Y \leq \frac{1}{\beta} \|B^*\psi\|_{X^*} \qquad \text{for all } \psi \in Y\,. \tag{1.6}$$

Let B be surjective. Hence,

$$\inf_{\psi \in Y \setminus \{0\}} \sup_{\varphi \in X \setminus \{0\}} \frac{\langle B^*\psi, \varphi\rangle_{X^*,X}}{\|\psi\|_Y \|\varphi\|_X} = \inf_{\psi \in Y \setminus \{0\}} \frac{\|B^*\psi\|_{X^*}}{\|\psi\|_Y} \overset{(1.6)}{\geq} \beta\,,$$

which gives (1.5). Conversely, the inf-sup condition (1.5) yields

$$\beta \|\psi\|_Y \leq \sup_{\varphi \in X \setminus \{0\}} \frac{\langle B^*\psi, \varphi\rangle_{X^*,X}}{\|\varphi\|_X} = \|B^*\psi\|_{X^*} \qquad \text{for all } \psi \in Y$$

so that (1.6) follows. □

3. Function Spaces

Now we introduce the function spaces, which we shall need in our work, and present some preliminaries.

3.1. Classical Function Spaces. Let Ω denote an open and bounded set in $\mathbb{R}^n$. A point of $\mathbb{R}^n$ is written as $x = (x_1, \dots, x_n)$. If $\mathbf{a} = (a_1, \dots, a_n)$ is an n-tuple of nonnegative integers a_j, we call $\mathbf{a}$ a *multi-index* and denote by $x^{\mathbf{a}}$ the monomial $x_1^{a_1} \cdots x_n^{a_n}$, which has degree $|\mathbf{a}| = \sum_{j=1}^n a_j$. Similarly,

$$D^{\mathbf{a}} = \frac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \frac{\partial^{a_n}}{\partial x_n^{a_n}}$$

denotes a differential operator of order $|\mathbf{a}|$. With this notation $D^{(0,\dots,0)}\varphi = \varphi$ for a function φ on Ω. We shall write $G \subset\subset \Omega$ provided $\overline{G} \subset \Omega$ and $\overline{G}$ is a compact subset of $\mathbb{R}^n$. If φ is a function defined on G, we define the *support* of φ as

$$\operatorname{supp} \varphi = \overline{\{x \in G : \varphi(x) \neq 0\}}\,.$$

We say that φ has *compact support* in Ω if $\operatorname{supp} \varphi \subset\subset \Omega$. For any nonnegative integer m let $C^m(\Omega)$ be the vector space consisting of all functions φ which, together with all their partial derivatives $D^{\mathbf{a}}\varphi$ of order $|\mathbf{a}| \leq m$, are continuous on Ω. We abbreviate $C^0(\Omega) = C(\Omega)$. Let $C^\infty(\Omega) = \bigcap_{m=1}^\infty C^m(\Omega)$. The subspace $C_0^\infty(\Omega)$ consists of all those functions in $C^\infty(\Omega)$, which have compact support in Ω.

We define the linear space $C^m(\overline{\Omega})$ to consist of all those functions $\varphi \in C^m(\Omega)$ for which $D^{\mathbf{a}}\varphi$ is bounded and uniformly continuous on Ω for $0 \leq |\mathbf{a}| \leq m$. The linear space $C^m(\overline{\Omega})$ is complete with the norm

$$\|\varphi\|_{C^m(\overline{\Omega})} = \max_{0\leq|\mathbf{a}|\leq m} \sup_{x\in\Omega} |D^{\mathbf{a}}\varphi(x)| .$$

To shorten notation we write $\|\cdot\|_{C^m}$ instead of $\|\cdot\|_{C^m(\overline{\Omega})}$, where no confusion of domain may occur.

3.2. Lebesgue Spaces. We denote by $L^p(\Omega)$ the class of all measurable functions φ, defined on Ω, for which

$$\int_\Omega |\varphi|^p \, dx < \infty .$$

The functional $\|\cdot\|_{L^p(\Omega)}$ defined by

$$\|\varphi\|_{L^p(\Omega)} = \Big(\int_\Omega |\varphi|^p \, dx \Big)^{\frac{1}{p}}$$

is a norm on $L^p(\Omega)$ provided $1 \leq p < \infty$.

A function φ, measurable on Ω, is called *essentially bounded* on Ω provided there exists a constant C for which $|\varphi(x)| \leq C$ almost everywhere (a.e. for short) on Ω. The greatest lower bound of such constants is called the *essential supremum* of φ on Ω and is denoted by $\operatorname{ess\,sup}_{x\in\Omega} |\varphi(x)|$. We denote by $L^\infty(\Omega)$ the vector space consisting of all functions φ that are essentially bounded on Ω. The functional

$$\|\varphi\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |\varphi(x)|$$

is a norm on $L^\infty(\Omega)$. In situations, where no confusion of domains may occur, we shall write $\|\cdot\|_{L^p}$ in place of $\|\cdot\|_{L^p(\Omega)}$.

Let $\varphi \in L^p(\Omega)$, $\phi \in L^q(\Omega)$ and $\psi \in L^r(\Omega)$, where $1 \leq p, q, r \leq \infty$ and

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$$

hold. We infer from the Hölder inequality [**GT77**, p. 138] that

$$\int_\Omega \varphi\phi\psi \, dx \leq \|\varphi\|_{L^p} \, \|\phi\|_{L^q} \, \|\psi\|_{L^r} \tag{1.7}$$

holds.

3.3. Sobolev Spaces. Let us define a functional $\|\cdot\|_{W^{m,p}(\Omega)}$, where m is a nonnegative integer and $1 \leq p \leq \infty$, as follows

$$\begin{aligned} \|\varphi\|_{W^{m,p}(\Omega)} &= \Big(\sum_{0\leq|\mathbf{a}|\leq m} \|D^{\mathbf{a}}\varphi\|_{L^p}^p \Big)^{\frac{1}{p}} \qquad \text{if } 1 \leq p < \infty , \\ \|\varphi\|_{W^{m,\infty}(\Omega)} &= \max_{0\leq|\mathbf{a}|\leq m} \|D^{\mathbf{a}}\varphi\|_{L^\infty} \end{aligned}$$

for every function φ for which the right side makes sense. In situations, where no confusion of domains may occur, we shall write $\|\cdot\|_{W^{m,p}}$ in place of $\|\cdot\|_{W^{m,p}(\Omega)}$. We define to every given values of m and p the *Sobolev spaces*

$$\begin{aligned} W^{m,p}(\Omega) \quad &= \quad \text{the completion of } \{\varphi \in C^m(\Omega) : \|\varphi\|_{W^{m,p}} < \infty\} \\ &\qquad \text{with respect to the norm } \|\cdot\|_{W^{m,p}(\Omega)}\,, \\ W_0^{m,p}(\Omega) \quad &= \quad \text{the closure of } C_0^\infty(\Omega) \text{ in the space } W^{m,p}(\Omega). \end{aligned}$$

LEMMA 3.1. *The Sobolev space $W^{m,2}(\Omega)$ is a separable Hilbert space endowed with the inner product*

$$\langle \varphi, \psi \rangle_{W^{m,2}} = \sum_{0 \le |\mathbf{a}| \le m} \int_\Omega D^{\mathbf{a}}\varphi \, D^{\mathbf{a}}\psi \, dx\,.$$

For the proof we refer the reader to [**Ada75**, p. 45 and p. 47].

REMARK 3.2. We shall write $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$ and $H_0^m(\Omega)$ instead of $W_0^{m,2}(\Omega)$. Moreover, we have $H^0(\Omega) = L^2(\Omega)$.

We define the linear space $\mathcal{H}$ by

$$\mathcal{H} = \{\varphi \in H^1(0,1) : \varphi(0) = 0\}\,. \tag{1.8}$$

It is obvious that $\mathcal{H}$ is a closed subspace of $H^1(0,1)$. For the proof of the next lemma we refer the reader to [**Maz79**, p. 26].

LEMMA 3.3. *The linear space $\mathcal{H}$ is a Hilbert space endowed with the inner product*

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_0^1 \varphi' \psi' \, dx \qquad \textit{for } \varphi, \psi \in \mathcal{H} \tag{1.9}$$

and the associated norm

$$\|\varphi\|_{\mathcal{H}} = \Big(\int_0^1 |\varphi'|^2 \, dx \Big)^{\frac{1}{2}} \qquad \textit{for } \varphi \in \mathcal{H}\,. \tag{1.10}$$

We shall use frequently the inequality of the following lemma both in Chapter 4 and Chapter 5.

LEMMA 3.4. *For all $\varphi \in \mathcal{H}$ and $1 \le p \le \infty$ we have*

$$\|\varphi\|_{L^p} \le \|\varphi\|_{\mathcal{H}}\,.$$

PROOF. Let $\varphi \in \{\phi \in C^1[0,1] \mid \phi(0) = 0\}$. By Hölder's inequality it follows that

$$|\varphi(x)| \le \int_0^x |\varphi'| \, dt \le \Big(\int_0^1 |\varphi'|^2 \, dt \Big)^{\frac{1}{2}} = \|\varphi\|_{\mathcal{H}}\,. \tag{1.11}$$

We infer from (1.11) that

$$\|\varphi\|_{L^p} = \Big(\int_0^1 |\varphi|^p \, dx \Big)^{\frac{1}{p}} \le \Big(\int_0^1 \|\varphi\|_{\mathcal{H}}^p \, dx \Big)^{\frac{1}{p}} = \|\varphi\|_{\mathcal{H}} \quad \text{for } 1 \le p < \infty$$

and

$$\|\varphi\|_{L^\infty} = \operatorname{ess}\sup_{x\in(0,1)} |\varphi(x)| \leq \|\varphi\|_{\mathcal{H}} .$$

Since $\{\phi \in C^1[0,1] \mid \phi(0) = 0\}$ is dense in $\mathcal{H}$, the claim follows for all $\varphi \in \mathcal{H}$ and $1 \leq p \leq \infty$. □

3.4. Young's and Agmon's Inequality. Because the following inequality are used frequently in this work, we shall give complete formulations of them here and only refer to their names whenever necessary.

Young's inequality [**Alt92**, p. 28]*:* Let $a, b \in [0, \infty)$ and $\varepsilon > 0$. Then we have

$$ab \leq \frac{1}{\varepsilon^p}\frac{a^p}{p} + \varepsilon^q \frac{b^q}{q}, \qquad \text{where } 1 < p < \infty \text{ and } \frac{1}{p} + \frac{1}{q} = 1 .$$

Agmon's inequality [**Tem88**]*:* For every $\varphi \in H^1(0,1)$, there exists a constant $c_A > 0$ such that

$$\|\varphi\|_{L^\infty} \leq c_A \|\varphi\|_{L^2}^{1/2} \|\varphi\|_{H^1}^{1/2} .$$

3.5. Embedding Results. The normed linear space V is called *continuously embedded* in the normed linear space W, and we write $V \hookrightarrow W$ to designate this embedding, provided that

a) V is a linear subspace of W, and
b) the identity operator I defined on V into W by $Iv = v$ for all $v \in V$ is continuous.

V is *compactly embedded* in W, $V \hookrightarrow\hookrightarrow W$, if the embedding operator I introduced in b) is compact.

LEMMA 3.5. *The Hilbert space $H^1(0,1)$ is compactly embedded in $L^p(0,1)$ for every $p \in [1, \infty]$.*

For the proof we refer the reader to [**Ada75**, p. 144].

REMARK 3.6. The embedding operator maps weakly convergent sequences in $H_0^1(0,1)$ into norm convergent sequences in $L^p(0,1)$ for $1 \leq p \leq \infty$, [**RS80**, p. 199]. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence in $H_0^1(0,1)$ such that $\varphi_n \rightharpoonup \varphi \in H_0^1(0,1)$, i.e.

$$\lim_{n\to\infty} \langle f, \varphi_n \rangle_{H^{-1},H_0^1} = \langle f, \varphi \rangle_{H^{-1},H_0^1} \qquad \text{for all } f \in H^{-1}(0,1) .$$

Here, $H^{-1}(0,1)$ denotes the dual of $H_0^1(0,1)$. Then it follows that

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{L^p} = 0$$

for all p with $1 \leq p \leq \infty$. We emphasize that Lemma 3.5 is only true in the one dimensional case.

LEMMA 3.7. *If $j \geq 0$ and $m > \frac{1}{2} + j$, then $H^m(0,1)$ is compactly embedded into $C^j[0,1]$.*

For the proof we refer the reader to [**Ada75**, p. 144].

REMARK 3.8. In particular, Lemma 3.7 yields $\varphi \in C^j[0,1]$ for any $\varphi \in H^{j+1}(0,1)$ and $j \geq 0$.

3.6. The Bochner Integral. To deal with the unsteady Burgers equation we shall make use of the following function spaces. Let $T > 0$. We denote by $L^p(0,T;B)$ the Banach space of (equivalence classes of) functions φ measurable on $(0,T)$ into the Banach space B such that

a) φ is measurable for dt,
b)

$$\left\{ \begin{array}{ll} \|\varphi\|_{L^p(0,T;B)} = \Big(\displaystyle\int_0^T \|\varphi(t,\cdot)\|_B \, dt \Big)^{\frac{1}{p}} < \infty & \text{if } 1 \leq p < \infty\,, \\ \|\varphi\|_{L^\infty(0,T;B)} = \operatorname{ess} \sup\limits_{t\in(0,T)} \|\varphi(t,\cdot)\|_B < \infty & \text{if } p = \infty\,. \end{array} \right. \tag{1.12}$$

When there is no risk of confusion, we shall utilize the simplified notation $L^p(B)$.

REMARK 3.9. a) Let B_1 and B_2 be two Banach spaces with $B_1 \hookrightarrow B_2$. Then the Banach space $L^p(B_1)$ is continuously embedded into $L^p(B_2)$ for $1 \leq p \leq \infty$, [**DL92**, p. 471].
b) Let X be a Hilbert space with dual X^* and $0 < T < \infty$. Then we can identify the dual of $L^2(X)$ with the space $L^2(X^*)$, [**Zei90**, p. 411].

Let B_1 be a Banach space. The next theorem deals with the compact embedding of an appropriate function space into the Banach space $L^p(B_1)$. For the proof we refer the reader to [**Tem79**, p. 271].

THEOREM 3.10. *Let B_0, B_1 and B_2 be three Banach spaces such that*

$$B_0 \hookrightarrow\hookrightarrow B_1 \hookrightarrow B_2$$

and $p, q \in (1,\infty)$. Then the space

$$\mathcal{W} = \{\varphi \in L^p(B_0),\ \varphi_t \in L^q(B_2)\}$$

equipped with the norm

$$\|\varphi\|_{\mathcal{W}} = \|\varphi\|_{L^p(B_0)} + \|\varphi_t\|_{L^q(B_2)}$$

is compactly embedded into $L^p(B_1)$.

Let B be a Banach space and $0 < T < \infty$. The space $C([0,T];B)$ consists of all continuous functions $\varphi : [0,T] \to B$ with the norm

$$\|\varphi\|_{C([0,T];B)} = \max_{t\in[0,T]} \|\varphi(t,\cdot)\|_B\,.$$

Contents permitting $[0,T]$ in the notation of $C([0,T];B)$ is omitted.

Let us consider two real, separable Hilbert spaces V and H. It is supposed that V is dense in H so that, by identifying H and its dual H^*, we have

$$V \hookrightarrow H = H^* \hookrightarrow V^*, \tag{1.13}$$

each embedding being dense. The space $W(0,T;V)$ is defined by

$$W(0,T;V) = \left\{\varphi : \varphi \in L^2(V),\, \varphi_t \in L^2(V^*)\right\}.$$

For brevity we write $W(V)$ in place of $W(0,T;V)$.

LEMMA 3.11. *The space $W(0,T;V)$ endowed with the norm*

$$\|\varphi\|_{W(V)} = \sqrt{\|\varphi\|^2_{L^2(V)} + \|\varphi_t\|^2_{L^2(V^*)}}$$

is a Hilbert space.

For the proof we refer the reader to [**DL92**, p. 473].

LEMMA 3.12. *For $T > 0$ every $\varphi \in W(V)$ is almost everywhere equal to a continuous function of $(0,T)$ in H. Further, $W(V)$ is continuously embedded into $C(H)$.*

For the proof we refer the reader to [**DL92**, p. 473].

REMARK 3.13. Lemma 3.12 implies that there exists an embedding constant $c_E > 0$ such that

$$\|\varphi\|_{C(H)} \le c_E \|\varphi\|_{W(V)} \qquad \text{for all } \varphi \in W(V).$$

4. Linear Boundary Value Problems of Second Order

In this section we review some standard facts on Sturm-Liouville systems describing linear boundary value problems of second order for ordinary differential equations. If there exists a unique solution of the boundary value problem, we shall see in Theorem 4.4 that this solution is given by an integral formula. By easy computation this representation allows us to derive an estimate for the solution in the L^∞- or in the $\mathcal{H}$-norm. We shall discuss some applications, which will be needed in Chapter 4.

4.1. Sturm-Liouville Systems. Let us consider the differential equation

$$(pv')' + qv = f \tag{1.14a}$$

in the interval $(0,1)$ with the boundary conditions

$$B_0 v = \alpha v(0) + \alpha_1 v'(0) = 0, \tag{1.14b}$$

$$B_1 v = \beta v(1) + \beta_1 v'(1) = 0, \tag{1.14c}$$

where $p \in C^1[0,1]$ with $p > 0$, $q \in C[0,1]$, $f \in L^2(0,1)$, α, α_1, β and β_1 are real numbers and $\alpha^2 + \alpha_1^2 > 0$, $\beta^2 + \beta_1^2 > 0$.

Problem (1.14) is called a *Sturm-Liouville system.* Here we will discuss solutions of the Sturm-Liouville system by relating the system to a certain compact self-adjoint integral operator.

We define

$$D_0 = \left\{\varphi \in H^2(0,1) \,:\, B_0\varphi = 0\right\} .$$

The set D_1 is defined similarly but each $\varphi \in D_1$ satisfies (1.14c) instead of (1.14b). We put $D = D_0 \cap D_1$.

Let us define the linear unbounded operator $\Lambda : D \to L^2(0,1)$ by

$$\Lambda v = (pv')' + qv \,. \tag{1.15}$$

The operator Λ is called a *Sturm-Liouville operator.* Equation (1.14a) can thus be written in the form $\Lambda v = f$.

It is well-known that a solution v to the homogeneous equation $\Lambda v = 0$ together with the boundary conditions (1.14b) and (1.14c) is a *classical solution*, i.e. $v \in C[0,1] \cap C^2(0,1)$ holds [**GT77**, p. 177].

The following result is taken from the theory of ordinary differential equations [**Heu91**, p. 377].

LEMMA 4.1. *Let $\varphi_0, \varphi_1 \in D$ denote two linear independent fundamental solutions of the homogeneous equation $\Lambda v = 0$. Then problem (1.14) is unique solvable if and only if*

$$\det\begin{pmatrix} B_0\varphi_0 & B_0\varphi_1 \\ B_1\varphi_0 & B_1\varphi_1 \end{pmatrix} \neq 0 \,. \tag{1.16}$$

REMARK 4.2. Let φ_0 and φ_1 denote two fundamental solutions of $\Lambda v = 0$ satisfying (1.16). Setting

$$\psi_0 = (B_0\varphi_1)\,\varphi_0 - (B_0\varphi_0)\,\varphi_1 \quad \text{and} \quad \psi_1 = (B_1\varphi_1)\,\varphi_0 - (B_1\varphi_0)\,\varphi_1$$

we obtain again two fundamental solutions satisfying $B_0\psi_0 = 0$ and $B_1\psi_1 = 0$. Thus, we can assume that $\varphi_0 \in D_0$ and $\varphi_1 \in D_1$.

We introduce the *Wronskian* of φ_0 and φ_1 by

$$W(x) = \det\begin{pmatrix} \varphi_0(x) & \varphi_1(x) \\ \varphi_0'(x) & \varphi_1'(x) \end{pmatrix} = \varphi_0(x)\varphi_1'(x) - \varphi_0'(x)\varphi_1(x) \,.$$

Note that

$$(pW)' = p'W + pW' = (p\varphi_1')'\varphi_0 - (p\varphi_0')'\varphi_1 = q\varphi_1\varphi_0 - q\varphi_0\varphi_1 = 0 \,.$$

Hence, $p(x)W(x) = p(0)W(0)$ for all $x \in [0,1]$.

Now we define $g : [0,1] \times [0,1] \to \mathbb{R}$ by

$$g(x,t) = \begin{cases} \dfrac{\varphi_0(x)\varphi_1(t)}{p(0)W(0)}\,, & 0 \le x \le t \le 1\,, \\[2ex] \dfrac{\varphi_0(t)\varphi_1(x)}{p(0)W(0)}\,, & 0 \le t \le x \le 1\,. \end{cases} \tag{1.17}$$

The function g is called the *Green function* for the operator Λ. From the definition of g we infer directly the next result.

LEMMA 4.3. *The function g defined in (1.17) is real-valued, continuous, and $g(x,t) = g(t,x)$.*

THEOREM 4.4. *Let $p \in C^1[0,1]$ with $p > 0$, $q \in C[0,1]$, $f \in L^2(0,1)$, α, α_1, β and β_1 be real numbers and $\alpha^2 + \alpha_1^2 > 0$, $\beta^2 + \beta_1^2 > 0$. Let Λ the Sturm-Liouville operator defined in (1.15) and $\varphi_0 \in D_0$ and $\varphi_1 \in D_1$ two fundamental solutions for $\Lambda v = 0$ satisfying (1.16). If g is the Green function for Λ defined by (1.17), we introduce the integral operator $K : L^2(0,1) \to L^2(0,1)$ by*

$$(Kf)(x) = \int_0^1 g(x,t) f(t)\, dt\,.$$

Then the operator K is compact and self-adjoint. Further, we have $\operatorname{ran} K = D$, *$\Lambda(Kf) = f$ for all $f \in L^2(0,1)$, and $K\Lambda\varphi = \varphi$ for all $\varphi \in D$.*

For the proof we refer the reader to [**Con78**, p. 52].

REMARK 4.5. We infer directly from Theorem 4.4 that v belongs to the set D and solves the boundary value problem (1.14). Since φ_0 and φ_1 satisfy condition (1.16), the homogeneous problem

$$\Lambda v = 0\,, \quad B_0 v = B_1 v = 0$$

has only the trivial solution $v = 0$. Consequently, the solution of (1.14) is unique, and the unique solution of problem (1.14) is given by

$$v(x) = \int_0^1 g(x,t) f(t)\, dt\,,$$

where g is the Green function for Λ.

4.2. Applications. Let $y \in C[0,1]$ and $\nu > 0$. We choose

$$p(x) = \exp\left(\frac{1}{\nu}\int_0^x y(s)\, ds\right)\,, \quad q = 0\,, \quad \alpha = 1\,, \quad \alpha_1 = 0\,.$$

Thus, we consider the differential equation

$$v'' + \frac{y}{\nu} v' = f \tag{1.18a}$$

in the interval $(0,1)$ with homogeneous Dirichlet boundary conditions

$$B_0 v = v(0) = 0\,, \tag{1.18b}$$

$$B_1 v = \beta v(1) + \beta_1 v'(1) = 0\,, \tag{1.18c}$$

where we assume that $\beta^2 + \beta_1^2 > 0$.

LEMMA 4.6. *Let $y \in C[0,1]$. Then the two functions $\phi_0(x) = 1$ and*

$$\phi_1(x) = \int_0^x \exp\left(-\frac{1}{\nu}\int_0^t y(s)\,ds\right) dt$$

are linear independent fundamental solutions of

$$v'' + \frac{y}{\nu}\,v' = 0 \qquad \textit{in } (0,1)\,. \tag{1.19}$$

PROOF. Clearly, ϕ_0 solves (1.19). From

$$\phi_1'(x) = \exp\left(-\frac{1}{\nu}\int_0^x y(s)\,ds\right)$$

and

$$\phi_1''(x) = -\frac{y(x)}{\nu}\,\exp\left(-\frac{1}{\nu}\int_0^x y(s)\,ds\right)$$

we see that ϕ_1 also solves (1.19). Finally, it is obvious that ϕ_0 and ϕ_1 are both linear independent. $\square$

REMARK 4.7. Since $B_0\phi_0 = 1$ and $B_0\phi_1 = 0$ are valid, we calculate

$$\begin{aligned}
\det\begin{pmatrix} B_0\phi_0 & B_0\phi_1 \\ B_1\phi_0 & B_1\phi_1 \end{pmatrix} &= B_1\phi_1 \\
&= \beta\int_0^1 \exp\left(-\frac{1}{\nu}\int_0^t y(s)\,ds\right) dt \\
&\quad +\beta_1 \exp\left(-\frac{1}{\nu}\int_0^1 y(s)\,ds\right).
\end{aligned}$$

If $B_1\phi_1 \neq 0$ holds, then problem (1.18) is unique solvable by Lemma 4.1. As $v = 0$ solves (1.19), there exists no nontrivial solution of (1.19).

Let

$$\begin{aligned}
\varphi_0(x) &= -\int_0^x \exp\left(-\frac{1}{\nu}\int_0^t y(s)\,ds\right) dt\,, \\
\varphi_1(x) &= B_1\phi_1 - \beta\int_0^x \exp\left(-\frac{1}{\nu}\int_0^t y(s)\,ds\right) dt\,.
\end{aligned}$$

Observe that φ_0 and φ_1 are two fundamental solutions of (1.19) with $\varphi \in D_0$ and $\varphi_1 \in D_1$. The Green function is given by

$$g(x,t) = \begin{cases} \dfrac{\varphi_0(x)\varphi_1(t)}{B_1\phi_1}\,, & 0 \le x \le t \le 1\,, \\[2ex] \dfrac{\varphi_0(t)\varphi_1(x)}{B_1\phi_1}\,, & 0 \le t \le x \le 1\,. \end{cases} \tag{1.20}$$

The next result follows directly from Remark 4.5.

LEMMA 4.8. *Let $f \in L^2(0,1)$, $\beta^2 + \beta_1 > 0$, $B_1\phi_1 \neq 0$ and let g denote the Green function given by (1.20). Then there exists a unique solution $v \in D \subset \mathcal{H}$ of (1.18), which possesses the integral representation*

$$v(x) = \int_0^1 g(x,t) f(t)\, dt \qquad \text{for } x \in [0,1]\,. \tag{1.21}$$

REMARK 4.9. Lemma 4.8 allows us to give an L^∞-estimate for the solution of (1.18):

$$\begin{aligned} \|v\|_{L^\infty} &\leq \frac{1}{|B_1\phi_1|} \|\varphi_0\, \varphi_1\|_{L^\infty} \|f\|_{L^2} \\ &\leq \exp\left(\frac{1}{\nu} \|y\|_{L^1}\right) \left(1 + \frac{|\beta|}{|B_1\phi_1|} \exp\left(\frac{1}{\nu} \|y\|_{L^1}\right)\right) \|f\|_{L^2}\,. \end{aligned}$$

Now let us consider a second application. For this purpose we assume that $y \in C^1[0,1]$ and $\nu > 0$. We choose

$$p(x) = \exp\left(\frac{1}{\nu} \int_0^x y(s)\, ds\right)\,, \quad q = y'\,, \quad \alpha = 1\,, \quad \alpha_1 = 0\,.$$

Thus, we consider the differential equation

$$v'' - \frac{1}{\nu}(yv)' = f \tag{1.22a}$$

in the interval $(0,1)$ with the boundary conditions

$$\begin{aligned} B_0 v &= v(0) = 0\,, && (1.22b)\\ B_1 v &= \beta v(1) + \beta_1 v'(1) = 0\,, && (1.22c) \end{aligned}$$

where $\beta^2 + \beta_1^2 > 0$ is satisfied.

LEMMA 4.10. *Let $y \in C^1[0,1]$. Then the two functions*

$$\phi_0(x) = \exp\left(\frac{1}{\nu} \int_0^x y(s)\, ds\right)$$

and

$$\phi_1(x) = \exp\left(\frac{1}{\nu} \int_0^x y(s)\, ds\right) \int_0^x \exp\left(-\frac{1}{\nu} \int_0^t y(s)\, ds\right) dt$$

are linear independent fundamental solutions of

$$v'' - \frac{1}{\nu}(yv)' = 0\,, \qquad 0 < x < 1\,. \tag{1.23}$$

PROOF. We begin by calculating the first and second derivatives of ϕ_0 and ϕ_1:

$$\phi_0' = \frac{y}{\nu}\phi_0\,, \quad \phi_0'' = \left(\left(\frac{y}{\nu}\right)^2 + \frac{y'}{\nu}\right)\phi_0$$

and

$$\phi_1' = \frac{y}{\nu}\,\phi_1 + 1\,, \quad \phi_1'' = \left(\left(\frac{y}{\nu}\right)^2 + \frac{y'}{\nu}\right)\phi_1 + \frac{y}{\nu}\,.$$

Hence,

$$\phi_0'' - \frac{1}{\nu}\,(y\phi_0)' = \left(\left(\frac{y}{\nu}\right)^2 + \frac{y'}{\nu} - \left(\frac{y}{\nu}\right)^2 - \frac{y'}{\nu}\right)\phi_0 = 0$$

and

$$\phi_1'' - \frac{1}{\nu}\,(y\phi_1)' = \left(\left(\frac{y}{\nu}\right)^2 + \frac{y'}{\nu} - \left(\frac{y}{\nu}\right)^2 - \frac{y'}{\nu}\right)\phi_1 + \frac{y}{\nu} - \frac{y}{\nu} = 0\,.$$

Obviously, ϕ_0 and ϕ_1 are linear independent such that the claim follows. $\square$

REMARK 4.11. Since $B_0\phi_0 = 1$ and $B_0\phi_1 = 0$ holds, we calculate

$$\begin{aligned}\det\begin{pmatrix} B_0\phi_0 & B_0\phi_1 \\ B_1\phi_0 & B_1\phi_1 \end{pmatrix} &= B_1\phi_1 \\ &= \left(\beta + \beta_1\,\frac{y(1)}{\nu}\right)\int_0^1 \exp\left(\frac{1}{\nu}\int_t^1 y(s)\,ds\right)dt + \beta_1\,.\end{aligned}$$

If $B_1\phi_1 \neq 0$ holds, problem (1.22) is unique solvable by Lemma 4.1. As $v = 0$ solves (1.23), there exists no nontrivial solution of (1.23).

Let

$$\begin{aligned}\varphi_0(x) &= -\exp\left(\frac{1}{\nu}\int_0^x y(s)\,ds\right)\int_0^x \exp\left(-\frac{1}{\nu}\int_0^t y(s)\,ds\right)dt \\ \varphi_1(x) &= B_1\phi_1 \exp\left(\frac{1}{\nu}\int_0^x y(s)\,ds\right) \\ &\quad - \left[\left(\beta + \beta_1\,\frac{y(1)}{\nu}\right)\exp\left(\frac{1}{\nu}\int_0^1 y(s)\,ds\right)\right] \\ &\quad \cdot \exp\left(\frac{1}{\nu}\int_0^x y(s)\,ds\right)\int_0^x \exp\left(-\frac{1}{\nu}\int_0^t y(s)\,ds\right)dt\,.\end{aligned}$$

If $B_1\phi_1 \neq 0$ holds, Remark 4.11 implies that φ_0 and φ_1 are also fundamental solutions of (1.22a) with $\varphi_0 \in D_0$ and $\varphi_1 \in D_1$, respectively. Let the Green function associated with problem (1.22) be given by

$$g(x,t) = \begin{cases} \dfrac{\varphi_0(x)\varphi_1(t)}{B_1\phi_1}\,, & 0 \le x \le t \le 1\,, \\[2ex] \dfrac{\varphi_0(t)\varphi_1(x)}{B_1\phi_1}\,, & 0 \le t \le x \le 1\,. \end{cases} \tag{1.24}$$

Applying Remark 4.5 we derive the following result.

LEMMA 4.12. *Let* $f \in L^2(0,1)$, $\beta^2 + \beta_1^2 > 0$, $B_1\phi_1 \neq 0$ *and let* g *denote the Green function given by (1.24). Then there exists a unique solution* $v \in D \subset \mathcal{H}$ *of (1.22) with the integral representation*

$$v(x) = \int_0^1 g(x,t) f(t)\, dt \qquad \textit{for } x \in [0,1]\,.$$

REMARK 4.13. Lemma 4.12 allows us to give an estimate for the solution of (1.22) in the $\mathcal{H}$-norm, which is introduced in (1.10):

$$\|v\|_{\mathcal{H}} \leq \max_{(x,t)\in[0,1]^2} |g_x(x,t)|\, \|f\|_{L^2} \leq c_\circ \|f\|_{L^2}\,,$$

where

$$c_\circ = \exp\left(\frac{3}{\nu}\|y\|_{L^1}\right)\left(1 + \frac{\|y\|_{L^\infty}}{\nu}\right)^2 \left[1 + \frac{|B_1\phi_0|}{|B_1\phi_1|}\exp\left(\frac{1}{\nu}\|y\|_{L^1}\right)\right]\,.$$

CHAPTER 2

The Augmented Lagrangian-SQP Method

1. Introduction

Augmented Lagrangian methods are hybrid methods between Lagrange-multiplier- and penalty-methods. Since these methods were introduced for the finite dimensional case by Hestenes [**Hes68**] and Powell [**Pow68**] there has come off a radical transformation of the area of Lagrange multipliers for constrained optimization. We refer also to the books of Bertsekas [**Ber82**] and Hestenes [**Hes75**].

The infinite dimensional case is studied much less. If the problem has equality constraints, Polyak and Tretyakov [**PT73**] give an elegant treatise of the augmented Lagrangian method. In the book of Fortin and Glowinski [**FG83**] augmented Lagrangian methods are developed systematically for equality constraints as a technique for solving nonlinear partial differential equations. In [**IK87**] Ito and Kunisch apply a technique realized by an augmented Lagrangian formulation for parameter estimation in elliptic partial differential equations. An augmented Lagrangian method for the minimization of a nonlinear functional in the presence of an equality and an affine inequality constraint is considered in [**IK90**]. Ito, Kroller and Kunisch discuss their numerical experience with an augmented Lagrangian method for estimating the coefficient in an elliptic equation from some given measurement [**IKK91**]. We refer to the paper [**IK91**], where Ito and Kunisch develop an estimation procedure for the diffusion coefficient in a parabolic partial differential equation from the knowledge of the state. In [**IK96a**] Ito and Kunisch continue their efforts in developing stable and efficient techniques to solve parameter estimation problems formulated as nonlinear optimization problems. Such problems are ill-posed in the sense of possible lack of continuous dependence of the minimizers with respect to perturbations of the problem. The authors concentrate on second-order methods and analyze the augmented Lagrangian-SQP technique with second-order update of the Lagrange multiplier in a general Hilbert space setting. Quadratical convergence for smooth optimization problems satisfying the second-order sufficient optimality condition is proved. In [**IK96b**] the second-order sufficient optimality condition is analyzed and numerical test examples are given. Kleis and Sachs [**KS95**] improve convergence estimates for the augmented Lagrangian-SQP method and present numerical results for a control problem. We refer to the paper [**KV97**], where Kunisch and Volkwein study two augmented Lagrangian-SQP algorithms for optimal control of partial differential equations. The approximation of one of these methods is analyzed. Examples will illustrate the theoretical investigations.

This chapter is organized as follows. In Section 2 we introduce the constrained minimization problem

$$(\mathbf{P}) \qquad \min J(y,u) \quad \text{subject to } e(y,u)=0,$$

where $J:\mathcal{Y}\times\mathcal{U}\to\mathbb{R}$, $e:\mathcal{Y}\times\mathcal{U}\to Y^*$ with $\mathcal{Y}$, $\mathcal{U}$ and Y real Hilbert spaces. The sets $\mathcal{Y}$ and $\mathcal{U}$ denote the state and the control space, respectively. We assume that $(\mathbf{P})$ has a local solution (y^*,u^*) and we denote by λ^* the associated Lagrange multiplier.

To solve $(\mathbf{P})$ we use an augmented Lagrangian-SQP method. This augmented Lagrangian-SQP method produces a sequence $\{(y^n,u^n,\lambda^n)\}_{n\in\mathbb{N}}$ converging quadratically to (y^*,u^*,λ^*). Since the algorithm cannot be executed in infinite dimensional spaces, problem $(\mathbf{P})$ is replaced in Section 3 by a family of approximations

$$(\mathbf{P_h}) \qquad \min J_h(y_h,u_h) \quad \text{subject to } e_h(y_h,u_h)=0,$$

– indexed by some mesh-size parameter h – where now $J_h:\mathcal{Y}_h\times\mathcal{U}_h\to\mathbb{R}$, $e_h:\mathcal{Y}_h\times\mathcal{U}_h\to Y_h^*$, $\{\mathcal{Y}_h\}_{h>0}$, $\{\mathcal{U}_h\}_{h>0}$ and $\{Y_h\}_{h>0}$ are given families of finite dimensional real Hilbert spaces. We propose the corresponding finite dimensional discretization of the augmented Lagrangian-SQP method. If $(\mathbf{P_h})$ possesses a solution (y_h^*,u_h^*) sufficiently close to (y^*,u^*) and if λ_h^* denotes the corresponding Lagrange multiplier, converge and rate of convergence of the finite dimensional method are proved. Moreover, the solutions $(y_h^*,u_h^*,\lambda_h^*)$ converge to (y^*,u^*,λ^*) as the mesh-size tends to zero. Since we have to solve a saddle-point problem at each level of the iteration process, this results depend on the discrete Babuška-Brezzi condition for the saddle-point problem. Under additional convergence properties the rate of convergence of the discrete iterates to the infinite dimensional ones is the same as the discretization error of the used approximation scheme. Furthermore, the existence of (y^*,u^*) solving $(\mathbf{P})$ implies the existence of a solution to $(\mathbf{P_h})$ for sufficiently small h.

2. Infinite Dimensional Method

2.1. Constrained Minimization Problem. We consider the constrained minimization problem

$$(\mathbf{P}) \qquad \min J(y,u) \quad \text{subject to } e(y,u)=0,$$

where $J:\mathcal{Y}\times\mathcal{U}\to\mathbb{R}$, $e:\mathcal{Y}\times\mathcal{U}\to Y^*$ with $\mathcal{Y}$, $\mathcal{U}$ and Y real Hilbert spaces and $\mathcal{Y}^*$, $\mathcal{U}^*$ and Y^* their duals. The sets $\mathcal{Y}$ and $\mathcal{U}$ denote the state and the control space, respectively. Let us emphasize that we consider the state variable y and the control variable u as independent variables in $(\mathbf{P})$. For brevity we set $X=\mathcal{Y}\times\mathcal{U}$ and $Z=X\times Y$. We identify the dual space of X

with $\mathcal{Y}^* \times \mathcal{U}^*$ and Z^* with $X^* \times Y^*$. The Fréchet-derivatives with respect to the variable $x = (y, u)$ will be denoted by primes.

We give an example for an optimal control problem which can be written in the form (**P**).

EXAMPLE 2.1. Let Ω be a bounded domain in $\mathbb{R}^d$, $d \le 3$, with $C^{1,1}$-boundary Γ or with Ω convex, and $f \in L^2(\Omega)$. For a control $u \in L^2(\Gamma)$ the state $y \in H^1(\Omega)$ is given by the weak solution to the elliptic problem

$$\begin{aligned} -\Delta y + y^3 - y &= f \quad \text{in } \Omega\,, \\ \frac{\partial y}{\partial n} &= u \quad \text{on } \Gamma\,. \end{aligned} \tag{2.1}$$

Problem (2.1) represents a simplified *Ginzburg-Landau model* for super-conductivity in the absence of internal magnetic fields with y the wave function [**Tin75**].

With every control u we associate the cost of tracking type

$$J(y,u) = \frac{1}{2}\int_\Omega |y - z|^2\,dx + \frac{\sigma}{2}\int_\Gamma |u|^2\,ds\,,$$

where $z \in L^2(\Omega)$ is a desired state and $\sigma > 0$ is fixed. We introduce the nonlinear operator

$$e : H^1(\Omega) \times L^2(\Gamma) \to H^1(\Omega)^*\,,$$

whose action is defined by

$$\langle e(y,u), \varphi\rangle_{(H^1)^*, H^1} = \int_\Omega \nabla y \cdot \nabla\varphi + \big(y^3 - y - f\big)\,\varphi\,dx - \int_\Gamma u\varphi\,ds$$

for $\varphi \in H^1(\Omega)$.

Putting $\mathcal{Y} = H^1(\Omega)$, $\mathcal{U} = L^2(\Gamma)$ and $Y = H^1(\Omega)$ the optimal control problem can be written in the form (**P**).

CONDITION 1. Problem (**P**) has a local solution (y^*, u^*). J and e are twice continuously Fréchet-differentiable, and the mappings J'' and e'' are Lipschitz-continuous in the neighborhood U_1 of (y^*, u^*).

EXAMPLE 2.2. In case of Example 2.1 the existence of a minimum (y^*, u^*) is proved in [**GHS91**] for all $\sigma > 0$. The regularity requirements are clearly met by the mappings J and e.

REMARK 2.3. By Lemma 1.8-a) on page 5 the dual operator of $e'(y,u)$ is a bounded linear operator satisfying $\|e'(y,u)^*\|_{L(Y,X^*)} = \|e'(y,u)\|_{L(X,Y^*)}$ for every $(y,u) \in U_1$.

The *Lagrangian functional* $L : Z \to \mathbb{R}$ associated with (**P**) is given by

$$L(y,u,\lambda) = J(y,u) + \langle e(y,u), \lambda\rangle_{Y^*,Y}\,.$$

We shall require the following regularity assumption.

CONDITION 2. The linearization $e'(y^*, u^*)$ of e at (y^*, u^*) is surjective.

The point (y^*, u^*) is thus assumed to be a *regular point* of e, [**Lue69**, p. 240].

EXAMPLE 2.4. In [**GHS91**] the validity of Condition 2 is shown for Example 2.1.

With Condition 2 holding there exists a Lagrange multiplier $\lambda^* \in Y$, such that the following *first-order necessary optimality condition* is satisfied [**Lue69**, p. 243]:

$$L'(y^*, u^*, \lambda^*) = 0\,, \quad e(y^*, u^*) = 0\,. \tag{2.2}$$

We also make use of the following *second-order sufficient optimality condition.*

CONDITION 3. The operator $L''(y^*, u^*, \lambda^*)$ is coercive on the kernel of $e'(y^*, u^*)$, i.e. there exists a constant $\kappa > 0$ such that

$$\begin{cases} \langle L''(y^*, u^*, \lambda^*)(v, q), (v, q)\rangle_{X^*,X} \geq \kappa \, \|(v, q)\|_X^2 \\ \text{for all } (v, q) \in \ker e'(y^*, u^*)\,. \end{cases}$$

Let J and e be twice continuously Fréchet-differentiable in U_1. With Conditions 2 and 3 holding the solution (y^*, u^*) of (2.2) solves (**P**), [**MZ79**].

2.2. Sensitivity Analysis. By Condition 1 the operator e is Fréchet-differentiable in U_1. Thus, we define the bounded linear operator

$$E(y, u) = e'(y, u)\, \mathfrak{J}_X\, e'(y, u)^* \quad \text{for all } (y, u) \in U_1\,. \tag{2.3}$$

The following two technical lemmas will be needed for the proof of Theorem 2.7.

LEMMA 2.5. *Let $(y, u) \in U_1$ be a point such that $e'(y, u)$ is surjective, and the operator $E(y, u)$ be given by (2.3). Then $E(y, u)^{-1}$ exists and belongs to $L(Y^*, Y)$. Moreover, $P_{\ker e'(y,u)}$ given by*

$$P_{\ker e'(y,u)} = I - \mathfrak{J}_X\, e'(y, u)^*\, E(y, u)^{-1}\, e'(y, u)$$

is an orthogonal projection onto the kernel of $e'(y, u)$.

PROOF. The first part of the claim follows directly from Lemma 1.11 on page 5. Thus, the operator $E(y, u)^{-1}$ exists and belongs to $L(Y^*, Y)$. Therefore, the operator $P_{\ker e'(y,u)}$ is well-defined. Furthermore,

$$\begin{aligned} P^2_{\ker e'(y,u)} &= P_{\ker e'(y,u)} - \mathfrak{J}_X e'(y, u)^* E(y, u)^{-1} e'(y, u) \\ &\qquad + \mathfrak{J}_X e'(y, u)^* E(y, u)^{-1} e'(y, u) \\ &= P_{\ker e'(y,u)} \end{aligned}$$

so that $P_{\ker e'(y,u)}$ is a projection [**RS80**, p. 187]. It can be seen that

$$e'(y, u) P_{\ker e'(y,u)} = 0\,.$$

Moreover, we derive for all $(v,q),(w,p) \in X$

$$\langle (v,q) - P_{\ker e'(y,u)}(v,q), P_{\ker e'(y,u)}(w,p)\rangle_X = 0$$

such that $P_{\ker e'(y,u)}$ is an orthogonal projection [**Zei90**, p. 267]. $\square$

LEMMA 2.6. *Let B be a Banach space, $a(\cdot\,,\cdot)$ be a continuous symmetric bilinear form on $B \times B$, $B_\circ$ a subset of B and $\kappa > 0$ with*

$$a(\varphi,\varphi) \geq \kappa \|\varphi\|_B^2 \quad \textit{for all } \varphi \in B_\circ\,.$$

Then there are $\kappa_\circ > 0$ and $\nu > 0$ such that

$$\begin{cases} a(\varphi+\psi,\varphi+\psi) \geq \kappa_\circ \|\varphi+\psi\|_B^2 \\ \textit{for all } \varphi \in B_\circ, \psi \in B \textit{ and } \|\psi\|_B \leq \nu \|\varphi\|_B\,. \end{cases}$$

For the proof we refer the reader to [**MZ79**].

THEOREM 2.7. *Let J and e be twice continuously Fréchet-differentiable in U_1. With Conditions 2 and 3 holding we obtain the following:*

a) *There exists a neighborhood U_2 of (y^*,u^*) such that the operator $e'(y,u)$ is surjective in U_2.*

b) *Moreover, there is a neighborhood U_3 of (y^*,u^*,λ^*) such that the operator $L''(y,u,\lambda)$ is coercive on the kernel of $e'(y,u)$ for all $(y,u,\lambda) \in U_3$, see (2.5).*

PROOF. a) From Lemma 1.11 on page 5 it follows that $E(y^*,u^*)$ is invertible. Since the set of the invertible mappings is open [**DL88**, p. 321], there exists a neighborhood $U_2 \subseteq U_1$ of (y^*,u^*) such that $E(y,u)$ is invertible in U_2. This implies that the operator $e'(y,u)$ is surjective for all $(y,u) \in U_2$.

b) For brevity we set $x = (y,u)$ and $x^* = (y^*,u^*)$. By part a), Condition 3 and the smoothness properties of J and e there exists a neighborhood U_3 of (x^*,λ^*) sufficiently small such that for every $(x,\lambda) \in U_3$ the operator $e'(x)$ is surjective and

$$\langle L''(x,\lambda)\varphi,\varphi\rangle_{X^*,X} \geq \frac{\kappa}{2}\|\varphi\|_X^2 \qquad \text{for all } \varphi \in \ker e'(x^*)\,. \tag{2.4}$$

From (2.4) and Lemma 2.6 we infer that there exists $\kappa_\circ > 0$ and $\nu > 0$ independent of (y,u,λ) such that

$$\begin{cases} \langle L''(x,\lambda)(\varphi+\psi),(\varphi+\psi)\rangle_{X^*,X} \geq \kappa_\circ \|\varphi+\psi\|_X^2 \\ \text{for all } \varphi \in \ker e'(x^*) \text{ and } \psi \in X \text{ with } \|\psi\|_X \leq \nu \|\varphi\|_X\,. \end{cases}$$

By Lemma 2.5 the operator $P_{\ker e'(x)}$ is an orthogonal projection onto the kernel of $e'(y,u)$ for all $(y,u) \in U_2$. Further, the mapping $x \mapsto E(x)^{-1}$ is continuous in U_2, [**DL88**, p. 321]. Possibly after reducing U_3 we have

$$\|\mathfrak{J}_X e'(x)^* E(x)^{-1} e'(x) - \mathfrak{J}_X e'(x^*)^* E(x^*)^{-1} e'(x^*)\|_{L(X)} \leq \frac{\nu}{\nu+1}$$

for all $(x, \lambda) \in U_3$. Hence, for given $\phi \in \ker e'(x)$ we obtain that

$$\begin{aligned}
&\|P_{\ker e'(x^*)^\perp}\phi\|_X \\
&\quad = \|\mathfrak{J}_X e'(x^*)^* E(x^*)^{-1} e'(x^*)\phi\|_X \\
&\quad = \|\mathfrak{J}_X e'(x^*)^* E(x^*)^{-1} e'(x^*)\phi - \mathfrak{J}_X e'(x)^* E(x)^{-1} e'(x)\phi\|_X \\
&\quad \le \frac{\nu}{\nu+1}\|\phi\|_X = \nu\left(\|\phi\|_X - \frac{\nu}{\nu+1}\|\phi\|_X\right) \\
&\quad \le \nu\left(\|\phi\|_X - \|\mathfrak{J}_X e'(x)^* E(x)^{-1} e'(x)\phi - \mathfrak{J}_X e'(x^*)^* E(x^*)^{-1} e'(x^*)\phi\|_X\right) \\
&\quad \le \nu\left(\|\phi - \mathfrak{J}_X e'(x^*)^* E(x^*)^{-1} e'(x^*)\phi\|_X\right) \\
&\quad = \nu\,\|P_{\ker e'(x^*)}\phi\|_X\,.
\end{aligned}$$

Setting $\phi = \varphi + \psi$ with $\varphi \in \ker e'(x^*)$ and $\psi \in (\ker e'(x^*))^\perp$ it follows from the estimate above that $\|\psi\|_X \le \nu\,\|\varphi\|_X$. Therefore,

$$\langle L''(x,\lambda)\phi, \phi\rangle_{X^*,X} \ge \kappa_\circ\,\|\phi\|_X^2 \qquad \text{for all } \phi \in \ker e'(x) \tag{2.5}$$

if $(x, \lambda) \in U_3$.

□

For any $c \ge 0$ the *augmented Lagrangian functional* is defined by

$$L_c(y,u,\lambda) = J(y,u) + \langle e(y,u), \lambda\rangle_{Y^*,Y} + \frac{c}{2}\,\|e(y,u)\|_{Y^*}^2\,.$$

Note that L_0 coincides with L.

LEMMA 2.8. *Let Conditions 1-3 hold. Then there exist a neighborhood U_4 of (y^*, u^*) and a constant $\bar{c} > 0$ such that the mapping $(y,u) \mapsto L_c''(y,u,\lambda^*)$ is coercive on the whole space X for all $(y,u) \in U_4$ and for all $c \ge \bar{c}$.*

For the proof we refer to [**IK96a**].

REMARK 2.9. Lemma 2.8 implies that $(y,u) \mapsto L_c(y,u,\lambda^*)$ can be bounded by a quadratic function. This fact is referred to as *augmentability* of L_c.

Henceforth we denote by $U = U_X \times U_Y$ with $U_X \subset X$ and $U_Y \subset Y$ a neighborhood of (y^*, u^*, λ^*) such that for all $(y,u,\lambda) \in U$

a) J and e are twice Fréchet-differentiable on $\overline{U_X}$ and their second Fréchet-derivatives are Lipschitz-continuous,
b) $e'(y,u)$ is surjective and
c) $L''(y,u,\lambda)$ is coercive on the kernel of $e'(y,u)$.

2.3. Algorithm. To find (y^*, u^*) numerically we observe that the first-order necessary optimality condition (2.2) implies that (y^*, u^*, λ^*) is a solution to

$$L_c'(y,u,\lambda) = L'(y,u,\lambda + c\mathfrak{J}e(y,u)) = 0\,, \qquad e(y,u) = 0\,, \tag{OS}$$

where $\mathfrak{J} \in L(Y^*, Y)$ denotes the Riesz isomorphism. We solve problem (**OS**) by the Newton method, where we avoid the explicit calculation of the augmented Lagrangian functional and its derivatives. It will be convenient to introduce the matrix of operators

$$M(y,u,\lambda) = \begin{pmatrix} L''(y,u,\lambda) & e'(y,u)^* \\ e'(y,u) & 0 \end{pmatrix} \qquad \text{for all } (y,u,\lambda) \in U\,. \tag{2.6}$$

Now we describe the *augmented Lagrangian-SQP method* introduced by Ito and Kunisch in [**IK96a**].

ALGORITHM 1. a) *Choose* $(y^0, u^0, \lambda^0) \in U$, $c \ge 0$ *and put* $n = 0$.
b) *Set* $\tilde{\lambda}^n = \lambda^n + c\,\mathfrak{J}e(y^n, u^n)$.
c) *Solve for* $(\bar{y}, \bar{u}, \bar{\lambda})$*:*

$$M(y^n, u^n, \tilde{\lambda}^n) \begin{pmatrix} (\bar{y},\bar{u}) - (y^n, u^n) \\ \bar{\lambda} - \tilde{\lambda}^n \end{pmatrix} = - \begin{pmatrix} L'(y^n, u^n, \tilde{\lambda}^n) \\ e(y^n, u^n) \end{pmatrix}. \tag{2.7}$$

d) *Set* $(y^{n+1}, u^{n+1}, \lambda^{n+1}) = (\bar{y}, \bar{u}, \bar{\lambda})$, $n = n+1$ *and goto* b).

REMARK 2.10. It can be shown that the update $(\bar{y}, \bar{u}, \bar{\lambda})$ of Algorithm 1 can equivalently be obtained from

$$\begin{pmatrix} L_c''(y^n, u^n, \lambda^n) & e'(y^n, u^n)^* \\ e'(y^n, u^n) & 0 \end{pmatrix} \begin{pmatrix} (\bar{y},\bar{u}) - (y^n, u^n) \\ \bar{\lambda} - \lambda^n \end{pmatrix} = - \begin{pmatrix} L_c'(y^n, u^n, \lambda^n) \\ e(y^n, u^n) \end{pmatrix}$$

which corresponds to a Newton step applied to (**OS**). This form of the iteration requires the implementation of the second derivative of $\|e(y,u)\|^2_{Y^*}$, whereas steps b) and c) of Algorithm 1 do not.

In each iteration of Algorithm 1 the infinite dimensional saddle-point problem (2.7) needs to be solved.

LEMMA 2.11. *Let J and e be twice continuously Fréchet-differentiable in U_1. With Conditions 2 and 3 holding the saddle-point problem*

$$M(y,u,\lambda) \begin{pmatrix} (\bar{y},\bar{u}) - (y,u) \\ \bar{\lambda} - \lambda \end{pmatrix} = - \begin{pmatrix} L'(y,u,\lambda) \\ e(y,u) \end{pmatrix}$$

has a unique solution $(\bar{y}, \bar{u}, \bar{\lambda})$ for all $(y,u,\lambda) \in U$.

PROOF. To apply the results presented in Chapter 1, Section 2, we set

$$\begin{aligned} A[y,u,\lambda] &= L''(y,u,\lambda)\,, \\ B[y,u] &= e'(y,u)\,, \\ f[y,u,\lambda] &= -L'(y,u,\lambda) + L''(y,u,\lambda)(y,u) + e'(y,u)^*\lambda\,, \\ g[y,u] &= -e(y,u) + e'(y,u)(y,u)\,, \\ a[y,u,\lambda](\varphi,\phi) &= \langle L''(y,u,\lambda)\varphi, \phi\rangle_{X^*,X}\,, \end{aligned}$$

where $(y,u,\lambda) \in U$ and $\varphi, \phi \in X$. By Lemma 2.7 there exists $\kappa_\circ > 0$ such that for every $(y,u,\lambda) \in U$

$$a[y,u,\lambda](\varphi,\varphi) \ge \kappa_\circ \|\varphi\|_X^2 \qquad \text{for all } \varphi \in \ker B[y,u]\,.$$

Let us consider

$$\left\{\begin{array}{rcl} A[y,u,\lambda](\bar{y},\bar{u}) + B[y,u]^*\bar{\lambda} & = & f[y,u,\lambda] \quad \text{in } X^*, \\ B[y,u](\bar{y},\bar{u}) & = & g[y,u] \quad \text{in } Y^* \end{array}\right. \tag{2.8}$$

for $(y,u,\lambda) \in U$. By Conditions 1 and 2 the operator $B[y,u]$ is surjective for all $(y,u) \in U_X$. Applying Lemma 2.4 on page 7 yields that the Babuška-Brezzi condition is satisfied. By Theorem 2.2 on page 6 problem (2.8) is well-posed, i.e. for every $(y,u,\lambda) \in U$ there exists a unique pair $(\bar{y},\bar{u},\bar{\lambda}) \in Z$ solving (2.8). □

REMARK 2.12. Under the hypotheses of Lemma 2.11 there exists a constant $\eta > 0$ satisfying $\|M^{-1}(y,u,\lambda)\|_{L(Z^*,Z)} \le \eta$ for all $(y,u,\lambda) \in U$.

2.4. Convergence Theorem. Now we are going to formulate our convergence result for Algorithm 1. For this purpose we introduce the following constants. Let $r_{\max}$ denote the largest radius of a ball centered at (y^*,u^*,λ^*) and contained in the neighborhood U, γ_e stands for the uniform Lipschitz-constant of e in U_X and γ_M denotes the uniform Lipschitz-constant of M in U. Moreover, we set

$$r^* = \min\left(\frac{r_{\max}}{\sqrt{\max(2, 1+2c^2\gamma_e^2)}}, \frac{2}{\gamma_M \eta \max(2, 1+2c^2\gamma_e^2)}\right). \tag{2.9}$$

For the proof of the next theorem we refer the reader to [**IK96a**].

THEOREM 2.13. *Let Conditions 1, 2 and 3 hold and let the start-up value (y^0,u^0,λ^0) belong to the open ball $B((y^*,u^*,\lambda^*); r^*)$. Then Algorithm 1 is well-defined and*

$$\|(y^{n+1},u^{n+1},\lambda^{n+1}) - (y^*,u^*,\lambda^*)\|_Z \le C\,\|(y^n,u^n,\lambda^n) - (y^*,u^*,\lambda^*)\|_Z^2,$$

where $C = \frac{1}{2}\gamma_M \eta \max(2, 1+2c^2\gamma_e^2)$.

REMARK 2.14. From (2.9) and Theorem 2.13 we derive that $(y^n,u^n) \in U_X$ holds for all $n = 0,1,\ldots$.

For optimal control problems with cost functional of tracking type, smallness of the cost or the residue helps for the second-order sufficient optimality condition to hold. This point is illustrated in the following example.

EXAMPLE 2.15. Let us turn to Example 2.1. We calculate

$$\langle L''(y,u,\lambda)(v,q),(v,q)\rangle^2_{X^*,X} = \|v\|^2_{\mathcal{Y}} + \sigma\,\|q\|^2_{\mathcal{U}} + 6\int_\Omega yv^2\lambda\,dx$$

for all $(v,q) \in H^1(\Omega) \times L^2(\Gamma)$. In [**IK96b**] it is proved that Condition 3 is satisfied if $\|(y^*-z,\sigma u^*)\|_X$ is sufficiently small. Other sufficient conditions that imply the second-order sufficient optimality condition are based on the type of nonlinearity in (2.1), [**IK96b**].

Let us mention that the differential equation $-\Delta y + y^3 - y = 0$ has three equilibria ± 1 and 0, of which ± 1 are stable and 0 is unstable. The globalizing effect of the augmented Lagrangian-SQP algorithm can be illustrated [**IK96b**] for instance by choosing for the desired state $z = 1$ and as start-up y^0 a value less than -1, with $u^0 = 0$. Then during the iteration the algorithm has to pass through the stable equilibrium at -1 and through the unstable equilibrium at 0 without terminating prematurely.

3. Approximation of the Infinite Dimensional Method

In many applications partial differential equations are approximated by one of the following techniques: the method of finite differences or the Rayleigh-Ritz-Galerkin method. We shall formulate these methods in functional analytic terms and apply them for approximation of smooth optimization problems.

3.1. Approximation of the Hilbert Spaces. In our work $\mathcal{Y}$, $\mathcal{U}$ and Y are infinite dimensional Hilbert spaces, which have to be discretized for numerical realization of (**P**). For this purpose we suppose that we are given a family $\{h\}$, $h > 0$, converging to zero and families $\{\mathcal{Y}_h\}_h$, $\{\mathcal{U}_h\}_h$ and $\{Y_h\}_h$ of Hilbert spaces with finite dimension. In practice, the parameter h is called to the mesh-size of the finite dimensional spaces and h varies over a sequence. To shorten notation we set $X_h = \mathcal{Y}_h \times \mathcal{U}_h$ and $Z_h = X_h \times Y_h$ for every h.

Let $p_h^X \in L(X_h, X)$ and $p_h^Y \in L(Y_h, Y)$ be given injective *prolongations*. In addition, we introduce surjective *restrictions* $r_h^X \in L(X, X_h)$ and $r_h^Y \in L(Y, Y_h)$. For brevity we set $p_h = (p_h^X, p_h^Y)$ and $r_h = (r_h^X, r_h^Y)$.

EXAMPLE 3.1. Let $Y = H^1(0,1)$. We set $h = \frac{1}{m}$, $x_j = jh$ for $j = 0, \ldots, m$ and introduce the piecewise linear functions

$$\varphi_0(x) = \begin{cases} \frac{1}{h}(x_1 - x) & \text{for } x \in [x_0, x_1]\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

$$\varphi_j(x) = \left\{ \begin{array}{ll} \frac{1}{h}(x - x_{j-1}) & \text{for } x \in (x_{j-1}, x_j]\,, \\ \frac{1}{h}(x_{j+1} - x) & \text{for } x \in (x_j, x_{j+1}]\,, \\ 0 & \text{otherwise}\,, \end{array} \right\} \qquad \text{for } 1 \le j \le m-1\,,$$

$$\varphi_m(x) = \begin{cases} \frac{1}{h}(x - x_{m-1}) & \text{for } x \in (x_{m-1}, x_m]\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

It follows that $\varphi_j \in H^1(0,1)$ holds [**GR92**, p. 119]. The *piecewise linear approximations* of $H^1(0,1)$ are defined by

a) $Y_h = \mathbb{R}^{m+1}$, $\lambda_h = (\lambda_h^0, \ldots, \lambda_h^m)^T \in Y_h$,

b) $(p_h^Y \lambda_h)(x) = \sum\limits_{j=0}^{m} \lambda_h^j \varphi_j(x)$ for $\lambda_h \in Y_h$,

c) $r_h^Y \lambda = (\lambda(x_0), \dots, \lambda(x_m))^T$ for $\lambda \in Y$.

From Lemma 3.7 on page 10 we conclude that the restriction operator is well-defined.

In particular, we can choose optimal restrictions associated with the prolongations.

DEFINITION 3.2. The bounded linear operator $\hat{r}_h^X : X \to X_h$ is called the optimal restriction associated with p_h^X in X if

$$\|(y,u) - p_h^X \hat{r}_h^X (y,u)\|_X = \inf_{(y_h,u_h)\in X_h} \|(y,u) - p_h^X (y_h,u_h)\|_X$$

for all $(y,u) \in X$. Analogously, if

$$\|\lambda - p_h^Y \hat{r}_h^Y \lambda\|_Y = \inf_{\lambda_h \in Y_h} \|\lambda - p_h^Y \lambda_h\|_Y$$

for all $\lambda \in Y$, we call the linear bounded operator $\hat{r}_h^Y : Y \to Y_h$ the optimal restriction associated with p_h^Y in Y. Finally, $\hat{r}_h = (\hat{r}_h^X, \hat{r}_h^Y)$ is the optimal restriction associated with p_h in Z if $\hat{r}_h^X$ and $\hat{r}_h^Y$ are optimal.

REMARK 3.3. Since the finite dimensional set $\operatorname{ran} p_h$ is closed in the Hilbert space Z, we infer from the projection theorem [**RS80**, p. 42] that for any $(y,u,\lambda) \in Z$ there exists a unique element $(\hat{y},\hat{u},\hat{\lambda}) \in \operatorname{ran} p_h$ satisfying

$$\begin{aligned} \|(y,u,\lambda) - (\hat{y},\hat{u},\hat{\lambda})\|_Z &= \inf_{(\varphi,\psi,\phi)\in \operatorname{ran} p_h} \|(y,u,\lambda) - (\varphi,\psi,\phi)\|_Z \\ &= \inf_{(\varphi_h,\psi_h,\phi_h)\in Z_h} \|(y,u,\lambda) - p_h(\varphi_h,\psi_h,\phi_h)\|_Z . \end{aligned}$$

Hence, the operator $p_h \hat{r}_h$ is uniquely defined by $p_h \hat{r}_h (y,u,\lambda) = (\hat{y},\hat{u},\hat{\lambda})$.

Optimal restrictions have the properties $\hat{r}_h p_h = I$ and $\|p_h \hat{r}_h\|_{L(Z)} \le 1$, [**Aub72**, p. 75]. In particular, optimal restrictions are surjective.

EXAMPLE 3.4. We consider the *cosine approximations* of the space $Y = H^1(0,1)$ associated with the basis

$$\varphi_j(x) = \left(\frac{2}{1+j^2\pi^2}\right)^{\frac{1}{2}} \cos(j\pi x)\,, \qquad j = 0,1,\dots .$$

It can be checked that $\{\varphi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $H^1(0,1)$. Therefore, if we set $h = \frac{1}{m}$ the cosine approximations associated with this basis is defined by

a) $Y_h = \mathbb{R}^{m+1}$, $\lambda_h = (\lambda_h^0, \dots, \lambda_h^m)^T \in Y_h$,

b) $(p_h^Y \lambda_h)(x) = \sum_{j=0}^{m} \lambda_h^j \varphi_j(x)$ for $\lambda_h \in Y_h$,

c) $\hat{r}_h^Y \lambda = (c_0(\lambda), \dots, c_m(\lambda))^T$ for $\lambda \in Y$, where

$$c_j(\lambda) = \langle \lambda, \varphi_j \rangle_{H^1} = \sqrt{2+2j^2\pi^2} \int_0^1 \cos(j\pi x)\, \lambda(x)\, dx$$

are the Fourier-coefficients of λ associated with the basis $\{\varphi_j\}_{j\in\mathbb{N}}$. Since Y is a Hilbert space, every $\lambda \in Y$ can be written as [**RS80**, p. 45]

$$\lambda = \sum_{j=0}^{\infty} c_j(\lambda)\,\varphi_j\,.$$

As $\{\varphi_j\}_{j\in\mathbb{N}}$ are orthogonal it follows that

$$\begin{aligned}
\|\lambda - p_h^Y \hat{r}_h^Y \lambda\|_Y^2 &= \|\sum_{j=m+1}^{\infty} c_j(\lambda)\varphi_j\|_Y^2 \\
&\le \|\sum_{j=m+1}^{\infty} c_j(\lambda)\varphi_j\|_Y^2 + \|\sum_{j=0}^{m}(c_j(\lambda) - \lambda_h^j)\varphi_j\|_Y^2 \\
&= \|\sum_{j=0}^{\infty} c_j(\lambda)\varphi_j - \sum_{j=0}^{m} \lambda_h^j \varphi_j\|_Y^2 \\
&= \|\lambda - p_h^Y \lambda_h\|_Y^2
\end{aligned}$$

for all $\lambda_h \in Y_h$. Since $\hat{r}_h^Y \lambda \in Y_h$ holds, we conclude from the estimate above that

$$\|\lambda - p_h^Y \hat{r}_h^Y \lambda\|_Y = \inf_{\lambda_h \in Y_h} \|\lambda - p_h^Y \lambda_h\|_Y$$

so that $\hat{r}_h^Y$ is the optimal restriction associated with the prolongation p_h^Y.

We define with the prolongations discrete inner products and discrete norms in the following way.

Definition 3.5. We introduce the discrete inner product in X_h by

$$\langle (y_h, u_h), (\bar{y}_h, \bar{u}_h)\rangle_{X_h} = \langle p_h^X(y_h, u_h), p_h^X(\bar{y}_h, \bar{u}_h)\rangle_X$$

and the discrete norm in X_h by

$$\|(y_h, u_h)\|_{X_h} = \|p_h^X(y_h, u_h)\|_X$$

for all $(y_h, u_h), (\bar{y}_h, \bar{u}_h) \in X_h$. Analogously, for all $\lambda_h, \bar{\lambda}_h \in Y_h$ we define by

$$\langle \lambda_h, \bar{\lambda}_h\rangle_{Y_h} = \langle p_h^Y \lambda_h, p_h^Y \bar{\lambda}_h\rangle_Y\,, \quad \|\lambda_h\|_{Y_h} = \|p_h^Y \lambda_h\|_Y$$

the discrete inner product and discrete norm in Y_h. In Z_h the corresponding inner product is the sum of the inner products $\langle\cdot\,,\cdot\rangle_{X_h}$ and $\langle\cdot\,,\cdot\rangle_{Y_h}$ and the discrete norm is

$$\|(y_h, u_h, \lambda_h)\|_{Z_h} = \|p_h(y_h, u_h, \lambda_h)\|_Z\,.$$

Remark 3.6. For these discrete norms the prolongations p_h^X, p_h^Y and p_h have norm 1.

The next lemma ensures existence of the optimal restrictions and gives us explicit formulas. For the proof we refer the reader to [**Aub72**, p. 77].

LEMMA 3.7. *Let the prolongations p_h^X and p_h^Y be given. The optimal restrictions associated with the prolongations have the following forms:*

$$\begin{aligned}\hat{r}_h^X &= \left((p_h^X)^* \mathfrak{J}_X^{-1} p_h^X\right)^{-1} (p_h^X)^* \mathfrak{J}_X^{-1}, \\ \hat{r}_h^Y &= \left((p_h^Y)^* \mathfrak{J}_Y^{-1} p_h^Y\right)^{-1} (p_h^Y)^* \mathfrak{J}_Y^{-1}\end{aligned}$$

and $\hat{r}_h = (\hat{r}_h^X, \hat{r}_h^Y)$.

For every h the data (X_h, p_h^X, r_h^X), (Y_h, p_h^Y, r_h^Y) and (Z_h, p_h, r_h) are called *approximations* of the spaces X, Y and Z, respectively.

3.2. Approximation of the Dual Spaces. We introduce approximations of the duals X^*, Y^* and Z^*. The data (X_h^*, q_h^X, s_h^X) are the *dual approximations* of (X_h, p_h^X, r_h^X), where $q_h^X = (r_h^X)^*$ is given by

$$\langle q_h^X f_h, (y,u)\rangle_{X^*,X} = \langle f_h, r_h^X(y,u)\rangle_{X_h^*,X_h} \tag{2.10a}$$

for all $f_h \in X_h^*$, $(y,u) \in X$ and $s_h^X = (p_h^X)^*$ is given by

$$\langle s_h^X f, (y_h,u_h)\rangle_{X_h^*,X_h} = \langle f, p_h^X(y_h,u_h)\rangle_{X^*,X} \tag{2.10b}$$

for all $f \in X^*$, $(y_h, u_h) \in X_h$. Analogously, (Y_h^*, q_h^Y, s_h^Y) are the dual approximations of (Y_h, p_h^Y, r_h^Y) and (Z_h^*, q_h, s_h) are the dual approximations of (Z_h, p_h, r_h).

REMARK 3.8. From Lemma 1.8-a) on page 5 we derive that the operators q_h and s_h are linear and bounded.

EXAMPLE 3.9. In case of Example 3.1 we have $Y_h^* = \mathbb{R}^{m+1}$, and the duality pairing is given by

$$\langle f_h, \lambda_h\rangle_{Y_h^*,Y_h} = \sum_{j=0}^{m} f_h^j \lambda_h^j \qquad \text{for } f_h \in Y_h^* \text{ and } \lambda_h \in Y_h\,.$$

Thus, the dual operator of r_h^Y is defined by

$$\langle q_h^Y f_h, \lambda\rangle_{Y^*,Y} = \sum_{j=0}^{m} f_h^j \lambda(x_j) \qquad \text{for } f_h \in Y_h^* \text{ and } \lambda \in Y\,.$$

LEMMA 3.10. *q_h^X and q_h^Y are injective, and s_h^X and s_h^Y are surjective.*

PROOF. Since p_h^X and p_h^Y are injective, the sets $\operatorname{ran} s_h^X$ and $\operatorname{ran} s_h^Y$ are dense in the finite dimensional spaces X_h and Y_h by Remark 1.10 on page 5. Hence, s_h^X and s_h^Y are surjective. Surjectivity of r_h^X and r_h^Y imply that both q_h^X and q_h^Y are injective by Lemma 1.9 on page 5. $\square$

Now we define the discrete inner products and discrete norms in the dual spaces X_h^*, Y_h^* and Z_h^*.

DEFINITION 3.11. We call the discrete inner product in X_h^*

$$\langle f_h, g_h \rangle_{X_h^*} = \langle q_h^X f_h, q_h^X g_h \rangle_{X^*} \qquad \text{for } f_h, g_h \in X_h^*$$

and the discrete norm in X_h^*

$$\|f_h\|_{X_h^*} = \|f_h\|_{L(X_h,\mathbb{R})} \qquad \text{for } f_h \in X_h^* .$$

Analogously, we define by

$$\langle f_h, g_h \rangle_{Y_h^*} = \langle q_h^Y f_h, q_h^Y g_h \rangle_{Y^*} , \qquad \|g_h\|_{Y_h^*} = \|g_h\|_{L(Y_h,\mathbb{R})} \qquad \text{for } f_h, g_h \in Y_h^*$$

the discrete inner product and the discrete norm in Y_h^*. In Z_h^* the corresponding inner product is the sum of the inner products $\langle \cdot\,, \cdot \rangle_{X_h}$ and $\langle \cdot\,, \cdot \rangle_{Y_h}$ and the discrete norm is given by

$$\|(f_h, g_h)\|_{Z_h^*} = \sqrt{\|f_h\|_{X_h^*}^2 + \|g_h\|_{Y_h^*}^2} \qquad \text{for } (f_h, g_h) \in Z_h^* .$$

REMARK 3.12. a) The restrictions s_h^X, s_h^Y and s_h have norm 1.

b) In general the norm on the dual space X_h^* is not induced by the inner product on X_h^*. If we use optimal prolongations $\hat{q}_h$, we shall see in Lemma 3.15 that

$$\|f_h\|_{X_h^*} = \sqrt{\langle \hat{q}_h f_h, \hat{q}_h f_h \rangle_{X_h^*}} \qquad \text{for all } f_h \in X_h^*$$

holds.

For every h we denote by $\mathfrak{J}_h$ the isometric isomorphism from the dual space Y_h^* onto Y_h. Then we infer [**Aub72**, p. 77]

$$\mathfrak{J}_h = r_h^Y \, \mathfrak{J}_Y \, q_h^Y . \tag{2.11}$$

DEFINITION 3.13. The operator $\hat{q}_h^X \in L(X_h^*, X^*)$ is called the optimal prolongation associated with s_h^X in X^* if

$$s_h^X \hat{q}_h^X = I , \quad \|f_h\|_{X_h^*} = \|\hat{q}_h^X f_h\|_{X^*} = \|\hat{q}_h^X s_h^X f\|_{X^*} \leq \|f\|_{X^*}$$

for all $f \in X^*$ such that $s_h^X f = f_h$. Analogously, if

$$s_h^Y \hat{q}_h^Y = I , \quad \|g_h\|_{Y_h^*} = \|\hat{q}_h^Y g_h\|_{Y^*} = \|\hat{q}_h^Y s_h^Y g\|_{Y^*} \leq \|g\|_{Y^*}$$

for all $g \in Y^*$ such that $s_h^Y g = g_h$ we say that the operator $\hat{q}_h^Y \in L(Y_h^*, Y^*)$ is the optimal prolongation associated with s_h^Y in Y^*. Finally, $\hat{q}_h = (\hat{q}_h^X, \hat{q}_h^Y)$ is called the optimal prolongation associated with s_h in Z^* if $\hat{q}_h^X$ and $\hat{q}_h^Y$ are optimal.

REMARK 3.14. Such prolongations have the properties [**Aub72**, p. 77]

$$\|f - \hat{q}_h^X s_h^X f\|_{X^*} = \inf_{f_h \in X_h^*} \|f - \hat{q}_h^X f_h\|_{X^*} \qquad \text{for all } f \in X^*$$

and

$$\|g - \hat{q}_h^Y s_h^Y g\|_{Y^*} = \inf_{g_h \in Y_h^*} \|g - \hat{q}_h^Y g_h\|_{Y^*} \qquad \text{for all } g \in Y^* .$$

In Definition 3.2 we have introduced optimal restrictions associated with given prolongations. Then we must ask, if the prolongation $(\hat{r}_h)^*$ associated with the restrictions s_h is optimal.

LEMMA 3.15. *Let the prolongations p_h^X, p_h^Y and p_h be given. Further, let $\hat{r}_h^X$, $\hat{r}_h^Y$ and $\hat{r}_h$ be optimal restrictions associated with these prolongations. Then the dual operators $\hat{q}_h^X = (\hat{r}_h^X)^*$, $\hat{q}_h^Y = (\hat{r}_h^Y)^*$ and $\hat{q}_h = (\hat{r}_h)^*$ are optimal prolongations associated with the restrictions s_h^X, s_h^Y and s_h and they have the following forms:*

$$\begin{aligned}\hat{q}_h^X &= \mathfrak{J}_X^{-1}\, p_h^X \left(s_h^X\, \mathfrak{J}_X^{-1}\, p_h^X\right)^{-1}\,,\\ \hat{q}_h^Y &= \mathfrak{J}_Y^{-1}\, p_h^Y \left(s_h^Y\, \mathfrak{J}_Y^{-1}\, p_h^Y\right)^{-1}\end{aligned}$$

and $\hat{q}_h = (\hat{q}_h^X, \hat{q}_h^Y)$. Further, the following commutation formulas hold

$$\mathfrak{J}_Z^{-1}\, p_h\, \hat{r}_h = \hat{q}_h\, s_h\, \mathfrak{J}_Z^{-1} \quad \text{and} \quad \mathfrak{J}_Z\, \hat{q}_h\, s_h = p_h\, \hat{r}_h\, \mathfrak{J}_Z\,.$$

Moreover, we have

$$\|(f_h, g_h)\|_{Z_h^*} = \|\hat{q}_h(f_h, g_h)\|_{Z^*} \qquad \text{for all } (f_h, g_h) \in Z_h^*\,,$$

i.e. the norm on Z_h^* is induced by the inner product $\langle\cdot\,,\cdot\rangle_{Z_h^*}$.

For the proof we refer the reader to [**Aub72**, p. 77].

REMARK 3.16. From Lemma 3.15 and (2.11) we derive that

$$\mathfrak{J}_h = (s_h^Y\, \mathfrak{J}_Y^{-1}\, p_h^Y)^{-1}$$

such that $\hat{q}_h^Y$ is also given by $\hat{q}_h^Y = \mathfrak{J}_Y^{-1}\, p_h^Y\, \mathfrak{J}_h$. Lemma 3.7 leads to $\hat{r}_h^Y = \mathfrak{J}_h\, s_h^Y\, \mathfrak{J}_Y^{-1}$. Using Definition 3.5 we see that $\lambda - p_h^Y \hat{r}_h^Y \lambda \in (\operatorname{ran} p_h^Y)^\perp$ for all $\lambda \in Y$ and analogously, $(y,u) - p_h^X \hat{r}_h^X (y,u) \in (\operatorname{ran} p_h^X)^\perp$ for all $(y,u) \in X$.

EXAMPLE 3.17. Let us turn to Example 3.4. Since $\{\varphi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of Y and

$$\lambda - p_h^Y \hat{r}_h^Y \lambda = \sum_{j=m+1}^{\infty} c_j(\lambda)\, \varphi_j \qquad \text{for all } \lambda \in Y$$

holds, it follows that $\lambda - p_h^Y \hat{r}_h^Y \lambda \in (\operatorname{ran} p_h^Y)^\perp$.

3.3. Convergent Approximations.

DEFINITION 3.18. The approximations (X_h, p_h^X, r_h^X) of X are called <u>convergent</u> if

$$\lim_{h\to 0} \|(y,u) - p_h^X r_h^X (y,u)\|_X = 0 \qquad \text{for all } (y,u) \in X\,.$$

Convergence of (Y_h, p_h^Y, r_h^Y) and (Z_h, p_h, r_h) and convergence of the dual approximations are defined analogously.

REMARK 3.19. Let $(y,u,\lambda) \in Z$ and $\hat{r}_h$ be the optimal restriction associated with the prolongation p_h. Then

$$\|(y,u,\lambda) - p_h\hat{r}_h(y,u,\lambda)\|_Z \le \|(y,u,\lambda) - p_h(y_h,u_h,\lambda_h)\|_Z$$

for all $(y_h,u_h,\lambda_h) \in Z_h$. Thus, the approximations $(Z_h,p_h,\hat{r}_h)$ of Z satisfy optimal properties of convergence.

EXAMPLE 3.20. The piecewise linear approximations of $Y = H^1(0,1)$ of Example 3.1 are convergent. It can be proved that

$$\lim_{h\to 0} \|\lambda - p_h^Y r_h^Y \lambda\|_{H^1} = 0 \qquad \text{for all } \lambda \in H^1(0,1)$$

holds [**Aub72**, p. 40-42].

EXAMPLE 3.21. In case of Example 3.4 we infer from the Parseval relation [**RS80**, p. 46] that

$$\|\lambda - p_h^Y \hat{r}_h^Y \lambda\|_{H^1} = \sum_{j=m+1}^{\infty} |\langle \lambda, \varphi_j\rangle_{H^1}|^2 \qquad \text{for all } \lambda \in H^1(0,1)\,.$$

Since the Fourier series of λ converges in $H^1(0,1)$ for every $\lambda \in H^1(0,1)$, [**RS80**, p. 45], we obtain

$$\lim_{m\to\infty} \sum_{j=m+1}^{\infty} |\langle \lambda, \varphi_j\rangle_{H^1}|^2 = 0 \qquad \text{for all } \lambda \in H^1(0,1)\,.$$

Hence, the cosine approximations of $H^1(0,1)$ are convergent.

Definition 3.18 leads directly to

COROLLARY 3.22. *The approximations (Z_h,p_h,r_h) of Z are convergent if and only if the approximations (X_h,p_h^X,r_h^X) of X and (Y_h,p_h^Y,r_h^Y) of Y are convergent.*

If we choose optimal restrictions associated with the given prolongations, the convergence of the approximations of the Hilbert spaces implies the convergence of the approximations of their duals.

LEMMA 3.23. *If the approximations $(Z_h,p_h,\hat{r}_h)$ of Z are convergent, then their dual approximations $(Z_h^*,\hat{q}_h,s_h)$ are convergent, where $\hat{q}_h$ is the optimal prolongation associated with the optimal restriction $\hat{r}_h$.*

For the proof we refer the reader to [**Aub72**, p. 78].

REMARK 3.24. The convergence of the approximations (Z_h,p_h,r_h) of Z does not imply the convergence of the dual approximations when r_h is no longer the optimal restriction associated with p_h.

LEMMA 3.25. *Let the approximations (Z_h,p_h,r_h) of Z be convergent. Then the dual approximations converge weakly:*

$$q_h s_h(f,g) \overset{h\to 0}{\rightharpoonup} (f,g) \ \text{in } Z^*, \ \text{uniformly over bounded sets of } Z^*\,.$$

For the proof we refer the reader to [**Aub72**, p. 79].

3.4. Internal Approximation of the Operators. Now we define the following *internal approximations* of the cost functional J and the constraint e, [**Aub72**, Chapter 3]:

$$J_h = J\, p_h^X : X_h \to \mathbb{R}, \quad e_h = s_h^Y\, e\, p_h^X : X_h \to Y_h^*\,. \tag{2.12}$$

Together with **(P)** we investigate the finite dimensional discrete problems

$$\min J_h(y_h, u_h) \quad \text{subject to } e_h(y_h, u_h) = 0. \tag{$\mathbf{P_h}$}$$

In many applications existence of a local solution of $(\mathbf{P_h})$ is proved by using standard techniques of finite dimensional optimization theory such that it is justified to assume that the finite dimensional problem $(\mathbf{P_h})$ has at least one solution.

CONDITION 4. For every h problem $(\mathbf{P_h})$ has a local solution (y_h^*, u_h^*). Moreover, there exist neighborhoods $\tilde{V}_h$ of (y_h^*, u_h^*) such that $p_h^X(\tilde{V}_h) \subseteq U_X$ for all h sufficiently small.

The Lagrangian for $(\mathbf{P_h})$ is given by

$$\begin{aligned} L_h(y_h, u_h, \lambda_h) &= J_h(y_h, u_h) + \langle e_h(y_h, u_h), \lambda_h \rangle_{Y_h^*, Y_h} \\ &\overset{(2.12)}{=} J(p_h^X(y_h, u_h)) + \langle s_h^Y e(p_h^X(y_h, u_h)), \lambda_h \rangle_{Y_h^*, Y_h} \\ &\overset{(2.10)}{=} J(p_h^X(y_h, u_h)) + \langle e(p_h^X(y_h, u_h)), p_h^Y \lambda_h \rangle_{Y^*, Y} \\ &= L(p_h(y_h, u_h, \lambda_h))\,. \end{aligned}$$

Since all restrictions and prolongations are linear, we are able to differentiate the Lagrangian L_h with respect to the variable $x_h = (y_h, u_h)$ for all $(y_h, u_h) \in \tilde{V}_h$ and $\lambda_h \in Y_h$:

$$L_h'(y_h, u_h, \lambda_h) = s_h^X L'(p_h(y_h, u_h, \lambda_h))\,. \tag{2.13}$$

Both $L_h'(y_h, u_h, \lambda_h)$ and $e_h(y_h, u_h)$ are also continuously Fréchet-differentiable with respect to the variable $x_h = (y_h, u_h)$ and their Fréchet-derivatives are Lipschitz-continuous. Explicitly, we get:

$$L_h''(y_h, u_h, \lambda_h) = s_h^X L''(p_h(y_h, u_h, \lambda_h)) p_h^X\,, \tag{2.14}$$

$$e_h'(y_h, u_h) = s_h^Y e'(p_h^X(y_h, u_h)) p_h^X \tag{2.15}$$

for all $(y_h, u_h) \in \tilde{V}_h$ and $\lambda_h \in Y_h$.

Now we calculate the adjoint of $e_h'(y_h, u_h)$. It follows from (2.10), (2.12) and Definition 1.6 on page 4 that

$$e_h'(y_h, u_h)^* = s_h^X e'(p_h^X(y_h, u_h))^* p_h^Y \qquad \text{for all } (y_h, u_h) \in \tilde{V}_h\,. \tag{2.16}$$

REMARK 3.26. Since $L''(y, u, \lambda)$ is a self-adjoint operator for all $(y, u) \in U_X$ and $\lambda \in Y$ we conclude that $L_h''(y_h, u_h, \lambda_h)$ is also self-adjoint for all $(y_h, u_h) \in \tilde{V}_h$ and $\lambda_h \in Y_h$.

3.5. Consistency of the Internal Approximations. In Section 3.4 we have introduced the internal approximations of the cost functional J, of the constraint e and of the Lagrangian L. Now we analyze if these approximations are consistent. We shall require the following assumption.

CONDITION 5. The approximations (Z_h, p_h, r_h) of the Hilbert space Z are convergent, and $p_h r_h(y,u,\lambda) \in U$ holds for $(y,u,\lambda) \in U$ and all h sufficiently small.

REMARK 3.27. With Condition 5 holding $\|p_h r_h(y,u,\lambda)\|_Z$ is bounded for all $(y,u,\lambda) \in Z$. According to the principle of uniform boundedness [**Wou79**, p. 112] there exists a constant $c_\circ > 0$ such that $\|p_h r_h\|_{L(Z)} \le c_\circ$ for all h. By Lemma 1.8-a) on page 5 this implies that $\|q_h s_h\|_{L(Z^*)} = \|(p_h r_h)^*\|_{L(Z)} \le c_\circ$, uniformly in h.

Let $(y,u) \in U_X$. Since J is Lipschitz-continuous in U_X, it follows that

$$\begin{aligned} |J(y,u) - J_h(r_h^X(y,u))| &= |J(y,u) - J(p_h^X r_h^X(y,u))| \\ &\le \gamma_J \,\|(y,u) - p_h^X r_h^X(y,u)\|_X\,, \end{aligned} \tag{2.17}$$

where γ_J denotes the uniform Lipschitz-constant of J in U_X. Applying Remark 3.12-a) and (2.12) we obtain

$$\begin{aligned} \|s_h^Y e(y,u) - e_h(r_h^X(y,u))\|_{Y_h^*} &= \|s_h^Y e(y,u) - s_h^Y e(p_h^X r_h^X(y,u))\|_{Y_h^*} \\ &\le \gamma_e \,\|(y,u) - p_h^X r_h^X(y,u)\|_X, \end{aligned} \tag{2.18}$$

where γ_e denotes the uniform Lipschitz-constant of e in U_X. From (2.17) and (2.18) we infer the following lemma.

LEMMA 3.28. *Let Condition 5 hold. Then the internal approximations of the operators J and e given by (2.12) are consistent, i.e.*

$$\begin{aligned} \lim_{h\to 0} |J(y,u) - J_h(r_h^X(y,u))| &= 0\,, \\ \lim_{h\to 0} \|s_h^Y e(y,u) - e_h(r_h^X(y,u))\|_{Y_h^*} &= 0 \end{aligned}$$

for all $(y,u) \in U_X$.

REMARK 3.29. Lemma 3.28 implies that the internal approximation L_h of the Lagrangian L is consistent, i.e.

$$\lim_{h\to 0} |L(y,u,\lambda) - L_h(r_h(y,u,\lambda))| = 0 \qquad \text{for all } (y,u,\lambda) \in U\,.$$

In the following lemma we shall see that the internal approximations of the first and second Fréchet-derivatives of L are consistent as well.

LEMMA 3.30. *Let Condition 5 hold. Then it follows that*

$$\begin{aligned} \lim_{h\to 0} \|s_h^X L'(y,u,\lambda) - L_h'(r_h(y,u,\lambda))\|_{X_h^*} &= 0\,, \\ \lim_{h\to 0} \|s_h^X L''(y,u,\lambda)(\bar y,\bar u) - L_h''(r_h(y,u,\lambda)) r_h^X(\bar y,\bar u)\|_{X_h^*} &= 0 \end{aligned}$$

for all $(y,u,\lambda) \in U$ and $(\bar y,\bar u) \in X$.

Proof. From Remark 3.12-a) and (2.13) we derive that

$$\|s_h^X L'(y,u,\lambda) - L_h'(r_h(y,u,\lambda))\|_{X_h^*} \leq \gamma_{L'} \, \|(y,u,\lambda) - p_h r_h(y,u,\lambda)\|_Z$$

for all $(y,u,\lambda) \in U$, where $\gamma_{L'}$ denotes the uniform Lipschitz-constant of L' in U. Applying Remark 3.12-a) and (2.14) we obtain

$$\begin{aligned}
&\|s_h^X L''(y,u,\lambda)(\bar{y},\bar{u}) - L_h''(r_h(y,u,\lambda)) r_h^X(\bar{y},\bar{u})\|_{X_h^*} \\
&\quad \leq \|L''(y,u,\lambda)(\bar{y},\bar{u}) - L''(p_h r_h(y,u,\lambda))(\bar{y},\bar{u})\|_{X^*} \\
&\qquad + \|L''(p_h r_h(y,u,\lambda))(\bar{y},\bar{u}) - L''(p_h r_h(y,u,\lambda)) p_h^X r_h^X(\bar{y},\bar{u})\|_{X^*} \\
&\quad \leq \gamma_{L''} \, \|(\bar{y},\bar{u})\|_X \, \|(y,u,\lambda) - p_h r_h(y,u,\lambda)\|_Z + \eta_{L''} \, \|(\bar{y},\bar{u}) - p_h^X r_h^X(\bar{y},\bar{u})\|_X
\end{aligned}$$

for all $(y,u,\lambda) \in U$ and $(\bar{y},\bar{u}) \in X$, where $\gamma_{L''}$ denotes the uniform Lipschitz-constant of L'' in U and $\eta_{L''}$ is a uniform upper bound of L'' in $\overline{U}$. Since the approximations (Z_h, p_h, r_h) of Z are convergent, the claim is proved. $\square$

The next lemma shows that the internal approximations of the operators $e'(y,u)$ and $e'(y,u)^*$ given by (2.15) and (2.16) are consistent as well.

Lemma 3.31. *Let Condition 5 be satisfied. Then it follows that*

$$\begin{aligned}
\lim_{h\to 0} \|s_h^Y e'(y,u)(\bar{y},\bar{u}) - e_h'(r_h^X(y,u)) r_h^X(\bar{y},\bar{u})\|_{Y_h^*} &= 0\,, \\
\lim_{h\to 0} \|s_h^X e'(y,u)^* \bar{\lambda} - e_h'(r_h^X(y,u))^* r_h^Y \bar{\lambda}\|_{X_h^*} &= 0
\end{aligned}$$

for all $(y,u) \in U_X$ and $(\bar{y},\bar{u},\bar{\lambda}) \in Z$.

Proof. We infer from Remark 3.12-a) and (2.15)

$$\begin{aligned}
\|s_h^Y e'(y,u)(\bar{y},\bar{u}) &- e_h'(r_h^X(y,u)) r_h^X(\bar{y},\bar{u})\|_{Y_h^*} \\
&\leq \gamma_{e'} \, \|(\bar{y},\bar{u})\|_X \, \|(y,u) - p_h^X r_h^X(y,u)\|_X \\
&\quad + \eta_{e'} \, \|(\bar{y},\bar{u}) - p_h^X r_h^X(\bar{y},\bar{u})\|_X \\
&\stackrel{h\to 0}{\longrightarrow} 0
\end{aligned}$$

for all $(y,u) \in U_X$ and $(\bar{y},\bar{u}) \in X$, where $\gamma_{e'}$ is the uniform Lipschitz-constant of e' in U_X and $\eta_{e'}$ denotes a uniform upper bound of e' in $\overline{U_X}$. From (2.16) we infer the second claim in the same way, where we additionally use the identities

$$\begin{aligned}
\|e'(y,u)^*\|_{L(Y,X^*)} &= \|e'(y,u)\|_{L(X,Y^*)}\,, \\
\|e'(y,u)^* - e'(\bar{y},\bar{u})^*\|_{L(Y,X^*)} &= \|e'(y,u) - e'(\bar{y},\bar{u})\|_{L(X,Y^*)}
\end{aligned} \tag{2.19}$$

for every $(y,u), (\bar{y},\bar{u}) \in U_X$ (see Remark 2.3). From (2.19) we infer that the uniform Lipschitz-constants of $e'(y,u)^*$ in U_X and the uniform bound of $e'(y,u)^*$ in $\overline{U_X}$ are given by $\gamma_{e'}$ and $\eta_{e'}$, respectively. $\square$

For every h the matrix of operators M given in (2.6) is approximated by

$$\begin{aligned} M_h(y_h,u_h,\lambda_h) &= s_h\, M(p_h(y_h,u_h,\lambda_h))\, p_h \\ &= \begin{pmatrix} L_h''(y_h,u_h,\lambda_h) & e_h'(y_h,u_h)^* \\ e_h'(y_h,u_h) & 0 \end{pmatrix} \end{aligned} \tag{2.20}$$

for all $(y_h,u_h) \in \tilde{V}_h$ and $\lambda_h \in Y_h$. Since $L_h''(y_h,u_h,\lambda_h)$, $e_h'(y_h,u_h)^*$ and $e_h'(y_h,u_h)$ are Lipschitz-continuous, the matrix $M_h(y_h,u_h,\lambda_h)$ is also Lipschitz-continuous.

COROLLARY 3.32. *Let Condition 5 hold. Then it follows that*

$$\lim_{h\to 0} \|s_h M(y,u,\lambda)(\bar{y},\bar{u},\bar{\lambda})^T - M_h(r_h(y,u,\lambda))[r_h(\bar{y},\bar{u},\bar{\lambda})]^T\|_{Z_h^*} = 0$$

for all $(y,u,\lambda) \in U$ *and* $(\bar{y},\bar{u},\bar{\lambda}) \in Z$.

PROOF. The statement follows directly from Lemmas 3.30 and 3.31. □

3.6. Internal Approximation of the Augmented Lagrangian-SQP Method. This section is devoted to the approximation of Algorithm 1 on page 25. For this purpose we use the internal approximations of the operators and their Fréchet-derivatives. The approximation of (**OS**) reads

$$L_h'(y_h,u_h,\lambda_h + c\,\mathfrak{J}_h e_h(y_h,u_h)) = 0\,, \quad e_h(y_h,u_h) = 0, \tag{$\mathbf{OS_h}$}$$

which is the first-order necessary optimality condition for ($\mathbf{P_h}$). We shall require the following assumption.

CONDITION 6. For sufficiently small h there exist a neighborhood $V_h \subseteq \tilde{V}_h$ of (y_h^*,u_h^*) and a uniform constant $r_{\min} > 0$ such that $B((y_h^*,u_h^*);r_{\min}) \subseteq V_h$ for all h and a *uniform discrete Babuška-Brezzi condition* is satisfied in V_h, i.e. there is a constant $\beta^* > 0$ independent of h with

$$\inf_{\bar{\lambda}_h \in Y_h} \sup_{(\bar{y}_h,\bar{u}_h)\in X_h} \frac{\langle e_h'(y_h,u_h)^*\bar{\lambda}_h, (\bar{y}_h,\bar{u}_h)\rangle_{X_h^*,X_h}}{\|(\bar{y}_h,\bar{u}_h)\|_{X_h}\, \|\bar{\lambda}_h\|_{Y_h}} \geq \beta^*$$

for all $(y_h,u_h) \in V_h$.

With Condition 6 holding Lemma 2.4 on page 7 implies the existence of a Lagrange multiplier $\lambda_h^* \in Y_h$ such that $z_h^* = (y_h^*,u_h^*,\lambda_h^*)$ solves ($\mathbf{OS_h}$).

CONDITION 7. For sufficiently small h there exists a neighborhood W_h of λ_h^* with $p_h^Y(W_h) \subset U_Y$ such that for $U_h = V_h \times W_h$

a) $B(z_h^*;r_{\min}) \subseteq U_h$ and
b) a *uniform second-order sufficient optimality condition* is satisfied in U_h, i.e. there exists a constant $\kappa^* > 0$ such that for all $(y_h,u_h,\lambda_h) \in U_h$

$$\begin{cases} \langle L_h''(y_h,u_h,\lambda_h)(\bar{y}_h,\bar{u}_h),(\bar{y}_h,\bar{u}_h)\rangle_{X_h^*,X_h} \geq \kappa^*\, \|(\bar{y}_h,\bar{u}_h)\|_{X_h}^2 \\ \text{for all } (\bar{y}_h,\bar{u}_h) \in \ker e_h'(y_h,u_h)\,. \end{cases}$$

Now we consider the internal approximation of Algorithm 1.

ALGORITHM 2. a) *Choose* $(y_h^0, u_h^0, \lambda_h^0) \in U_h$, $c \geq 0$ *and put* $n = 0$.
b) *Set* $\tilde{\lambda}_h^n = \lambda_h^n + c\,\mathfrak{J}_h e_h(y_h^n, u_h^n)$.
c) *Solve for* $(\bar{y}_h, \bar{u}_h, \bar{\lambda}_h)$*:*

$$M_h(y_h^n, u_h^n, \tilde{\lambda}_h^n) \begin{pmatrix} (\bar{y}_h, \bar{u}_h) - (y_h^n, u_h^n) \\ \bar{\lambda}_h - \tilde{\lambda}_h^n \end{pmatrix} = - \begin{pmatrix} L_h'(y_h^n, u_h^n, \tilde{\lambda}_h^n) \\ e_h(y_h^n, u_h^n) \end{pmatrix}. \tag{2.21}$$

d) *Set* $(y_h^{n+1}, u_h^{n+1}, \lambda_h^{n+1}) = (\bar{y}_h, \bar{u}_h, \bar{\lambda}_h)$, $n = n+1$ *and goto* b).

REMARK 3.33. Since the matrix M_h has the same structure as M we have to solve a saddle-point problem in (2.21). But now the linear system (2.21) has finite dimension.

LEMMA 3.34. *With Conditions 6 and 7 holding the saddle-point problem*

$$M_h(y_h, u_h, \lambda_h) \begin{pmatrix} (\bar{y}_h, \bar{u}_h) - (y_h, u_h) \\ \bar{\lambda}_h - \lambda_h \end{pmatrix} = - \begin{pmatrix} L_h'(y_h, u_h, \lambda_h) \\ e_h(y_h, u_h) \end{pmatrix}$$

has a unique solution $(\bar{y}_h, \bar{u}_h, \bar{\lambda}_h)$ *for every* $(y_h, u_h, \lambda_h) \in U_h$.

PROOF. The assumptions allow us to prove Lemma 3.34 in the same way as Lemma 2.11. □

REMARK 3.35. Since both κ^* and β^* do not dependent on h the inverse operators $M_h^{-1}(y_h, u_h, \lambda_h)$ are uniformly bounded in the operator norm.

3.7. Convergence Theorem. Now we are going to formulate our convergence theorem. The essential assumptions to guarantee the well-posedness and the second-order convergence of Algorithm 2 are given by the application of Theorem 2.13 to the finite dimensional case. Therefore, we have to assume that there exists a local solution of $(\mathbf{P_h})$, that the start-up value $z_h^0 = (y_h^0, u_h^0, \lambda_h^0)$ is sufficiently close to the solution z_h^* of the optimality system $(\mathbf{OS_h})$ and that both the Babuška-Brezzi- and the second-order sufficient optimality condition – Conditions 6 and 7 – are satisfied. If in addition Condition 8 is valid and the start-up value $z^0 = (y^0, u^0, \lambda^0)$ of Algorithm 1 is sufficiently close to $z^* = (y^*, u^*, \lambda^*)$ then we shall show that

$$\lim_{h \to 0} \|(y_h^n, u_h^n, \lambda_h^n) - r_h(y^n, u^n, \lambda^n)\|_{Z_h} = 0 \qquad \text{for all } n$$

and

$$\lim_{h \to 0} \|z_h^* - r_h z^*\|_{Z_h} = 0\,.$$

Under the hypotheses of Lemma 3.34 there exists a bound $\eta^* > 0$ which may depend on β^* and κ^* but is independent of h so that

$$\|M_h^{-1}(y_h, u_h, \lambda_h)\|_{L(Z_h^*, Z_h)} \leq \eta^* \tag{2.22}$$

for all $(y_h, u_h, \lambda_h) \in U_h$ and for sufficiently small h.

Using (2.12), Remark 3.12-a) and Definition 3.5 we derive

$$\|e_h(y_h,u_h) - e_h(\bar y_h,\bar u_h)\|_{Y_h^*} \le \gamma_e \,\|(y_h,u_h) - (\bar y_h,\bar u_h)\|_{X_h}$$

and from (2.20) we infer

$$\|M_h(y_h,u_h,\lambda_h) - M_h(\bar y_h,\bar u_h,\bar\lambda_h)\|_{L(Z_h,Z_h^*)} \le \gamma_M \,\|(y_h,u_h,\lambda_h) - (\bar y_h,\bar u_h,\bar\lambda_h)\|_{Z_h}$$

for all $(y_h,u_h,\lambda_h),(\bar y_h,\bar u_h,\bar\lambda_h) \in U_h$. As a consequence uniform Lipschitz-constants of e_h in V_h and M_h in U_h can be chosen of the uniform Lipschitz-properties of e and M in U_X and U, respectively.

The following lemma will be needed for the proof of Theorem 3.38. For the proof we refer the reader to [**OR70**, p. 73].

Lemma 3.36. *Let H be a finite dimensional Hilbert space and let $G : H \to H^*$ be continuously Fréchet-differentiable on a convex set $D \subseteq H$. Further, assume that the operator G' is Lipschitz-continuous in D with a Lipschitz-constant $\gamma_G > 0$. Then:*

$$\|G(\bar x) - G(x) - G'(x)(\bar x - x)\|_{H^*} \le \frac{\gamma_G}{2}\,\|\bar x - x\|_H^2 \qquad \text{for all } x,\bar x \in D\,.$$

The next condition will be used for the convergence analysis of Algorithm 2.

Condition 8. For all $(y,u) \in U_X$ we have

$$\lim_{h\to 0} \|r_h^Y \mathfrak{J} e(y,u) - \mathfrak{J}_h e_h(r_h^X(y,u))\|_{Y_h} = 0\,.$$

Remark 3.37. Condition 8 is satisfied if Conditions 5 holds and the dual approximation (Y_h^*, q_h^Y, s_h^Y) of Y^* are convergent: For all $(y,u) \in U$ we have

$$\begin{aligned}\|r_h^Y \mathfrak{J} e(y,u) - \mathfrak{J}_h e_h(r_h^X(y,u))\|_{Y_h} &\le c_\circ\,\|e(y,u) - q_h^Y s_h^Y e(y,u)\|_{Y^*}\\ &\quad + c_\circ^2\gamma_e\,\|(y,u) - p_h^X r_h^X(y,u)\|_X\\ &\xrightarrow{h\to 0} 0\end{aligned}$$

by (2.11) and Remark 3.27. Note that the dual approximations are convergent if we choose the optimal restriction $\hat r_h^Y$, see Lemma 3.23.

In Chapter 4, Section 6, we shall prove Condition 8 by using properties of the operator $\mathfrak{J}_h$.

Theorem 3.38. *Let Conditions 1-4 and 6-7 be satisfied. Moreover, let $c\,\|z_h^0 - z_h^*\|_{Z_h}$ be sufficiently small for all h. Then we obtain:*

a) *Algorithm 2 is well-defined and*

$$\|(y_h^{n+1},u_h^{n+1},\lambda_h^{n+1}) - z_h^*\|_{Z_h} \le C\,\|(y_h^n,u_h^n,\lambda_h^n) - z_h^*\|_{Z_h}^2$$

where $C = \frac{1}{2}\,\gamma_M \eta^* \max(2, 1+2c^2\gamma_e^2)$ is independent of h.

b) *Let z^0 be a start-up value of Algorithm 1 such that the term $c\,\|z^0 - z^*\|_Z$ is sufficiently small, and assume that*

$$\lim_{h\to 0} \left\| z_h^0 - r_h z^0 \right\|_{Z_h} = 0\,. \tag{2.23}$$

If in addition Conditions 5 and 8 holds then we obtain

$$\lim_{h\to 0} \|(y_h^n, u_h^n, \lambda_h^n) - r_h(y^n, u^n, \lambda^n)\|_{Z_h} = 0 \qquad \textit{for each } n\,,$$

where $(y_h^n, u_h^n, \lambda_h^n)$ and (y^n, u^n, λ^n) are the n-th iterates of the finite and infinite dimensional methods, respectively.

Proof. a) Due to Conditions 6 and 7 and the uniform Lipschitz properties of e_h in V_h and of M_h in U_h, the first claim follows directly from Theorem 2.13 provided that

$$\|z_h^0 - z_h^*\|_{Z_h} < \min\left(\frac{r_{\min}}{\sqrt{a}}, \frac{2}{a\eta^*\gamma_M}\right)\,, \tag{2.24}$$

holds, where $a = \max\,(2, 1 + 2c^2\gamma_e^2)$.

b) Let (2.24) hold and

$$\|z^0 - z^*\|_Z < \min\left(\frac{r_{\max}}{\sqrt{a}}, \frac{2}{a\eta\gamma_M}\right)\,, \tag{2.25}$$

where $r_{\max}$ denotes the largest radius for a ball centered at z^* and contained in U and $a = \max\,(2, 1{+}2c^2\gamma_e^2)$. From (2.25) and Conditions 1-3 we infer that Algorithm 1 is well-defined and converges to z^*. Similarly, (2.24) yields that Algorithm 2 is well-defined with start-up value z_h^0 and converges to z_h^*. By induction we prove that for all $n = 0, 1, \ldots$

$$\lim_{h\to 0} \|(y_h^n, u_h^n, \lambda_h^n) - r_h(y^n, u^n, \lambda^n)\|_{Z_h} = 0\,. \tag{2.26}$$

The induction hypothesis $n = 0$ coincides with (2.23). From Remark 3.27, Condition 8 and (2.26) we derive that

$$\begin{aligned}
&\|\tilde\lambda_h^n - r_h^Y \tilde\lambda^n\|_{Y_h} \\
&\quad\le\quad \|\lambda_h^n - r_h^Y\lambda^n\|_{Y_h} + c\,\|\mathfrak{J}_h e_h(y_h^n, u_h^n) - r_h^Y \mathfrak{J} e(y^n, u^n)\|_{Y_h} \\
&\quad\le\quad \|\lambda_h^n - r_h^Y\lambda^n\|_{Y_h} + \gamma_e c c_\circ\, \|(y_h^n, u_h^n) - r_h^X(y^n, u^n)\|_{X_h} \\
&\qquad\qquad\qquad\qquad + c\,\|\mathfrak{J}_h e_h(r_h^X(y^n, u^n)) - r_h^Y \mathfrak{J} e(y^n, u^n)\|_{Y_h} \\
&\quad\xrightarrow{h\to 0} 0
\end{aligned}$$

so that

$$\lim_{h\to 0} \|(y_h^n, u_h^n, \tilde\lambda_h^n) - r_h(y^n, u^n, \tilde\lambda^n)\|_{Z_h} = 0\,. \tag{2.27}$$

Using Algorithm 1 and Algorithm 2 we obtain the identity

$$\begin{pmatrix} (y_h^{n+1}, u_h^{n+1}) \\ \lambda_h^{n+1} \end{pmatrix} - \begin{pmatrix} r_h^X(y^{n+1}, u^{n+1}) \\ r_h^Y \lambda^{n+1} \end{pmatrix} = M_h^{-1}(y_h^n, u_h^n, \tilde\lambda_h^n) \cdot \sum_{i=1}^4 I_h^i\,, \tag{2.28}$$

where

$$\begin{aligned} I_h^1 &= M_h(y_h^n, u_h^n, \tilde\lambda_h^n)[(y_h^n, u_h^n, \tilde\lambda_h^n) - r_h(y^n, u^n, \tilde\lambda^n)]^T - F_h(y_h^n, u_h^n, \tilde\lambda_h^n) \\ &\quad + F_h(r_h(y^n, u^n, \tilde\lambda^n)) \\ I_h^2 &= [M_h(y_h^n, u_h^n, \tilde\lambda_h^n) - M_h(r_h(y^n, u^n, \tilde\lambda^n))] \\ &\quad \cdot r_h[M^{-1}(y^n, u^n, \tilde\lambda^n) F(y^n, u^n, \tilde\lambda^n)] \\ I_h^3 &= M_h(r_h(y^n, u^n, \tilde\lambda^n))\, r_h[M^{-1}(y^n, u^n, \tilde\lambda^n) F(y^n, u^n, \tilde\lambda^n)] \\ &\quad - s_h F(y^n, u^n, \tilde\lambda^n) \\ I_h^4 &= s_h F(y^n, u^n, \tilde\lambda^n) - F_h(r_h(y^n, u^n, \tilde\lambda^n)) \end{aligned}$$

and

$$F(y,u,\lambda) = \begin{pmatrix} L'(y,u,\lambda) \\ e(y,u) \end{pmatrix}, \quad F_h = s_h\, F\, p_h\,. \tag{2.29}$$

To estimate (2.28) we first proceed as in the proof of Theorem 2.13 in [**IK96a**] to conclude that $(y_h^n, u_h^n, \tilde\lambda_h^n) \in U_h$. By (2.22) this implies

$$\|M_h^{-1}(y_h^n, u_h^n, \tilde\lambda_h^n)\|_{L(Z_h^*, Z_h)} \le \eta^*\,.$$

Applying Lemma 3.36 yields

$$\|I_h^1\|_{Z_h^*} \le \frac{\gamma_M}{2}\, \|(y_h^n, u_h^n, \tilde\lambda_h^n) - r_h(y^n, u^n, \tilde\lambda^n)\|_{Z_h}^2\,.$$

Under the hypotheses of Theorem 2.13 we have

$$\|(y^{n+1}, u^{n+1}, \lambda^{n+1}) - z^*\|_Z \le \|(y^n, u^n, \lambda^n) - z^*\|_Z$$

and

$$\|(y^n, u^n, \tilde\lambda^n) - z^*\|_Z \le \sqrt{a}\, \|(y^n, u^n, \lambda^n) - z^*\|_Z\,.$$

Moreover, from c) of Algorithm 1 it follows that

$$M^{-1}(y^n, u^n, \tilde\lambda^n) F(y^n, u^n, \tilde\lambda^n) = [(y^{n+1}, u^{n+1}, \lambda^{n+1}) - (y^n, u^n, \tilde\lambda^n)]^T\,. \tag{2.30}$$

Using (2.30) and Remark 3.27 we derive

$$\begin{aligned} \|I_h^2\|_{Z_h^*} &\le \gamma_M c_\circ\, \|(y_h^n, u_h^n, \tilde\lambda_h^n) - r_h(y^n, u^n, \tilde\lambda^n)\|_{Z_h} \\ &\quad \cdot \|(y^{n+1}, u^{n+1}, \lambda^{n+1}) - (y^n, u^n, \tilde\lambda^n)\|_Z \\ &\le c_1\, \|z^0 - z^*\|_Z\, \|(y_h^n, u_h^n, \tilde\lambda_h^n) - r_h(y^n, u^n, \tilde\lambda^n)\|_{Z_h}\,, \end{aligned}$$

where $c_1 = \gamma_M c_\circ (1 + \sqrt{a})$. Setting

$$(\bar y, \bar u, \bar\lambda)^T = M^{-1}(y^n, u^n, \tilde\lambda^n) F(y^n, u^n, \tilde\lambda^n)$$

we obtain

$$\|I_h^3\|_{Z_h^*} = \|s_h M(y^n, u^n, \tilde\lambda^n)(\bar y, \bar u, \bar\lambda)^T - M_h(r_h(y^n, u^n, \tilde\lambda^n))[r_h(\bar y, \bar u, \bar\lambda)]^T\|_{Z_h^*}$$

and

$$\begin{aligned}\|I_h^4\|_{Z_h^*} &\le \|s_h^X L'(y^n,u^n,\tilde\lambda^n) - L_h'(r_h(y^n,u^n,\tilde\lambda^n))\|_{X_h^*}\\ &\quad + \|s_h^Y e(y^n,u^n) - e_h(r_h^X(y^n,u^n))\|_{Y_h^*}\,.\end{aligned}$$

With these majorizations in (2.28) we obtain from equation (2.27), Corollary 3.32 and from Lemmas 3.30 and 3.28 that

$$\lim_{h\to 0} \|(y_h^{n+1},u_h^{n+1},\lambda_h^{n+1}) - r_h(y^{n+1},u^{n+1},\lambda^{n+1})\|_{Z_h} = 0$$

such that (2.26) follows.

□

REMARK 3.39. a) Assumption (2.23) is satisfied if we take the start-up value $z_h^0 = r_h z^0$.

b) Note that Algorithm 2 is still well-defined for every $h > 0$ if both β^* and κ^* depend on h. In this case, η^* may also depend on h such that the constant C given by Theorem 3.38-a) need not to be independent of h.

COROLLARY 3.40. *Under the hypotheses of Theorem 3.38*

$$\lim_{h\to 0} \|z_h^* - r_h z^*\|_{Z_h} = 0$$

holds.

PROOF. Let $\varepsilon > 0$. Applying Theorem 2.13 there exists $n_1 \in \mathbb{N}$ such that

$$\|(y^{n_1},u^{n_1},\lambda^{n_1}) - z^*\|_Z < \frac{\varepsilon}{3c_\circ} \quad \text{for all } n \ge n_1\,,$$

where $c_\circ$ is given by Remark 3.27. From Theorem 3.38-a) we infer that

$$\|(y_h^n,u_h^n,\lambda_h^n) - z_h^*\|_{Z_h} \le \|z_h^0 - z_h^*\|_{Z_h} \left(\frac{1}{2}\, a\gamma_M \eta^*\, \|z_h^0 - z_h^*\|_{Z_h}\right)^{2^n - 1},$$

where $a = \max(2, 1+2c^2\gamma_e)$. Using (2.24) there is $n_2 \in \mathbb{N}$ such that for all h

$$\|(y_h^{n_2},u_h^{n_2},\lambda_h^{n_2}) - z_h^*\|_{Z_h} < \frac{\varepsilon}{3} \quad \text{for all } n \ge n_2\,.$$

Let $n_\circ = \max(n_1,n_2)$. By Theorem 3.38-b) there exists $h_\circ = h_\circ(n_\circ) > 0$ with

$$\|(y_h^{n_\circ},u_h^{n_\circ},\lambda_h^{n_\circ}) - r_h(y^{n_\circ},u^{n_\circ},\lambda^{n_\circ})\|_{Z_h} < \frac{\varepsilon}{3} \quad \text{for all } h \in (0,h_\circ]\,.$$

Thus, we derive from Remark 3.27 that

$$\begin{aligned}\|z_h^* - r_h z^*\|_{Z_h} &\le \|z_h^* - (y_h^{n_\circ},u_h^{n_\circ},\lambda_h^{n_\circ})\|_{Z_h}\\ &\quad + \|(y_h^{n_\circ},u_h^{n_\circ},\lambda_h^{n_\circ}) - r_h(y^{n_\circ},u^{n_\circ},\lambda^{n_\circ})\|_{Z_h}\\ &\quad + \|p_h r_h\left[(y^{n_\circ},u^{n_\circ},\lambda^{n_\circ}) - z^*\right]\|_Z\\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + c_\circ\,\frac{\varepsilon}{3c_\circ} = \varepsilon\end{aligned}$$

for all $h \in (0,h_\circ]$.

□

COROLLARY 3.41. *Let the hypotheses of Theorem 3.38 hold. Moreover, let* $\{(Z_{h(n)}, p_{h(n)}, r_{h(n)})\}_{n\in\mathbb{N}}$ *be a sequence of internal approximations satisfying* $\lim_{n\to\infty} h(n) = 0$. *Then*

$$\lim_{n\to\infty} \|(y^n_{h(n)}, u^n_{h(n)}, \lambda^n_{h(n)}) - r_{h(n)} z^*\|_{Z_{h(n)}} = 0$$

holds.

PROOF. The proof follows from estimating

$$\begin{aligned}\|(y^n_{h(n)}, u^n_{h(n)}, \lambda^n_{h(n)}) &- r_{h(n)}(y^n, u^n, \lambda^n)\|_{Z_{h(n)}} \\ &\leq \|(y^n_{h(n)}, u^n_{h(n)}, \lambda^n_{h(n)}) - z^*_{h(n)}\|_{Z_{h(n)}} + \|z^*_{h(n)} - r_{h(n)} z^*\|_{Z_{h(n)}}\end{aligned}$$

by Theorem 3.38 and Corollary 3.40. □

3.8. Special Convergence Properties. In Example 3.1 we introduced the piecewise linear approximations of $H^1(0,1)$. We mentioned in Example 3.20 that this discretization scheme is convergent. In particular, for a constant $\bar{c} > 0$ the following estimate holds [**Aub72**, p. 41]

$$\|\lambda - p^Y_h r^Y_h \lambda\|_Y \leq \bar{c}h\, \|\lambda\|_{H^2} \qquad \text{for all } \lambda \in H^2(0,1)\,.$$

Let us define $\rho : [0,1] \to [0,\infty)$ by $\rho(h) = \bar{c}h$. Then ρ is bounded and right-continuous at $h = 0$ with $\rho(0) = 0$ satisfying

$$\|\lambda - p^Y_h r^Y_h \lambda\|_Y \leq \rho(h) \qquad \text{for all } \lambda \in H^1(0,1) \text{ with } \|\lambda\|_{H^2} \leq 1\,.$$

In many applications it turns out that the solution (y^*, u^*, λ^*) of (**OS**) as well as the iterates (y^n, u^n, λ^n) of Algorithm 1 have "better smoothness" properties than the elements of Z. This is a motivation for the following assumption.

CONDITION 9. There are bounded subsets $V \subset X$ and $W \subset Y$ such that

$$(y^*, u^*),\ (y^n, u^n),\ (y^n, u^n) - (y^*, u^*),\ (y^{n+1}, u^{n+1}) - (y^n, u^n) \in V$$

and

$$\lambda^*,\ \lambda^n,\ \lambda^n - \lambda^*,\ \lambda^{n+1} - \lambda^n \in W$$

hold for all $n = 0, 1, \ldots$.

Moreover, there exists a bounded function $\rho : [0,1] \to [0,\infty)$ which is right-continuous at $h = 0$ with $\rho(0) = 0$ satisfying

$$\|(y, u, \lambda) - p_h r_h (y, u, \lambda)\|_Z \leq \rho(h)$$

for all $(y, u) \in V$ and $\lambda \in W$ and for all $h > 0$.

REMARK 3.42. Note that V and W are assumed to be only subsets of X and Y, respectively. For example, in our motivation at the beginning of this section we have $Y = H^1(0,1)$ and $W = \{\lambda \in H^2(0,1) : \|\lambda\|_{H^2(0,1)} \leq 1\}$, which is no subspace of Y.

Recall the operators F and F_h introduced in (2.29).

LEMMA 3.43. *With Condition 5 and 9 holding there exists a constant* $\hat{C} > 0$ *independent of* h *such that*

$$\|s_h F(y,u,\lambda) - F_h(r_h(y,u,\lambda))\|_{Z_h^*} \le \hat{C}\rho(h)$$

and

$$\|s_h M(y,u,\lambda)(\bar{y},\bar{u},\bar{\lambda})^T - M_h(r_h(y,u,\lambda))[r_h(\bar{y},\bar{u},\bar{\lambda})]^T\|_{Z_h^*} \le \hat{C}\rho(h)$$

for all $(y,u) \in V \cap U_X$, $\lambda \in W \cap U_Y$, $(\bar{y},\bar{u}) \in V$, $\bar{\lambda} \in W$ *and for all* $h > 0$.

PROOF. Let $(y,u), (\bar{y},\bar{u}) \in V$ and $\lambda, \bar{\lambda} \in W$. By Condition 9 we conclude

$$\begin{aligned} \|(y,u,\lambda) - p_h r_h(y,u,\lambda)\|_Z &\le \rho(h)\,, \\ \|(\bar{y},\bar{u},\bar{\lambda}) - p_h r_h(\bar{y},\bar{u},\bar{\lambda})\|_Z &\le \rho(h) \end{aligned}$$

and $\|(\bar{y},\bar{u},\bar{\lambda})\|_Z \le \bar{c}$ for a constant $\bar{c} > 0$. Setting $\hat{C} = \max(\gamma_F, \bar{c}\gamma_M + \eta_M)$, where $\gamma_F > 0$ denotes the Lipschitz-constant of F in U and η_M is a uniform upper bound of M in $\overline{U}$, we derive

$$\|s_h F(y,u,\lambda) - F_h(r_h(y,u,\lambda))\|_{Z_h^*} \le \gamma_F \|(y,u,\lambda) - p_h r_h(y,u,\lambda)\|_Z \le \hat{C}\rho(h)$$

and by triangle inequality

$$\begin{aligned} &\|s_h M(y,u,\lambda)(\bar{y},\bar{u},\bar{\lambda})^T - M_h(r_h(y,u,\lambda))[r_h(\bar{y},\bar{u},\bar{\lambda})]^T\|_{Z_h^*} \\ &\le \gamma_M \|(y,u,\lambda) - p_h r_h(y,u,\lambda)\|_Z \|(\bar{y},\bar{u},\bar{\lambda})\|_Z \\ &\qquad + \eta_M \|(\bar{y},\bar{u},\bar{\lambda}) - p_h r_h(\bar{y},\bar{u},\bar{\lambda})\|_Z \\ &\le \hat{C}\rho(h)\,. \end{aligned}$$

□

REMARK 3.44. Lemma 3.43 implies that

$$\|s_h L'(y,u,\lambda) - L_h'(r_h(y,u,\lambda))\|_{X_h^*} \le \hat{C}\rho(h)$$

and

$$\|s_h^Y e(y,u) - e_h(r_h^X(y,u))\|_{Y_h^*} \le \hat{C}\rho(h)$$

for all $(y,u) \in V \cap U_X$, $\lambda \in W \cap U_Y$, $(\bar{y},\bar{u}) \in V$, $\bar{\lambda} \in W$ and $h > 0$.

3.9. Further Convergence Results. In Theorem 3.38 we obtained converge and rate of convergence results for Algorithm 2 under the assumption that the optimality system $(\mathbf{OS_h})$ has a solution $(y_h^*, u_h^*, \lambda_h^*)$ for every h and that the start-up value $(y_h^0, u_h^0, \lambda_h^0)$ is chosen sufficiently close to this solution. However, Theorem 3.38 does not give answers to the following questions:

a) Can the existence of solutions $(y_h^*, u_h^*, \lambda_h^*)$ to $(\mathbf{OS_h})$ be ascertained directly from the existence of (y^*, u^*, λ^*) and the convergence of Algorithm 2?

b) When Algorithm 2 and Algorithm 1 are stopped after the n-th step, how can the difference between the iterates of finite dimensional method and the infinite dimensional ones be estimated?

Using Condition 9 we are able to answer these questions.

Since we shall not require that for every h the finite dimensional problem $(\mathbf{P_h})$ has at least a solution, we replace Condition 4 on page 34 by the following assumption.

CONDITION 10. For every h there exist neighborhoods V_h of $r_h^X(y^*, u^*)$ and W_h of $r_h^Y \lambda^*$ and a constant $r_{\min} > 0$ independent of h such that $p_h(U_h) \subset U$ and $B(r_h(y^*, u^*, \lambda^*); r_{\min})$ is a subset of $U_h = V_h \times W_h$ for all h.

We shall again need a uniform discrete Babuška-Brezzi and a uniform second-order sufficient optimality condition in V_h and U_h, respectively. Thus, we combine Conditions 6 and 7.

CONDITION 11. a) There exists a constant $\beta^* > 0$ independent of h with

$$\inf_{\bar\lambda_h \in Y_h} \sup_{(\bar y_h, \bar u_h) \in X_h} \frac{\langle e_h'(y_h, u_h)^* \bar\lambda_h, (\bar y_h, \bar u_h)\rangle_{X_h^*, X_h}}{\|(\bar y_h, \bar u_h)\|_{X_h} \, \|\bar\lambda_h\|_{Y_h}} \geq \beta^*$$

for all $(y_h, u_h) \in V_h$, and

b) there exists $\kappa^* > 0$ independent of h such that

$$\begin{cases} \langle L_h''(y_h, u_h, \lambda_h)(\bar y_h, \bar u_h), (\bar y_h, \bar u_h)\rangle_{X_h^*, X_h} \geq \kappa^* \, \|(\bar y_h, \bar u_h)\|_{X_h}^2 \\ \text{for all } (\bar y_h, \bar u_h) \in \ker e_h'(y_h, u_h) \end{cases}$$

for all $(y_h, u_h, \lambda_h) \in U_h$ and for all h.

With Condition 11 holding we infer from Lemma 3.34 that there exists a bound $\eta^* > 0$ which may depend on β^* or κ^* but is independent of h satisfying

$$\|M_h^{-1}(r_h(y^*, u^*, \lambda^*))\|_{L(Z_h^*, Z_h)} \leq \eta^* \qquad \text{for all } h\,. \tag{2.31}$$

By $\mathfrak{J}$ and $\mathfrak{J}_h$ we denote the Riesz isomorphisms mapping Y^* onto Y and Y_h^* onto Y_h, respectively. The following requirement on the restriction r_h^Y will be used in an essential way, compare Condition 8.

CONDITION 12. There exists a constant $c_{\mathfrak{J}} > 0$ such that the function ρ introduced in Condition 9 also satisfies

$$\|r_h^Y \mathfrak{J} e(y, u) - \mathfrak{J}_h e_h(r_h^X(y, u))\|_{Y_h} \leq c_{\mathfrak{J}}\, \rho(h)\,,$$

for all $(y, u) \in V \cap U_X$.

REMARK 3.45. If the optimal restriction $\hat r_h^Y$ is chosen, we derive from Remark 3.16 that $\hat r_h^Y = \mathfrak{J}_h\, s_h^Y\, \mathfrak{J}^{-1}$. Hence, Condition 9 gives

$$\|\hat r_h^Y\, \mathfrak{J}\, e(y, u) - \mathfrak{J}_h\, e_h(\hat r_h^X(y, u))\|_{Y_h} = \|s_h^Y\, e(y, u) - e_h(r_h^X(y, u))\|_{Y_h^*} \leq \gamma_e \rho(h)\,.$$

Setting $c_{\mathfrak{J}} = \gamma_e$ we conclude that Condition 9 implies Condition 12.

By γ_e and γ_M we denoted the Lipschitz-constants of e in U_X and of M in U, respectively.

Now we are going to formulate the convergence results. First of all we derive existence of a local solution of $(\mathbf{OS_h})$ from existence of (y^*, u^*, λ^*) solving $(\mathbf{OS})$ for sufficiently small h.

THEOREM 3.46. *Let Conditions 1-3, 5, 9-11 hold. Then there is a constant* $\bar{h} \in (0, 1]$ *so that* $(\mathbf{OS_h})$ *has a local solution satisfying*

$$(y_h^*, u_h^*, \lambda_h^*) = r_h(y^*, u^*, \lambda^*) + O(\rho(h)) \qquad \text{for all } h \in (0, \bar{h}]\,. \tag{2.32}$$

For the proof we refer the reader to Section 3.10.

REMARK 3.47. Theorem 3.46 gives an answer to the question a) posed at the beginning of this section. Since ρ is right-continuous at $h = 0$ with $\rho(0) = 0$, we derive from (2.32) that

$$\lim_{h \to 0} \|(y_h^*, u_h^*, \lambda_h^*) - r_h(y^*, u^*, \lambda^*)\|_{Z_h} = 0$$

holds.

From existence of $(y_h^*, u_h^*, \lambda_h^*)$ the second-order convergence of Algorithm 2 is proved for start-up values $(y_h^0, u_h^0, \lambda_h^0) = r_h(y^0, u^0, \lambda^0)$, where (y^0, u^0, λ^0) is the start-up value of Algorithm 1 chosen in an appropriate neighborhood of (y^*, u^*, λ^*). This is formulated precisely in the next theorem, which is proved in Section 3.11. We introduced the constant r^* in (2.9).

THEOREM 3.48. *Let the assumptions of Theorem 3.46 hold. Then there exist* $\tilde{h} \in (0, \bar{h}]$ *and* $r_1 \in (0, r^*]$ *such that Algorithm 2 converges to* $(y_h^*, u_h^*, \lambda_h^*)$ *for start-up values* $(y_h^0, u_h^0, \lambda_h^0) = r_h(y^0, u^0, \lambda^0)$ *with* $(y^0, u^0, \lambda^0) \in (V \times W) \cap B((y^*, u^*, \lambda^*); r_1)$*, and the iterates satisfy*

$$\|(y_h^{n+1}, u_h^{n+1}, \lambda_h^{n+1}) - (y_h^*, u_h^*, \lambda_h^*)\|_{Z_h} \le C\, \|(y_h^n, u_h^n, \lambda_h^n) - (y_h^*, u_h^*, \lambda_h^*)\|_{Z_h}^2$$

for a constant $C > 0$ *independent of* $h \in (0, \tilde{h}]$ *and* n.

The following theorem answers the second question posed at the beginning of this section. When Algorithm 2 and Algorithm 1 are stopped after the n-th step, the difference between the iterates of finite dimensional method and the infinite dimensional ones can be estimated by the discretization error of the used approximation scheme.

THEOREM 3.49. *Let the assumptions of Theorem 3.48 be satisfied. By* $\{(y^n, u^n, \lambda^n)\}_{n \in \mathbb{N}}$ *we denote the sequence generated by Algorithm 1. If Condition 12 holds, there exist* $\hat{h} \in (0, \tilde{h}]$ *and a constant* $r_2 \in (0, r_1]$ *such that*

$$\begin{aligned}
(y_h^n, u_h^n, \lambda_h^n) &= r_h(y^n, u^n, \lambda^n) + O(\rho(h))\,, && (2.33)\\
F_h(y_h^n, u_h^n, \tilde{\lambda}_h^n) &= s_h F(y^n, u^n, \tilde{\lambda}^n) + O(\rho(h))\,, && (2.34)\\
(y_h^n - y_h^*, u_h^n - u_h^*, \lambda_h^n - \lambda_h^*) &= r_h(y^n - y^*, u^n - u^*, \lambda^n - \lambda^*) && (2.35)\\
&\quad + O(\rho(h))
\end{aligned}$$

for $n = 0, 1, \ldots$, *for start-up values* $(y^0, u^0, \lambda^0) \in (V \times W) \cap B((y^*, u^*, \lambda^*); r_2)$, *for the sequence* $\{(y_h^n, u_h^n, \lambda_h^n)\}_{n\in\mathbb{N}}$ *generated by Algorithm 2 using the start-up value* $r_h(y^0, u^0, \lambda^0)$ *with* $h \in (0, \hat{h}]$.

For the proof we refer the reader to Section 3.12.

3.10. Proof of Theorem 3.46. The proof will be based on the theory of nonlinear majorants. For more details we refer the reader to [**OR70**, Part V].

Let H be a finite dimensional Hilbert space, $z^0 \in H$ and $G : D \subseteq H \to H$. We introduce the fixed point iteration

$$z^{n+1} = Gz^n\,, \qquad n = 0, 1, \ldots\,. \tag{2.36}$$

To show that the sequence $\{z^n\}_{n\in\mathbb{N}}$ generated by (2.36) is well-defined and converges to a fixed point of G we shall look for a majorizing sequence for $\{z^n\}_{n\in\mathbb{N}}$.

Definition 3.50. Let $\{z^n\}_{n\in\mathbb{N}}$ be any sequence in a finite dimensional Hilbert space H. Then a sequence $\{t^n\}_{n\in\mathbb{N}} \subset [0, \infty)$ for which

$$\|z^{n+1} - z^n\|_H \le t^{n+1} - t^n\,, \qquad n = 0, 1, \ldots\,,$$

holds is a majorizing sequence for $\{z^n\}_{n\in\mathbb{N}}$.

Note that any majorizing sequence is necessarily monotonically increasing. In the proof of Theorem 3.46 a majorizing sequence will arise as the solution of a certain nonlinear difference equation which in turn is based on an estimate for $G(Gz) - Gz$. The idea is given by the following lemma.

Lemma 3.51. *Let* H *be a finite dimensional Hilbert space,* $G : D \subseteq H \to H$ *and* $\varphi : J_1 \times J_2 \subset \mathbb{R}^2 \to [0, \infty)$ *where each* J_i *is an interval of the form* $[0, \alpha]$, $[0, \alpha)$ *or* $[0, \infty)$ *and* φ *is monotonically increasing in each variable. Suppose that there are a set* $D_\circ \subset D$ *and an element* $z^0 \in D_\circ$ *such that*

$$\|G(Gz) - Gz\|_H \le \varphi\left(\|Gz - z\|_H, \|Gz - z^0\|_H\right) \tag{2.37}$$

holds whenever $z, Gz \in D_\circ$, *and that with* $t^0 = 0$, $t^1 = \|Gz^0 - z^0\|_H$ *the solution of the difference equation*

$$t^{n+1} - t^n = \varphi(t^n - t^{n-1}, t^n)\,, \qquad n = 1, 2, \ldots \tag{2.38}$$

exists and converges to $t^* < \infty$. *Finally, assume that* $\overline{B(z^0, t^*)} \subset D_\circ$. *Then the iterates* $z^{n+1} = Gz^n$, $n = 0, 1, \ldots$, *are well-defined, lie in* $\overline{B(z^0, t^*)}$, *converge to some* $z^* \in \overline{B(z^0, t^*)}$ *and satisfy*

$$\|z^* - z^n\|_H \le t^* - t^n\,, \qquad n = 0, 1, \ldots\,.$$

If $z^* \in D_\circ$ *and* G *is continuous at* z^*, *then* $z^* = Gz^*$.

For the proof we refer the reader to [**OR70**, p. 416].

REMARK 3.52. In the proof of Lemma 3.51 it is shown that $\{z^n\}_{n\in\mathbb{N}}$ is majorized by $\{t^n\}_{n\in\mathbb{N}}$.

In the proof of Theorem 3.46 below the difference equation (2.38) will arise in the form

$$t^{n+1} - t^n = \frac{c_1(t^n - t^{n-1})^2 + 8(t^n - t^{n-1})}{2(7 - c_1 t^n)}, \quad n = 1, 2, \dots, \tag{2.39}$$

where $c_1 = 8\eta^*\gamma_M(1 + c\gamma_e)^2$.

LEMMA 3.53. *The solution of* (2.39) *is given by*

$$t^{n+1} = \psi(t^n), \qquad t^0 = 0, \qquad n = 0, 1, \dots, \tag{2.40}$$

where

$$\psi(t) = t + \frac{c_1 t^2 - 6t + 14\,\|Gz^0 - z^0\|_H}{2(7 - c_1 t)}.$$

PROOF. First we conclude that $t^0 = 0$ and $t^1 = \|Gz - z^0\|_H$ hold. Multiplying (2.40) by $2(7 - c_1 t^n)$ leads to

$$\begin{aligned} 14t^{n+1} - 2c_1 t^n t^{n+1} &= 14t^n - 2c_1(t^n)^2 + c_1(t^n)^2 - 6t^n + 14\,\|Gz^0 - z^0\|_H \\ &= -c_1(t^n)^2 + 8t^n + 14\,\|Gz^0 - z^0\|_H \end{aligned}$$

so that we obtain

$$c_1(t^{n+1})^2 - 14t^{n+1} + 14\,\|Gz^0 - z^0\|_H = c_1(t^{n+1} - t^n)^2 - 8t^n \tag{2.41}$$

and, hence, that

$$\begin{aligned} t^{n+1} - t^n &= \psi(t^n) - t^n \\ &\overset{(2.40)}{=} \frac{c_1(t^n)^2 - 6t^n + 14\,\|Gz^0 - z^0\|_H}{2(7 - c_1 t^n)} \\ &\overset{(2.41)}{=} \frac{c_1(t^n - t^{n-1})^2 + 8(t^n - t^{n-1})}{2(7 - c_1 t^n)}. \end{aligned}$$

□

The following lemma gives sufficient conditions that the solution of (2.39) converges. For the proof we refer the reader to [**OR70**, p. 424].

LEMMA 3.54. *Let* $\|Gz^0 - z^0\|_H \in [0, \frac{9}{14c_1}]$. *Then the sequence* $\{t^n\}_{n\in\mathbb{N}}$ *of* (2.39) *with initial values* $t^0 = 0$ *and* $t^1 = \|Gz^0 - z^0\|_H$ *is strictly increasing unless* $\|Gz^0 - z^0\|_H = 0$, *and*

$$\lim_{n\to\infty} t^n = t^* = \frac{1}{c_1}\left(3 - \sqrt{9 - 14c_1\,\|Gz^0 - z^0\|_H}\right).$$

The following *perturbation lemma* is a consequence of Neumann's lemma. It will be needed in the proof of Theorem 3.46.

LEMMA 3.55. *Let $A, B \in L(H, H^*)$ hold, where H is a finite dimensional Hilbert space, and let the operator A^{-1} exist with $\|A^{-1}\|_{L(H^*,H)} \le \alpha$. If $\|A - B\|_{L(H^*,H)} \le \sigma$ and $\alpha\sigma < 1$, then B is also invertible, and*

$$\|B^{-1}\|_{L(H,H^*)} \le \frac{\alpha}{1-\alpha\sigma}\,.$$

For the proof we refer the reader to [**OR70**, p. 45].

PROOF OF THEOREM 3.46. To prove the theorem we shall proceed as in the proof of the Newton-Kantorovich theorem [**OR70**, p. 421]. Before we shall start with our proof, the idea of the proof will be summarized. For every h we shall introduce a nonlinear mapping $G_h : U_h \to Z_h$ such that Algorithm 2 will take the form (2.36). By choosing the start-up value

$$(y_h^0, u_h^0, \lambda_h^0) = r_h(y^*, u^*, \lambda^*) \tag{2.42}$$

and setting $z_h^n = (y_h^n, u_h^n, \lambda_h^n)$ for all n we shall investigate the sequence $\{z_h^n\}_{n\in\mathbb{N}}$ generated by Algorithm 2. To apply Lemma 3.51 we derive an estimate of the form

$$\|G_h(G_h z_h) - G_h z_h\|_{Z_h} \le \varphi\left(\|G_h z_h - z_h\|_{Z_h}, \|G_h z_h - z_h^0\|_{Z_h}\right)\,,$$

where φ is a nonlinear function. This will lead to the difference equation

$$t_h^{n+1} - t_h^n = \frac{c_1(t_h^n - t_h^{n-1})^2 + 8(t_h^n - t_h^{n-1})}{2(7 - c_1 t_h^n)}\,, \quad n = 1, 2, \dots\,,$$

where $c_1 = 8\eta^*\gamma_M(1+c\gamma_e)^2$. By Lemmas 3.53 and 3.54 the sequence $\{t_h^n\}_{n\in\mathbb{N}}$ exists and converges to some $t_h^* \in (0, r_{\min}]$ for all h sufficiently small. Finally, applying Lemma 3.51 will complete our proof.

Throughout the proof we will write $x_h = (y_h, u_h)$, $\tilde\lambda_h = \lambda_h + c\mathfrak{J}_h e_h(y_h, u_h)$, $z_h = (x_h, \lambda_h)$ and $\tilde z_h = (x_h, \tilde\lambda_h)$ for $(y_h, u_h) \in X_h$ and $\lambda_h \in Y_h$. Further we set $x^* = (y^*, u^*)$ and $z^* = (x^*, \lambda^*)$.

Let $G_h : U_h \to Z_h$ be defined by

$$G_h z_h = \tilde z_h - \left[M_h^{-1}(\tilde z_h)\, F_h(\tilde z_h)\right]^T \tag{2.43}$$

and let z_h^0 be given by (2.42). Using (2.43) Algorithm 2 can be written as a fixed point iteration:

a) Set $z_h^0 = r_h z^*$, $c \ge 0$, and put $n = 0$.
b) Compute $z_h^{n+1} = G_h z_h^n$.
c) Set $n = n+1$ and goto b).

Using (2.42) and the identity $e(x^*) = 0$ we infer from Remark 3.6 and Conditiontion 9 that

$$\begin{aligned}\|\tilde z_h - z_h^0\|_{Z_h} &\le \|z_h - z_h^0\|_{Z_h} + c\,\|e_h(x_h) - e_h(x_h^0)\|_{Y_h^*}\\ &\quad + c\,\|s_h^Y e(p_h^X r_h^X x^*) - s_h^Y e(x^*)\|_{Y_h^*}\\ &\le (1+c\gamma_e)\|z_h - z_h^0\|_{Z_h} + c\gamma_e\rho(h).\end{aligned} \tag{2.44}$$

Let the set $D_h \subset U_h$ be given by

$$D_h = B\left(z_h^0; \min\left\{\frac{r_{\min}}{2(1+c\gamma_e)}, \frac{3}{8\eta^*\gamma_M(1+c\gamma_e)^2}\right\}\right), \tag{2.45}$$

where the radius $r_{\min} > 0$ was introduced in Condition 10. Note that ρ is right-continuous at $h = 0$. If we choose $h_1 \in (0,1]$ with

$$\rho(h) \le \frac{r_{\min}}{2c\gamma_e} \text{ for all } h \in (0, h_1]$$

we conclude from (2.44) that $\tilde{z}_h \in U_h$ for every $z_h \in D_h$ and for all $h \le (0, h_1]$. Hence $F_h(\tilde{z}_h)$ and $M_h(\tilde{z}_h)$ are defined for any $z_h \in D_h$ and for all $h \in (0, h_1]$.

Writing $(y_h, \mu_h) = G(z_h)$ for $z_h \in U_h$ we have

$$\widetilde{G_h z_h} = (y_h, \tilde{\mu}_h) = G_h z_h + (0, ce_h(y_h)). \tag{2.46}$$

Analogously to estimate (2.44) we infer that

$$\|M_h(\widetilde{G_h z_h}) - M_h(z_h^0)\|_{\mathcal{L}(Z_h, Z_h^*)} \le \gamma_M(1+c\gamma_e)\,\|G_h z_h - z_h^0\|_{Z_h} + c\gamma_e\gamma_M\rho(h)$$

for all $G_h z_h \in D_h$. Let $h_2 \in (0, h_1]$ be given such that

$$\rho(h) \le \frac{1}{8c\eta^*\gamma_e\gamma_M(1+c\gamma_e)} \text{ for all } h \in (0, h_2]\,. \tag{2.47}$$

By using (2.45) and (2.47) we derive that

$$\|M_h(\widetilde{G_h z_h}) - M_h(z_h^0)\|_{\mathcal{L}(Z_h, Z_h^*)} \le \frac{3}{8\eta^*} + \frac{1}{8\eta^*} < \frac{1}{\eta^*}$$

for all $G_h z_h \in D_h$ and $h \in (0, h_2]$. From Lemma 3.55 with $A = M_h(z_h^0)$ and $B = M_h(\widetilde{G_h z_h})$ we infer that $M_h(\widetilde{G_h z_h})$ is invertible for all $G_h z_h \in D_h$ and $h \in (0, h_2]$, and that

(2.48)

$$\begin{aligned}
\|M_h^{-1}(\widetilde{G_h z_h})\|_{\mathcal{L}(Z_h^*, Z_h)} &\le \frac{\eta^*}{1 - \eta^*[\gamma_M(1+c\gamma_e)\,\|G_h z_h - z_h^0\|_{Z_h} + c\gamma_e\gamma_M\rho(h)]} \\
&\overset{(2.47)}{\le} \frac{\eta^*}{\frac{7}{8} - \eta^*\gamma_M(1+c\gamma_e)\,\|G_h z_h - z_h^0\|_{Z_h}} \\
&= \frac{8\eta^*}{7 - c_1\,\|G_h z_h - z_h^0\|_{Z_h}},
\end{aligned}$$

where we set

$$c_1 = 8\,\eta^*\gamma_M(1+c\gamma_e)^2\,. \tag{2.49}$$

Equation (2.43) yields

$$\widetilde{G_h z_h} - \tilde{z}_h = (0, ce_h(y_h)) - [M_h(\tilde{z}_h)^{-1}F_h(\tilde{z}_h)]^T\,. \tag{2.50}$$

Hence, we conclude from (2.50) and Lemma 3.36 that

$$\begin{aligned}
&\|M_h(\tilde{z}_h)(0, ce_h(y_h))^T - F_h(\widetilde{G_h z_h})\|_{Z_h^*} \\
&\qquad = \|F_h(\widetilde{G_h z_h}) - F_h(\tilde{z}_h) - M_h(\tilde{z}_h)(\widetilde{G_h z_h} - \tilde{z}_h)^T\|_{Z_h^*} \qquad (2.51)\\
&\qquad \le \frac{\gamma_M}{2} \|\widetilde{G_h z_h} - \tilde{z}_h\|^2_{Z_h}.
\end{aligned}$$

Note that

$$\widetilde{G_h z_h} - \tilde{z}_h = G_h z_h - z_h + c(0, e_h(y_h) - e_h(x_h))$$

holds such that we have

$$\|\widetilde{G_h z_h} - \tilde{z}_h\|_{Z_h} \le (1 + c\gamma_e) \|G_h z_h - z_h\|_{Z_h}. \qquad (2.52)$$

Now we estimate the difference $G_h(G_h z_h) - G_h z_h$ for $z_h, G_h z_h \in D_h$. By (2.46) and (2.43) we have

$$\begin{aligned}
&\|G_h(G_h z_h) - G_h z_h\|_{Z_h} \\
&\quad = \|\widetilde{G_h z_h} - [M_h^{-1}(\widetilde{G_h z_h}) F_h(\widetilde{G_h z_h})]^T - G_h z_h\|_{Z_h} \\
&\quad = \|(0, ce_h(y_h))^T - M_h^{-1}(\widetilde{G_h z_h}) F_h(\widetilde{G_h z_h})\|_{Z_h} \\
&\quad \le \|M_h^{-1}(\widetilde{G_h z_h})\|_{\mathcal{L}(Z_h^*, Z_h)} \|M_h(\widetilde{G_h z_h})(0, ce_h(y_h))^T - F_h(\widetilde{G_h z_h})\|_{Z_h^*} \\
&\quad \le \|M_h^{-1}(\widetilde{G_h z_h})\|_{\mathcal{L}(Z_h, Z_h^*)} \\
&\qquad \cdot \|[M_h(\widetilde{G_h z_h}) - M_h(\tilde{z}_h) + M_h(\tilde{z}_h)](0, ce_h(y_h))^T - F_h(\widetilde{G_h z_h})\|_{Z_h^*}.
\end{aligned}$$

Using (2.51) we obtain that

$$\begin{aligned}
&\|G_h(G_h z_h) - G_h z_h\|_{Z_h} \\
&\quad \le \|M_h^{-1}(\widetilde{G_h z_h})\|_{\mathcal{L}(Z_h, Z_h^*)} \\
&\qquad \cdot \Big(\|[M_h(\widetilde{G_h z_h}) - M_h(\tilde{z}_h)](0, ce_h(y_h))^T\|_{Z_h^*} + \frac{\gamma_M}{2} \|\widetilde{G_h z_h} - \tilde{z}_h\|^2_{Z_h} \Big) \\
&\quad \le \|M_h^{-1}(\widetilde{G_h z_h})\|_{\mathcal{L}(Z_h, Z_h^*)} \\
&\qquad \cdot \Big(c\gamma_M \|\widetilde{G_h z_h} - \tilde{z}_h\|_{Z_h} \|e_h(y_h)\|_{Y_h} + \frac{\gamma_M}{2} \|\widetilde{G_h z_h} - \tilde{z}_h\|^2_{Z_h} \Big) \\
&\quad \le \|M_h^{-1}(\widetilde{G_h z_h})\|_{\mathcal{L}(Z_h, Z_h^*)} \\
&\qquad \cdot \Big(c\gamma_M \|e_h(y_h)\|_{Y_h^*} + \frac{\gamma_M}{2} \|\widetilde{G_h z_h} - \tilde{z}_h\|_{Z_h} \Big) \|\widetilde{G_h z_h} - \tilde{z}_h\|_{Z_h} \\
&\quad = \|M_h^{-1}(\widetilde{G_h z_h})\|_{\mathcal{L}(Z_h, Z_h^*)} \|\widetilde{G_h z_h} - \tilde{z}_h\|_{Z_h} \\
&\qquad \cdot \Big(c\gamma_e \gamma_M [\|y_h - x_h^0\|_{X_h} + \|x_h^0 - r_h^X x^*\|_{X_h}] + \frac{\gamma_M}{2} \|\widetilde{G_h z_h} - \tilde{z}_h\|_{Z_h} \Big) \\
&\quad \le \|M_h^{-1}(\widetilde{G_h z_h})\|_{\mathcal{L}(Z_h, Z_h^*)} \|\widetilde{G_h z_h} - \tilde{z}_h\|_{Z_h} \\
&\qquad \cdot \Big(c\gamma_e \gamma_M [\|y_h - x_h^0\|_{X_h} + \rho(h)] + \frac{\gamma_M}{2} \|\widetilde{G_h z_h} - \tilde{z}_h\|_{Z_h} \Big)
\end{aligned}$$

such that (2.47) and (2.48) lead to

$$
\begin{aligned}
&\|G_h(G_h z_h) - G_h z_h\|_{Z_h} \\
&\quad \le \|M_h^{-1}(\widetilde{G_h z_h})\|_{\mathcal{L}(Z_h, Z_h^*)} \, \|\widetilde{G_h z_h} - \tilde{z}_h\|_{Z_h} \left(\frac{1}{2\eta^*(1+c\gamma_e)} + \frac{\gamma_M}{2} \|\widetilde{G_h z_h} - \tilde{z}_h\|_{Z_h} \right) \\
&\quad \le \frac{8\eta^* \, \|\widetilde{G_h z_h} - \tilde{z}_h\|_{Z_h}}{7 - c_1 \|G_h z_h - z_h^0\|_{Z_h}} \left(\frac{1}{2\eta^*(1+c\gamma_e)} + \frac{\gamma_M}{2} \|\widetilde{G_h z_h} - \tilde{z}_h\|_{Z_h} \right) \\
&\quad = \frac{\frac{4}{1+c\gamma_e} \|\widetilde{G_h z_h} - \tilde{z}_h\|_{Z_h} + 4\eta^* \gamma_M \|G_h z_h - z_h\|_{Z_h}^2}{7 - c_1 \|G_h z_h - z_h^0\|_{Z_h}} \\
&\quad \le \frac{4\,\|G_h z_h - z_h\|_{Z_h} + 4\eta^*\gamma_M(1+c\gamma_e)^2 \|G_h z_h - z_h\|_{Z_h}^2}{7 - c_1 \|G_h z_h - z_h^0\|_{Z_h}} \\
&\quad = \varphi(\|G_h z_h - z_h\|_{Z_h}, \|G_h z_h - z_h^0\|_{Z_h}) \,,
\end{aligned}
$$

where $\varphi : [0,\infty) \times [0, \frac{7}{c_1}) \to [0,\infty)$ is defined by

$$
\varphi(s,t) = \frac{c_1 s^2 + 8s}{2(7 - c_1 t)} \,.
$$

We infer from (2.45) that $\|G_h z_h - z_h^0\|_{Z_h} < \frac{7}{c_1}$ for all $G_h z_h \in U_h$. Clearly, φ is monotonically increasing in $[0,\infty) \times [0, \frac{7}{c_1})$. To apply Lemma 3.51 we thus consider the difference equation

$$
\text{(2.53)} \qquad t_h^{n+1} - t_h^n = \varphi(t_h^n - t_h^{n-1}, t_h^n) \,, \quad t_h^0 = 0 \,, \quad t_h^1 = \delta_h \,, \quad n = 1, 2, \ldots \,,
$$

where δ_h is given by

$$
\text{(2.54)} \qquad \delta_h = \|G z_h^0 - z_h^0\|_{Z_h} = \|(0, c\mathfrak{J}_h e_h(x_h^0))^T - M_h^{-1}(\tilde{z}_h^0)\, F_h(\tilde{z}_h^0)\|_{Z_h} \,.
$$

In case of $\delta_h = 0$ it follows from (2.54) that

$$
\text{(2.55)} \qquad c e_h'(x_h^0)^* \mathfrak{J}_h e_h(x_h^0) = L_h'(\tilde{z}_h^0) \quad \text{and} \quad 0 = e_h(x_h^0) \,.
$$

Therefore, we derive $F_h(\tilde{z}_h^0) = 0$ such that z_h^0 is already a solution to $(\mathbf{OS_h})$ and the claim of the theorem is proved. Henceforth, let $\delta_h > 0$. By Lemma 3.53 there exists a solution $t^{n+1} = \psi(t^n)$, $t^0 = 0$ for $n = 0, 1, \ldots$, where

$$
\psi(t) = t + \frac{c_1 t^2 - 6t + \|G z^0 - z^0\|_H}{2(7 - c_1 t)} \,.
$$

Lemma 3.54 implies that the sequence $\{t_h^n\}_{n\in\mathbb{N}}$ of (2.53) is strictly increasing, and

$$
\lim_{n\to\infty} t_h^n = t_h^* = \frac{1}{c_1} \left(3 - \sqrt{9 - 14 c_1 \delta_h} \right) \,,
$$

if

$$
\text{(2.56)} \qquad \delta_h \le \frac{9}{14 c_1} \overset{(2.49)}{=} \frac{9}{112 \eta^* \gamma_M (1 + c\gamma_e)^2} \,.
$$

We shall prove that (2.56) can be achieved if h is sufficiently small. Since

$$\|M_h(\tilde{z}_h^0) - M_h(z_h^0)\|_{L(Z_h,Z_h^*)} \le \gamma_M \|\tilde{z}_h^0 - z_h^0\|_{Z_h} \le c\gamma_e\gamma_M\rho(h) \overset{(2.47)}{\le} \frac{1}{8\eta^*}$$

for every $h \in (0, h_1]$ we infer from Lemma 3.55 that

$$\|M_h^{-1}(\tilde{z}_h^0)\|_{L(Z_h,Z_h^*)} \le \frac{7\eta^*}{8}\,.$$

Using Condition 9 we estimate δ_h as follows:

$$\begin{aligned}
\delta_h &\le c\,\|e_h(x_h^0)\|_{Y_h^*} + \|M_h^{-1}(\tilde{z}_h^0)\|_{L(Z_h,Z_h^*)}\,\|F_h(\tilde{z}_h^0)\|_{Z_h^*} \\
&\le c\gamma_e\rho(h) + \frac{7\eta^*}{8}\,\|F_h(\tilde{z}_h^0) - F_h(r_h z^*)\|_{Z_h^*} \\
&\quad + \frac{7\eta^*}{8}\,\|F_h(r_h z^*) - s_h F(z^*)\|_{Z_h^*} \\
&\le c\gamma_e\rho(h) + \frac{7\eta^*}{8}\Big(c\gamma_F\,\|e_h(x_h^0)\|_{Y_h^*} + \gamma_F\rho(h)\Big) \\
&\le \underbrace{\left(c\gamma_e + \frac{7\eta^*\gamma_F}{8}(c\gamma_e+1)\right)}_{=c_2}\rho(h)\,,
\end{aligned} \tag{2.57}$$

where we use the identities $e(x^*) = 0$ and $F(z^*) = 0$. Let $h_2 \in (0, h_1]$ such that

$$\rho(h) \le \frac{9}{14c_1c_2} \qquad \text{for all } h \in (0, h_2]\,.$$

Then δ_h satisfies (2.56) for $h \in (0, h_2]$. We conclude that

$$t_h^* = \frac{14\delta_h}{3+\sqrt{9-14c_1\delta_h}} < \frac{14\delta_h}{3} \overset{(2.57)}{\le} \frac{14c_2}{3}\,\rho(h)\,. \tag{2.58}$$

If we choose $\bar{h} \in (0, h_2]$ such that

$$\rho(h) \le \frac{3r_{\min}}{14c_2} \qquad \text{for every } h \in (0, \bar{h}]$$

holds, we infer from Condition 10 that

$$\overline{B(z_h^0; t_h^*)} = \left\{z_h \in Z_h : \|z_h - z_h^0\|_{Z_h} \le t_h^*\right\} \subset U_h$$

for $h \in (0, \bar{h}]$. Due to Lemma 3.51 the sequence $\{z_h^n\}_{n\in\mathbb{N}}$ converges to some $z_h^* = (y_h^*, u_h^*, \lambda_h^*) \in \overline{B(z_h^0; t_h^*)}$. Since G_h is continuous in U_h, we have $z_h^* = G_h(z_h^*)$ and therefore $F_h(z_h^*) = 0$. Hence, z_h^* is a solution of the optimality system ($\mathbf{OS_h}$). Estimate (2.32) follows from

$$\|z_h^* - r_h z^*\|_{Z_h} \le t_h^* = O\left(\rho(h)\right)\,.$$

□

3.11. Proof of Theorem 3.48. In this section we shall use the same notations as introduced in Section 3.10. Moreover, the constants c_1 and c_2 are given by (2.49) and (2.57), respectively.

PROOF OF THEOREM 3.48. From Theorem 3.46 we conclude the existence of z_h^* solving $(\mathbf{OS_h})$ and of $\bar{h} > 0$ such that

$$\|z_h^* - r_h z^*\|_{Z_h} \overset{(2.58)}{\leq} \frac{14c_2}{3}\,\rho(h) \qquad \text{for every } h \in (0, \bar{h}]\,. \tag{2.59}$$

Applying Lemma 3.55 and (2.45) we obtain that

$$\|M_h^{-1}(z_h)\|_{L(Z_h^*, Z_h)} \leq \frac{\eta^*}{1 - \eta^* \gamma_M \|z_h - r_h z^*\|_{Z_h}} < \frac{8\eta^*(1 + c\gamma_e)}{5 + 8c\gamma_e} = c_3$$

for all $z_h \in U_h$. Let $z^0 \in U$ denote a start-up value of Algorithm 1 satisfying

$$\|z^0 - z^*\|_Z < \min\left(r^*, \frac{3r_{\min}}{8c_\circ\sqrt{\max(2, 1 + c^2\gamma_e^2)}}, \frac{1}{\gamma_M c_\circ c_3 \max(2, 1 + c^2\gamma_e^2)}\right), \tag{2.60}$$

where the constant $c_\circ$ is specified in Remark 3.27. Since ρ is right-continuous at $h = 0$ there exists $\tilde{h} \in (0, \bar{h}]$ such that

$$\rho(h) \leq \min\left(\frac{3r_{\min}}{56c_2}, \frac{9r_{\min}}{112c_2\sqrt{\max(2, 1 + c^2\gamma_e^2)}}, \frac{3}{14\gamma_M c_2 c_3 \max(2, 1 + c^2\gamma_e^2)}\right) \tag{2.61}$$

for all $h \in (0, \tilde{h}]$. Applying (2.60) yields

$$\|z^0 - z^*\|_Z < \frac{3r_{\min}}{8c_\circ\sqrt{\max(2, 1 + c^2\gamma_e^2)}} < \frac{r_{\min}}{2c_\circ}\,. \tag{2.62}$$

From (2.59), (2.61) and (2.62) we infer that

$$\|z_h^0 - z_h^*\|_{Z_h} + \|z_h^* - r_h z^*\|_{Z_h} \leq c_\circ \|z^0 - z^*\|_Z + 2\,\|z_h^* - r_h z^*\|_{Z_h} \leq r_{\min}\,.$$

By Condition 10 it follows that $B(z_h^*; \|z_h^0 - z_h^*\|_{Z_h}) \subseteq U_h$. To apply Theorem 2.13 we set

$$\bar{r}^* = \min\left(\frac{3r_{\min}}{4\sqrt{\max(2, 1 + c^2\gamma_e^2)}}, \frac{2}{\gamma_M c_3 \max(2, 1 + c^2\gamma_e^2)}\right)$$

(compare with (2.9)). From (2.59) and (2.61) we conclude that

$$\|z_h^* - r_h z^*\|_{Z_h} \leq \frac{r_{\min}}{4}$$

for every $h \in (0, \bar{h}]$. Moreover, it follows that $\bar{r}^* < \frac{3}{4}\,r_{\min}$. Consequently, $B(z_h^*; \bar{r}^*) \subset U_h$. Then Theorem 2.13 yields $\lim_{n\to\infty} \|z_h^n - z_h^*\|_{Z_h} = 0$ and

$$\|z_h^{n+1} - z_h^*\|_{Z_h} \leq \frac{1}{2}\,\gamma_M c_3 \max(2, 1 + c^2\gamma_e^2)\,\|z_h^n - z_h^*\|_{Z_h}^2$$

for the sequence $\{z_h^n\}_{n\in\mathbb{N}}$ generated by Algorithm 2 with initial value $r_h z^0$ if $\|r_h z^0 - z_h^*\|_{Z_h} < \bar{r}^*$, i.e.

$$\|r_h z^0 - z_h^*\|_{Z_h} < \frac{3r_{\min}}{4\sqrt{\max(2, 1+c^2\gamma_e^2)}} \tag{2.63}$$

and

$$\|r_h z^0 - z_h^*\|_{Z_h} < \frac{2}{\gamma_M c_3 \max(2, 1+c^2\gamma_e^2)}\,. \tag{2.64}$$

Using Remark 3.27, (2.60), (2.59) and (2.61) we derive that

$$\begin{aligned} \|r_h z^0 - z_h^*\|_{Z_h} &\le \|r_h[z^0 - z^*]\|_{Z_h} + \|r_h z^* - z_h^*\|_{Z_h} \\ &\le c_\circ \|z^0 - z^*\|_Z + \frac{14}{3} c_2 \rho(h) \\ &< \frac{3r_{\min}}{4\sqrt{\max(2, 1+c^2\gamma_e^2)}}\,. \end{aligned} \tag{2.65}$$

Analogously, estimate (2.64) follows. □

REMARK 3.56. The constant r_1 of Theorem 3.48 is given by

$$r_1 = \min\left(r^*, \frac{3r_{\min}}{8c_\circ\sqrt{\max(2, 1+c^2\gamma_e^2)}}, \frac{1}{\gamma_M c_\circ c_3 \max(2, 1+c^2\gamma_e^2)}\right)\,.$$

3.12. Proof of Theorem 3.49. By induction we shall prove that the estimate

$$\|z_h^n - r_h z^n\|_{Z_h} \le \tau_h \qquad \text{for } n = 0, 1, \ldots \tag{2.66}$$

holds, where the quantity $\tau_h > 0$ is of order $O(\rho(h))$. Thus (2.66) implies (2.33). Using (2.66) we shall deduce both (2.34) and (2.35).

PROOF OF THEOREM 3.49. We shall use the same notations introduced in Section 3.10 and Section 3.11. To define the quantity τ_h in (2.67) we set

$$\begin{aligned} c_4 &= c_3(1+c\gamma_e)\gamma_M c_\circ\left(1+\sqrt{\max(2, 1+c^2\gamma_e^2)}\right)\,, \\ c_5 &= c_3\gamma_M(1+c\gamma_e)^2\,, \\ c_6 &= c_3\left(c_\circ(r^* + \|z^*\|_Z) + \gamma_F + c^2 c_{\mathfrak{Z}}^2\gamma_M\right)\,, \\ c_7 &= c_3\gamma_M(1+cc_\circ c_{\mathfrak{Z}})\left(1+\sqrt{\max(2, 1+c^2\gamma_e^2)}\right)\,, \end{aligned}$$

where r^* is defined in (2.9) and $c_{\mathfrak{Z}}$ is given by Condition 12 on page 45. We put

$$\tau_h = \frac{2a\rho(h)}{b + \sqrt{b^2 - 4ac_5\,\rho(h)}}\,, \tag{2.67}$$

where

$$a = c_6 + c_7\|z^0 - z^*\|_Z\,, \qquad b = 1 - c_4\|z^0 - z^*\|_Z\,.$$

Let $z^0 \in U$ denote a start-up value of Algorithm 1 satisfying

$$\|z^0 - z^*\|_Z < \min\left(r_1, \frac{1}{2c_4}\right) = r_2\,, \tag{2.68}$$

where the constant r_1 is introduced in Remark 3.56. From (2.68) we infer that $b > 0$ holds. Since ρ is right-continuous at $h = 0$ there exists $\hat{h} \in (0, \tilde{h}]$ such that

$$\rho(h) \le \left(1, \frac{1}{16c_5(c_6 + c_7 r_2)}\right) \qquad \text{for all } h \in (0, \hat{h}] \tag{2.69}$$

As a consequence of (2.68) and (2.69) we have

$$\left(1 - c_4\,\|z^0 - z^*\|_Z\right)^2 > (1 - c_4\,r_2)^2 \ge \frac{1}{4} \ge 4c_5\,(c_6 + c_7 r_2)\,\rho(h)\,,$$

which implies $b^2 \ge 4c_5 a\rho(h)$. Hence for all $h \in (0, \hat{h}]$ and $z^0 \in B(z^*; r_2)$ the quantity τ_h is well-defined and positive for all h.

By induction we prove that for all $n = 0, 1, \dots$

$$\|z_h^n - r_h z^n\|_{Z_h} \le \tau_h \tag{2.70}$$

holds. The induction hypothesis $n = 0$ follows from $z_h^0 = r_h z^0$. Suppose, that (2.70) holds for iteration level $n > 0$. From Algorithm 1 and Algorithm 2 we obtain the identity

$$[z_h^{n+1} - r_h z^{n+1}]^T = M_h^{-1}(\tilde{z}_h^n) \sum_{i=1}^{4} I_h^i\,, \tag{2.71}$$

where

$$\begin{aligned}
I_h^1 &= M_h(\tilde{z}_h^n)[\tilde{z}_h^n - r_h \tilde{z}^n]^T - F_h(\tilde{z}_h^n) + F_h(r_h \tilde{z}^n)\,,\\
I_h^2 &= [M_h(\tilde{z}_h^n) - M_h(r_h \tilde{z}^n)]\, r_h[M^{-1}(\tilde{z}^n)F(\tilde{z}^n)]\,,\\
I_h^3 &= M_h(r_h \tilde{z}^n)\, r_h[M^{-1}(\tilde{z}^n)F(\tilde{z}^n)] - s_h F(\tilde{z}^n)\,,\\
I_h^4 &= s_h F(\tilde{z}^n) - F_h(r_h \tilde{z}^n)\,.
\end{aligned}$$

To estimate (2.71) we first proceed as in the proof of Theorem 2.13 in [**IK96a**] to conclude that $\|\tilde{z}_h^n - z_h^*\|_{Z_h} \le r_{\min}$ and thus $\tilde{z}_h^n \in U_h$. Thus, $\|M_h^{-1}(\tilde{z}_h^n)\|_{L(Z_h^*, Z_h)} \le c_3$. Applying Lemma 3.36 we find that

$$\|I_h^1\|_{Z_h^*} \le \frac{\gamma_M}{2}\,\|\tilde{z}_h^n - r_h \tilde{z}^n\|_{Z_h}^2\,.$$

Under the hypotheses of Theorem 2.13 we have

$$\begin{aligned}
\|z^{n+1} - z^*\|_Z &\le \|z^n - z^*\|_Z\,,\\
\|\tilde{z}^n - z^*\|_Z &\le \sqrt{\max(2, 1 + c^2\gamma_e^2)}\,\|z^n - z^*\|_Z\,.
\end{aligned} \tag{2.72}$$

Moreover, from c) of Algorithm 1 it follows that

$$M^{-1}(\tilde{z}^n)F(\tilde{z}^n) = [z^{n+1} - \tilde{z}^n]^T\,. \tag{2.73}$$

Using (2.72) and (2.73) we derive that

$$\begin{aligned}\|I_h^2\|_{Z_h^*} &\leq \gamma_M \|\tilde{z}_h^n - r_h\tilde{z}^n\|_Z \, c_\circ \|z^{n+1} - \tilde{z}^n\|_Z \\ &\leq c_\circ\gamma_M \underbrace{\left(1 + \sqrt{\max(2, 1 + c^2\gamma_e^2)}\right)}_{=:c_8} \|z^0 - z^*\|_Z \|\tilde{z}_h^n - r_h\tilde{z}^n\|_Z \,.\end{aligned}$$

As in the proof of Lemma 3.43 we obtain from (2.73) that

$$\begin{aligned}\|I_h^3\|_{Z_h^*} &= \|s_h M(\tilde{z}^n)[z^{n+1} - \tilde{z}^n]^T - M_h(r_h\tilde{z}^n)r_h[z^{n+1} - \tilde{z}^n]^T\|_{Z_h^*} \\ &\leq \left(\gamma_M \|z^{n+1} - \tilde{z}^n\|_Z + c_\circ\|\tilde{z}^n\|_Z\right) \rho(h)\,.\end{aligned}$$

From (2.72) we infer that $\|z^{n+1} - \tilde{z}^n\|_Z \leq c_8 \|z^0 - z^*\|_Z$. Moreover, the proof of Theorem 2.13 in [**IK96a**] yields

$$\|\tilde{z}^n\|_Z \leq \|\tilde{z}^n - z^*\|_Z + \|z^*\|_Z \leq r^* + \|z^*\|_Z \,.$$

Thus, we have

$$\|I_h^3\|_{Z_h^*} \leq \left(\gamma_M c_8 \|z^0 - z^*\|_Z + c_\circ(r^* + \|z^*\|_Z)\right) \rho(h)\,.$$

At last, we derive from Condition 9 that $\|I_h^4\|_{Z_h^*} \leq \gamma_F \rho(h)$. With these majorizations in (2.71) we obtain that

$$\begin{aligned}\|z_h^{n+1} - r_h z^{n+1}\|_{Z_h} \leq\ & \frac{c_3\gamma_M}{2} \|\tilde{z}_h^n - r_h\tilde{z}^n\|_{Z_h}^2 \\ &+ c_\circ c_3 c_8 \gamma_M \|z^0 - z^*\|_Z \|\tilde{z}_h^n - r_h\tilde{z}^n\|_Z \\ &+ c_3 c_8 \gamma_M \|z^0 - z^*\|_Z \rho(h) \\ &+ c_\circ c_3 (r^* + \|z^*\|_Z)\rho(h) + c_3\gamma_F\rho(h)\,.\end{aligned} \tag{2.74}$$

Applying (2.11) and Condition 12 we obtain

$$\begin{aligned}\|\tilde{z}_h^n - r_h\tilde{z}^n\|_{Z_h} &\leq \|z_h^n - r_h z^n\|_{Z_h} + c\,\|\mathfrak{J}_h e_h(x_h^n) - r_h^Y \mathfrak{J} e(x^n)\|_{Y_h} \\ &\leq \|z_h^n - r_h z^n\|_{Z_h} + c\,\|\mathfrak{J}_h e_h(x_h^n) - \mathfrak{J}_h e_h(r_h^X x^n)\|_{Y_h} \\ &\quad + c\,\|\mathfrak{J}_h e_h(r_h^X x^n) - r_h^Y \mathfrak{J} e(x^n)\|_{Y_h} \\ &\leq (1 + c\gamma_e)\,\|z_h^n - r_h z^n\|_{Z_h} + c c_{\mathfrak{J}}\rho(h) \\ &\leq (1 + c\gamma_e)\tau_h + c c_{\mathfrak{J}}\rho(h)\end{aligned} \tag{2.75}$$

and thus

$$\|\tilde{z}_h^n - r_h z^n\|_{Z_h}^2 \overset{(2.69)}{\leq} 2(1 + c\gamma_e)^2\,\tau_h^2 + 2c^2 c_{\mathfrak{J}}^2\rho(h)\,.$$

From (2.74) we infer

$$\begin{aligned}\|z_h^{n+1} - r_h z^{n+1}\|_{Z_h} \leq\ & c_5\,\tau_h^2 + c_4\,\|z^0 - z^*\|_Z\,\tau_h + c_6\rho(h) \\ &+ c_7\,\|z^0 - z^*\|_Z\,\rho(h)\,.\end{aligned} \tag{2.76}$$

Let us consider the quadratic equation

$$c_5\,\tau_h^2 + c_4\,\|z^0 - z^*\|_Z\,\tau_h + \left(c_6 + c_7\,\|z^0 - z^*\|_Z\right)\rho(h) = \tau_h\,, \tag{2.77}$$

which leads to

$$\tau_h^2 - \frac{1 - c_4 \|z^0 - z^*\|_Z}{c_5} \tau_h + \frac{c_6 + c_7 \|z^0 - z^*\|_Z}{c_5} \rho(h) = 0 \,.$$

Since τ_h given by (2.67) is the smallest solution of the quadratic equation (2.77), it is readily verified that the right-hand side of (2.76) equals τ_h. This completes the induction and hence the proof of (2.70). Moreover, from $b > \frac{1}{2}$ and $\|z^0 - z^*\|_Z < r_2$ we deduce the estimate

$$\|z_h^n - r_h z^n\|_{Z_h} \leq \tau_h < \frac{2a}{b} \rho(h) \leq 4\,(c_6 + c_7 r_2)\,\rho(h) \qquad n = 0, 1, \dots\,, \tag{2.78}$$

which gives (2.33).

Finally, we prove (2.34) and (2.35). Applying Condition 9 we infer (2.34) from

$$\begin{aligned} \|F_h(\tilde{z}_h^n) - s_h F(\tilde{z}^n)\|_{Z_h^*} &\leq \gamma_F \,\|p_h \tilde{z}_h^n - \tilde{z}^n\|_Z \\ &\leq \gamma_F (\|\tilde{z}_h^n - r_h \tilde{z}^n\|_{Z_h} + \|p_h r_h \tilde{z}^n - \tilde{z}^n\|_Z) \\ &\overset{(2.75)}{\leq} \gamma_F ((1 + c\gamma_e)\tau_h + c c_{\mathfrak{J}} \rho(h) + \rho(h)) \\ &\overset{(2.78)}{\leq} \gamma_F (4(1 + c\gamma_e)(c_6 + c_7 r_2) + c c_{\mathfrak{J}} + 1)\rho(h). \end{aligned} \tag{2.79}$$

Further, it follows from (2.78) and (2.59) that

$$\begin{aligned} \|(z_h^n - z_h^*) - r_h(z^n - z^*)\|_{Z_h} &\leq \|z_h^n - r_h z^n\|_{Z_h} + \|z_h^* - r_h z^*\|_{Z_h} \\ &\leq \left(4(c_6 + c_7 r_2) + \frac{14}{3}\, c_4\right) \rho(h) \end{aligned} \tag{2.80}$$

such that (2.35) is satisfied. □

CHAPTER 3

Mesh-Independence

1. Introduction

If one solves an infinite dimensional optimization problem by introducing discretizations and using a solution method to the resulting finite dimensional problem, one often observes the very stable behavior of the method with respect to varying discretizations. The most striking observation can be the constancy of the number of iterates needed to satisfy a given stopping criterion. This phenomenon is called the *mesh-independence principle*. Thus, the mesh-independence principle asserts that, when a solution method is applied to a nonlinear equation as well as to some finite dimensional discretization of that equation, then the behavior of the discretized process is asymptotically the same as that for the original iteration and, consequently, the number of steps required by the two processes to converge within a given tolerance is essentially the same.

Mesh-independence is important for two reasons. First it allows us to predict the convergence of the method applied to the discretized problem when the method has been analyzed for the infinite dimensional problem. Secondly, it can be used to improve the performance of the method: Since we are interested in the solution of an infinite dimensional problem, it is usually necessary to choose reasonably fine discretizations. This leads to a large number of variables in the discrete minimization problem and therefore to a large amount of work per iteration. If the method is fixed, the only possibility for reducing the total about of work consists in the improvement of the start-up value. For these problems it is obvious that we must use information from the coarse discretizations to obtain good starting points for the finer grids. This leads to mesh refinement strategies. Mesh-independence is a theoretical justification for mesh-refinement strategies and, moreover, it can be used to design the refinement process and to predict the overall performance of the method.

In this chapter we shall prove mesh-independence results for the augmented Lagrangian-SQP method applied to the first-order necessary optimality condition (**OS**) and to its approximation, which is posed in Chapter 2, Section 3. The following section is devoted to the mesh-independence in its sharpest form, where estimates are given which are uniform with respect to the iteration count. The presented result is a consequence of the convergence theorems of Chapter 2, Section 3.9, where special convergence properties are required (see Condition 9 on page 43 and Condition 12 on page 45). The assertion of Theorem 2.1 will be that for all $\varepsilon > 0$ there exists $h^* \in (0, 1]$, for which an explicit representation can be given, such that

$$\begin{aligned} |\min\{n_0|\text{ for } n \geq n_0 : \|(y^n, u^n, \lambda^n) - (y^*, u^*, \lambda^*)\|_Y < \varepsilon\} \\ - \min\{n_0|\text{ for } n \geq n_0 : \|(y_h^n, u_h^n, \lambda_h^n) - (y_h^*, u_h^*, \lambda_h^*)\|_{Z_h} < \varepsilon\}| \leq 1 \end{aligned}$$

for all $h \in (0, h^*]$ and for all start-up values (y^0, u^0, λ^0) sufficiently close to (y^*, u^*, λ^*). Here, (y^n, u^n, λ^n) and $(y_h^n, u_h^n, \lambda_h^n)$ denote the iterates generated by Algorithm 1 on page 25 and Algorithm 2 on page 38, respectively. In Section 3 an asymptotic mesh-independence result will be proved, which means that Algorithm 2 behaves essentially the same for all sufficiently fine discretizations. This is formulated in Theorem 3.4, which says that for all $\varepsilon > 0$ there exists $h_\varepsilon \in (0, 1]$ such that

$$\begin{aligned} |\min\{m_0|\text{ for } m \geq m_0 : \|F(y^m, u^m, \tilde{\lambda}^m)\|_{Z^*} < \varepsilon\} \\ - \min\{m_0|\text{ for } m \geq m_0 : \|F_h(y_h^m, u_h^m, \tilde{\lambda}_h^m)\|_{Z_h^*} < \varepsilon\}| \leq 1 \end{aligned}$$

for all $h_\varepsilon \in (0, h^\star]$ and for all start-up values (y^0, u^0, λ^0) sufficiently close to (y^*, u^*, λ^*), where F and F_h are introduced in (2.29). Note that Theorem 3.4 only guarantees the existence of h_ε, but and gives no explicit upper bound. The proof is based on the convergence theorem in Chapter 3, Section 3.7.

For some classes of nonlinear boundary value problems the mesh-independence principle for Newton's method was proved in [**AM78**] and [**AMP79**], and in [**AM78**], [**ABM81**] and [**McC78**] it was used for the construction of some mesh-refinement strategies. In [**ABPR86**] a proof of the mesh-independence principle for Newton's method was presented in its sharpest formulation for a general class of discretizations of nonlinear operator equations under fairly general and natural conditions on the operator and the discretizations. The paper combined the results of [**AB87**] with the procedure in [**PR84**]. Argyros extended the validity of the mesh-independence principle for nonlinear equations and their discretizations to include operators whose derivatives are only Hölder-continuous [**Arg90**]. In [**Arg92**] Argyros proved a mesh-independence principle for operator equations and the secant method. A mesh-independence result of Newton's method for generalized equations is achieved by Alt in [**Alt95**]. Kelley and Sachs prove a mesh-independence result for the BFGS (Broyden-Fletcher-Goldfarb-Shanno) method in Hilbert spaces and apply it to a class of unconstrained optimal control problems in [**KS87**]. In [**KS91**] Kelley and Sachs consider globally convergent modifications of Newton's method, such as the Armijo rule. It is shown that for proper discretization schemes the convergence behavior of the iteration is the same for the discrete problems as it is for the infinite dimensional problem. Optimal control problems and their discretizations are considered by Kelley and Sachs in [**KS92**]. To solve such discretized problems by the gradient projection method the finite identification of active constraints will be necessary.

These authors prove a mesh-independence result using a proper condition on the convergence of the gradients. In [**Hei93**] Heinckenschloss extended the results of [**ABPR86**] to norm constrained Gauss-Newton method using a somewhat different discretization scheme based on Galerkin approximations. Deuflhard and Potra presented a theoretical characterization of the asymptotic mesh-independence of Newton's method [**DP92**]. The theory does not need any uniform Lipschitz assumptions. The results are obtained by means of a refined Newton-Mysovskii theorem in an affine invariant formulation.

A theoretical refinement discussion was given in [**Rhe80**]. Further, refinement strategies were presented in [**Axe93**] for Newton-type methods, in [**KS90**] for quasi-Newton methods and in [**HLS91**] for the Gauss-Newton method.

2. Strong Mesh-Independence

In this section we present the mesh-independence principle in its sharpest formulation, which allows us to give exact conditions for the mesh-size h such that the result hold. The proof is based on the convergence theorems of Chapter 2, Section 3.9, where special convergence properties are required (see Condition 9 and Condition 12).

Let $\varepsilon > 0$. We define

$$\begin{aligned} n(\varepsilon) &= \min\{n_0 \mid \text{for } n \geq n_0 : \|(y^n, u^n, \lambda^n) - (y^*, u^*, \lambda^*)\|_Z < \varepsilon\}, \\ n_h(\varepsilon) &= \min\{n_0 \mid \text{for } n \geq n_0 : \|(y_h^n, u_h^n, \lambda_h^n) - (y_h^*, u_h^*, \lambda_h^*)\|_{Z_h} < \varepsilon\}. \end{aligned}$$

Let us point out that $n(\varepsilon)$ and $n_h(\varepsilon)$ depend on the start-up values (y^0, u^0, λ^0) and $(y_h^0, u_h^0, \lambda_h^0) = r_h(y^0, u^0, \lambda^0)$ of the infinite and finite dimensional method.

THEOREM 2.1. *Suppose that the hypotheses of Theorems 3.46, 3.48 and 3.49 in Chapter 2 hold. Let $\varepsilon > 0$ be chosen. Then there exist $h^* \in (0, \hat{h}]$ and $r_3 \in (0, r_2]$, which are explicitly given by (3.4), so that*

$$|n(\varepsilon) - n_h(\varepsilon)| \leq 1 \tag{3.1}$$

for all $h \in (0, h^*]$ and $(y^0, u^0, \lambda^0) \in (V \times W) \cap B((y^*, u^*, \lambda^*); r_3)$.

PROOF. We shall use the notations and constants, which are introduced in Chapter 2, Sections 3.10-3.12.

Let $\varepsilon > 0$ and let $j = j(\varepsilon)$ be the unique integer defined by

$$\|z^{j+1} - z^*\|_Z < \varepsilon \leq \|z^j - z^*\|_Z. \tag{3.2}$$

Since we derive from Condition 5 on page 35 that

$$\lim_{h \to 0} \|r_h z^n\|_{Z_h} = \|z^n\|_Z, \qquad \lim_{h \to 0} \|r_h z^*\|_{Z_h} = \|z^*\|_Z$$

for each $n \in \mathbb{N}$, there exist $h_1 \in (0, \hat{h}]$ such that

$$\|r_h[z^j - z^*]\|_{Z_h} \geq \frac{1}{2} \|z^j - z^*\|_Z \tag{3.3}$$

for all $h \in (0, h_1]$. We prove that the claim of the theorem holds for

$$r_3 = \min\left(r_2, \frac{\beta}{4c_\circ C}\right), \tag{3.4a}$$

$$\beta = \min\left(\frac{1}{c_\circ}, \frac{1}{2}, c_\circ\right) \tag{3.4b}$$

and for some $h^* \in (0, h_1]$ such that

$$\rho(h) \leq \min\left(\tfrac{3\beta\varepsilon}{2(12(c_6+c_7r_2)+14)}, \tfrac{3\beta}{56c_2C}\right) \qquad \text{for all } h \in (0, h^*]\,. \tag{3.4c}$$

From (3.2) and (3.4c) we derive

$$\begin{aligned} \|z_h^{j+1} - z_h^*\|_{Z_h} &\overset{(2.80)}{\leq} \|r_h[z^{j+1} - z^*]\|_{Z_h} \\ &\qquad + \left(4(c_6 + c_7r_2) + \frac{14}{3}\,c_4\right)\rho(h) \\ &\leq c_\circ\varepsilon + \frac{\beta\varepsilon}{2} \overset{(3.4b)}{<} 2c_\circ\varepsilon \end{aligned} \tag{3.5}$$

and, from Theorem 3.48 on page 46 and (3.4), that

$$\begin{aligned} \|z_h^{j+2} - z_h^*\|_{Z_h} &\leq C\,\|z_h^{j+1} - z_h^*\|_{Z_h}^2 \\ &\leq C\,\|z_h^0 - z_h^*\|_{Z_h}\,\|z_h^{j+1} - z_h^*\|_{Z_h} \\ &\overset{(2.65)}{\leq} \left(c_\circ C r_3 + \frac{14}{3}\,c_2 C\rho(h)\right)\|z_h^{j+1} - z_h^*\|_{Z_h} \\ &\overset{(3.5)}{<} \left(c_\circ C r_3 + \frac{14}{3}\,c_2 C\rho(h)\right)2c_\circ\varepsilon \\ &\leq c_\circ\beta\varepsilon \overset{(3.4b)}{<} \varepsilon\,. \end{aligned} \tag{3.6}$$

From estimating

$$\begin{aligned} \varepsilon &\overset{(3.2)}{\leq} \|z^j - z^*\|_Z \overset{(3.3)}{\leq} 2\,\|r_h[z^j - z^*]\|_{Z_h} \\ &\overset{(2.80)}{\leq} 2\left(\|z_h^j - z_h^*\|_{Z_h} + \left(4(c_6 + c_7r_2) + \frac{14}{3}\,c_2\right)\rho(h)\right) \end{aligned}$$

we conclude that

$$\|z_h^j - z_h^*\|_{Z_h} \geq \frac{\varepsilon}{2} - \left(4(c_6 + c_7r_2) + \frac{14}{3}\,c_2\right)\rho(h) \overset{(3.4c)}{\geq} \frac{\varepsilon}{2} - \frac{\beta\varepsilon}{2} \overset{(3.4b)}{=} \frac{\varepsilon}{4}\,. \tag{3.7}$$

If $\|z_h^{j-1} - z_h^*\|_{Z_h} < \varepsilon$ then as in (3.6) we obtain the inequality

$$\|z_h^j - z_h^*\|_{Z_h} \leq C\left(c_\circ r_3 + \frac{14}{3}\,c_2\,\rho(h)\right)\|z_h^{j-1} - z_h^*\|_{Z_h} \overset{(3.4c)}{<} \frac{\beta\varepsilon}{2} \overset{(3.4b)}{\leq} \frac{\varepsilon}{4}$$

which contradicts (3.7). Therefore we must have

$$\|z_h^{j-1} - z_h^*\|_{Z_h} \geq \varepsilon\,. \tag{3.8}$$

Since the sequence $\{\|z_h^n - z_h^*\|_{Z_h}\}_{n\in\mathbb{N}}$ decreases strictly, we infer from (3.8) that

$$\|z_h^i - z_h^*\|_{Z_h} \geq \varepsilon \qquad \text{for } i = 0, \dots, j-1\,. \tag{3.9}$$

Hence, (3.2), (3.6) and (3.9) imply (3.1). □

REMARK 2.2. As we can see in (3.3) and (3.4c) the upper bound h^* for the mesh-size depends on ε. Further, h^* is influenced by the approximation scheme ($c_\circ$), by the Lipschitz-constants of e, F and M (γ_e, γ_F and γ_M), by the uniform upper bound of the operator M_h and its inverse in U_h (c_3 and η^*) and by r^* and the norm of z^*.

3. Asymptotic Mesh-Independence

The mesh-independence principle presented in this section describes the fact that the infinite and the finite dimensional method have asymptotically the same behavior. Hence, Algorithm 2 behaves essentially the same as for sufficiently small mesh-sizes. This fact will be observed numerically in the applications presented in Chapters 4 and 5. To prove the result we only need the assumptions of Theorem 3.38.

Lemma 3.1 gives a sufficient condition that $\{\|F(y^n, u^n, \tilde{\lambda}^n)\|_{Z^*}\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence. Note that we have already use — for example see (3.2) — that the sequence $\{\|(y^n, u^n, \tilde{\lambda}^n) - (y^*, u^*, \lambda^*)\|_Z\}_{n\in\mathbb{N}}$ is also strictly decreasing, which follows directly from

$$\|(y^0, u^0, \lambda^0) - (y^*, u^*, \lambda^*)\|_Z < r^*$$

and Theorem 2.13 on page 26.

LEMMA 3.1. *Let Conditions 1-3 hold and let (y^0, u^0, λ^0) be sufficiently close to (y^*, u^*, λ^*). Then the sequence $\{\|F(y^m, u^m, \tilde{\lambda}^m)\|_{Z^*}\}_{m\in\mathbb{N}}$ is strictly decreasing.*

PROOF. Throughout the proof we will write $z^n = (y^n, u^n, \lambda^n)$, $\tilde{z}^n = (y^n, u^n, \tilde{\lambda}^n)$ for all n and $z^* = (y^*, u^*, \lambda^*)$.

Let $m \in \mathbb{N}$ be arbitrary. The equation $F(z^*) = 0$ yields

$$\begin{aligned} \|F(\tilde{z}^m)\|_{Z^*} &= \|F(\tilde{z}^m) - F(z^*)\|_{Z^*} \\ &\leq \gamma_F\,\|\tilde{z}^m - z^*\|_Z \\ &\leq \underbrace{\gamma_F\sqrt{\max(2, 1 + c^2\gamma_e^2)}}_{=c_9}\,\|z^m - z^*\|_Z\,. \end{aligned}$$

By the Taylor expansion we have

$$F(\tilde{z}^m) = M(z^*)[\tilde{z}^m - z^*]^T + \Phi(\tilde{z}^m)\,, \tag{3.10}$$

where the mapping $\Phi : U \to Z^*$ satisfies

$$\lim_{\|z-z^*\|_Z \to 0} \frac{\|\Phi(z)\|_{Z^*}}{\|z-z^*\|_Z} = 0$$

Then there exists a constant $\delta > 0$ such that

$$\frac{\|\Phi(z)\|_{Z^*}}{\|z-z^*\|_Z} \leq \frac{1}{2} \qquad \text{for all } z \text{ with } \|z-z^*\|_Z \leq \delta.$$

Let the start-up value satisfies the estimate $\|z^0 - z^*\|_Z \leq \delta$. Since the sequence $\{\|\tilde{z}^n - z^*\|_Z\}_{n\in\mathbb{N}}$ is strictly decreasing, we have $\|\tilde{z}^m - z^*\|_Z \leq \delta$ and

$$\|\Phi(\tilde{z}^m)\|_{Z^*} \leq \frac{1}{2} \|\tilde{z}^m - z^*\|_Z .$$

Hence,

$$\begin{aligned} \|F(\tilde{z}^m)\|_{Z^*} &\geq \frac{1}{\eta} \|M^{-1}(z^*)\|_{L(Z^*,Z)} \|F(\tilde{z}^m)\|_{Z^*} \\ &\geq \frac{1}{\eta} \|M^{-1}(z^*)F(\tilde{z}^m)\|_Z \\ &\overset{(3.10)}{\geq} \frac{1}{\eta} \|\tilde{z}^m - z^*\|_Z - \frac{1}{2\eta} \|\tilde{z}^m - z^*\|_Z \\ &\geq \frac{1}{2\eta} \|\tilde{z}^m - z^*\|_Z \end{aligned} \tag{3.11}$$

With

$$\|z^0 - z^*\|_Z < \min\left(r^*, \frac{1}{2c_9\eta C\sqrt{\max(2, 1+c^2\gamma_e^2)}} \right) ,$$

where $C > 0$ is introduced in Theorem 2.13 on page 26 and r^* is defined in 2.9, we find

$$\|z^m - z^*\|_Z < \frac{1}{c_9\eta C\sqrt{\max(2, 1+c^2\gamma_e^2)}} \quad \text{for all } m \tag{3.12}$$

and

$$\begin{aligned} \|F(\tilde{z}^{m+1})\|_{Z^*} &\leq c_9 \|z^{m+1} - z^*\|_Z \\ &\leq c_9 C \|z^m - z^*\|_Z^2 \\ &\overset{(3.12)}{<} \frac{\|z^m - z^*\|_Z}{2\eta\sqrt{\max(2, 1+c^2\gamma_e^2)}} \\ &\leq \frac{1}{2\eta} \|\tilde{z}^m - z^*\|_Z \\ &\overset{(3.11)}{\leq} \|F(\tilde{z}^m)\|_{Z^*} . \end{aligned}$$

Hence, $\{\|F(\tilde{z}^m)\|_{Z^*}\}_{m\in\mathbb{N}}$ is a strictly decreasing sequence. □

Let $\varepsilon > 0$. We define

$$\begin{aligned} m(\varepsilon) &= \min\{m_0 | \text{ for } m \geq m_0 : \|F(y^m, u^m, \tilde{\lambda}^m)\|_{Z^*} < \varepsilon\}\,, \\ m_h(\varepsilon) &= \min\{m_0 | \text{ for } m \geq m_0 : \|F_h(y_h^m, u_h^m, \tilde{\lambda}_h^m)\|_{Z_h^*} < \varepsilon\}\,. \end{aligned}$$

Note that $m(\varepsilon)$ and $m_h(\varepsilon)$ depend again on the start-up values (y^0, u^0, λ^0) and $(y_h^0, u_h^0, \lambda_h^0)$, respectively.

To prove Lemma 3.3 we need the following condition.

CONDITION 13. For all $(y, u, \lambda) \in U$ we have

$$\lim_{h\to 0} \|r_h^X \mathfrak{J}_X L(y, u, \lambda) - \mathfrak{J}_{X_h} L_h(r_h(y, u, \lambda))\|_{X_h} = 0\,.$$

REMARK 3.2. With Condition 8 on page 35 holding Condition 13 implies that for all $(y, u, \lambda) \in U$ we have

$$\lim_{h\to 0} \|r_h \mathfrak{J}_Z F(y, u, \lambda) - \mathfrak{J}_{Z_h} F_h(r_h(y, u, \lambda))\|_{Z_h} = 0\,.$$

The following relation holds between $m(\varepsilon)$ and $m_h(\varepsilon)$.

LEMMA 3.3. *Let the assumptions of Theorem 3.38 on page 39 and the hypotheses of Lemma 3.1 be valid. If in addition Condition 13 holds, then for each $\varepsilon > 0$ there exists $h_\varepsilon > 0$ such that*

$$m(\varepsilon + \delta) \leq m_h(\varepsilon) \leq m(\varepsilon) \tag{3.13}$$

for every $h \in (0, h_\varepsilon]$ and every $\delta > 0$.

PROOF. Throughout the proof we will write $\tilde{z}^m = (y^m, u^m, \tilde{\lambda}^m)$ for all m and $\tilde{z}_h^m = (y_h^m, u_h^m, \tilde{\lambda}_h^m)$ for all m and h.

First we shall prove that

$$\lim_{h\to 0} |\, \|F_h(\tilde{z}_h^m)\|_{Z_h^*} - \|F(\tilde{z}^m)\|_{Z^*}\,| = 0 \qquad \text{for each } m\,. \tag{3.14}$$

For every h we denote by $\mathfrak{J}_{Z_h}$ the Riesz isomorphism from the dual space Z_h^* onto Z_h. Then we infer [**Aub72**, p. 77]

$$\mathfrak{J}_{Z_h} = r_h\, \mathfrak{J}_Z\, q_h\,, \tag{3.15}$$

where $\mathcal{J}_Z$ is the Riesz isomorphism from Z^* onto Z. Then we derive from Remark 3.27 on page 35 that for each $m \in \mathbb{N}$

$$\begin{aligned} |\, \|s_h F(\tilde{z}^m)\|_{Z_h^*} - \|F(\tilde{z}^m)\|_{Z^*}\,| &= |\, \|\mathfrak{J}_{Z_h} s_h F(\tilde{z}^m)\|_{Z_h} - \|\mathfrak{J}_Z F(\tilde{z}^m)\|_Z\,| \\ &\leq \|p_h \mathfrak{J}_{Z_h} s_h F(\tilde{z}^m) - \mathfrak{J}_Z F(\tilde{z}^m)\|_Z \\ &\overset{(3.15)}{=} \|p_h r_h \mathfrak{J}_Z q_h s_h F(\tilde{z}^m) - \mathfrak{J}_Z F(\tilde{z}^m)\|_Z \\ &\leq c_\circ^2 \gamma_F\, \|\tilde{z}^m - p_h r_h \tilde{z}^m\|_Z \\ &\quad + \|\mathfrak{J}_{Z_h} F_h(r_h \tilde{z}^m) - r_h \mathfrak{J}_Z F(\tilde{z}^m)\|_{Z_h} \\ &\quad + \|p_h r_h \mathfrak{J}_Z F(\tilde{z}^m) - \mathfrak{J}_Z F(\tilde{z}^m)\|_Z \end{aligned}$$

such that Condition 5 on page 35 and Remark 3.2 lead to

$$\lim_{h\to 0} \|s_h F(\tilde{z}^m)\|_{Z_h^*} = \|F(\tilde{z}^m)\|_{Z^*} \quad \text{for each } m \in \mathbb{N}. \tag{3.16}$$

For every m we have

$$\begin{aligned} | \, \|F_h(\tilde{z}_h^m)\|_{Z_h^*} - \|F(\tilde{z}^m)\|_{Z^*} \, | \;\le\; & \|F_h(\tilde{z}_h^m) - s_h F(p_h r_h \tilde{z}^m)\|_{Z_h^*} \\ & + \|s_h F(p_h r_h \tilde{z}^m) - s_h F(\tilde{z}^m)\|_{Z_h^*} \\ & + | \, \|s_h F(\tilde{z}^m)\|_{Z_h^*} - \|F(\tilde{z}^m)\|_{Z^*} \, | \\ \le\; & \gamma_F \, \|\tilde{z}_h^m - r_h \tilde{z}^m\|_{Z_h} \\ & + \gamma_F \, \|p_h r_h \tilde{z}^m - \tilde{z}^m\|_Z \\ & + | \, \|s_h F(\tilde{z}^m)\|_{Z_h^*} - \|F(\tilde{z}^m)\|_{Z^*} \, | \, . \end{aligned}$$

By Theorem 3.38 on page 39 it follows that

$$\lim_{h\to 0} \|\tilde{z}_h^m - r_h \tilde{z}^m\|_{Z_h} = 0 \qquad \text{for each } m \, ,$$

and from Condition 5 on page 35 we infer that

$$\lim_{h\to 0} \|p_h r_h \tilde{z}^m - \tilde{z}^m\|_Z \qquad \text{for each } m \, .$$

Using (3.16) equation (3.14) follows.

Let $\varepsilon > 0$. If $\|F(\tilde{z}^m)\|_{Z^*} < \varepsilon$, then there exists $h_\varepsilon > 0$ such that $\|F_h(\tilde{z}_h^m)\|_{Z^*} < \varepsilon$ for any $h \in (0, h_\varepsilon]$. This leads to

$$m_h(\varepsilon) \le m(\varepsilon) \qquad \text{for all } h \in (0, h_\varepsilon] \, .$$

Hence, the right inequality of (3.13) follows.

We next prove the inequality $m(\varepsilon + \delta) \le m_h(\varepsilon)$ for every $h \in (0, h_\varepsilon]$ and every $\delta > 0$. We proceed by contradiction and suppose that for a given $\varepsilon_\circ > 0$ there are sequences $\{h_l\}_{l\in\mathbb{N}}$ and $\{\delta_l\}_{l\in\mathbb{N}}$ with $\lim_{l\to\infty} h_l = 0$ and $\delta_l > 0$, respectively, satisfying

$$m_{h_l}(\varepsilon_\circ) < m(\varepsilon_\circ + \delta_l) \qquad \text{for all } l \, . \tag{3.17}$$

Since $\delta_l > 0$ for all l holds, it follows that

$$\|F(\tilde{z}^j)\|_{Z^*} > \varepsilon_\circ \quad \text{for } j = 1, \dots, m(\varepsilon_\circ + \delta_l) - 1 \text{ and all } l \, .$$

From (3.14) we conclude that there exists $l_\circ \in \mathbb{N}$ such that

$$\|F_{h_l}(\tilde{z}_{h_l}^j)\|_{Z_{h_l}^*} \ge \varepsilon_\circ \quad \text{for } j = 1, \dots, m(\varepsilon_\circ + \delta_l) - 1 \text{ and all } l \ge l_\circ \, .$$

Thus, $m_{h_l}(\varepsilon_\circ) \ge m(\varepsilon_\circ + \delta_l)$ for all $l \ge l_\circ$, which contradicts (3.17). ☐

THEOREM 3.4. *Let the assumptions of Lemma 3.3 holds. If the term* $\|(y^0, u^0, \lambda^0) - (y^*, u^*, \lambda^*)\|_Z$ *is sufficiently small then for each* $\varepsilon > 0$ *there exists* $h_\varepsilon > 0$ *such that*

$$m(\varepsilon) - 1 \le m_h(\varepsilon) \le m(\varepsilon) \tag{3.18}$$

for any $h \in (0, h_\varepsilon]$.

PROOF. Throughout the proof we will write $\tilde{z}^m = (y^m, u^m, \tilde{\lambda}^m)$ for all m.

The right inequality of (3.18) follows directly from Lemma 3.3. We shall derive that for $\delta > 0$ sufficiently small $m(\varepsilon + \delta) \geq m(\varepsilon) - 1$ holds, which implies (3.18).

By Lemma 3.1 the sequence $\{\|F(\tilde{z}^m)\|_{Z^*}\}_{m\in\mathbb{N}}$ is a strictly decreasing if $\|(y^0, u^0, \lambda^0) - (y^*, u^*, \lambda^*)\|_Z$ is sufficiently small. Let $\varepsilon > 0$ and $m = m(\varepsilon)$ be the unique integer defined by

$$\|F(\tilde{z}^m)\|_{Z^*} < \varepsilon \leq \|F(\tilde{z}^{m-1})\|_{Z^*} .$$

Then,

$$\begin{aligned} \|F(\tilde{z}^{m-1})\|_{Z^*} &= \|F(\tilde{z}^{m-1})\|_{Z^*} - \|F(\tilde{z}^m)\|_{Z^*} + \|F(\tilde{z}^m)\|_{Z^*} \\ &< \underbrace{\|F(\tilde{z}^{m-1})\|_{Z^*} - \|F(\tilde{z}^m)\|_{Z^*}}_{>0} + \varepsilon \end{aligned}$$

implies that

$$m(\varepsilon) - 1 \geq m(\varepsilon + \|F(\tilde{z}^{m-1})\|_{Z^*} - \|F(\tilde{z}^m)\|_{Z^*})$$

Hence, $0 < \delta \leq \|F(\tilde{z}^{m-1})\|_{Z^*} - \|F(\tilde{z}^m)\|_{Z^*}$ yields $m(\varepsilon) - 1 \geq m(\varepsilon + \delta)$. $\square$

REMARK 3.5. Let us point out that relation (3.18) gives an estimate for the number of iterates necessary to achieve a certain stopping criterion for the finite dimensional problem that is independent of the dimension and is only determined by the iteration number of the infinite dimensional problem.

CHAPTER 4

Control of the Stationary Burgers Equation

1. Introduction

In the recent past there has been given considerable attention to the problem of active control of fluids or combustion, where nonlinear effects actually can improve mixing. Many investigations are concerned with the study of asymptotic or steady state properties of solutions of nonlinear distributed parameter systems, such as Navier-Stokes equations, which contains both diffusion terms and nonlinear convection terms.

A one dimensional simple model for convection-diffusion phenomena is the Burgers equation, such as shock waves, supersonic flow about airfoils, traffic flows, acoustic transmission etc. We mention that for small viscosity parameter the solution of Burgers' equation produces steep gradients due to the nonlinear nature of the convection.

Burgers introduced the equation

$$y_t + yy_x = \nu y_{xx} \tag{4.1}$$

as a simple model for turbulence [**Bur40, Bur48, Bur72**], where $\nu > 0$ denotes a viscosity parameter. In [**Col51**] general properties of solutions to (4.1) are studied and related to some applications.

In this chapter we simplify equation (4.1) and consider the stationary Burgers equation

$$yy' = \nu y'' + f\,,$$

where f is given.

As an application of the convergence results and mesh-independence principle presented in Chapters 2 and 3, respectively, we analyze control problems governed by the stationary Burgers equation. Let $\sigma > 0$ and $C \in L(H^1(0,1),\mathcal{V})$, where $\mathcal{V}$ is a real Hilbert space. We consider optimal distributed control minimizing

$$J(y,u) = \frac{1}{2}\,\|Cy - z\|_{\mathcal{V}}^2 + \frac{\sigma}{2}\,\|u\|_{L^2}^2$$

subject to

$$\begin{cases} -\nu y'' + yy' = f + u & \text{in } (0,1)\,, \\ y(0) = \alpha,\ y(1) = \beta\,, \end{cases} \tag{4.2}$$

where $\alpha, \beta \in \mathbb{R}$ and $f \in L^2(0,1)$, $\nu > 0$ denotes the viscosity parameter and $u \in L^2(0,1)$ is the control variable. Further, we study optimal boundary control problems minimizing

$$J(y,u) = \frac{1}{2}\, \|Cy - z\|_{\mathcal{V}}^2 + \frac{\sigma}{2}\, u^2$$

subject to

$$\left\{ \begin{array}{ll} -\nu y'' + yy' = f & \text{in } (0,1)\,, \\ y(0) = \alpha,\; y'(1) = u\,, & \end{array} \right. \tag{4.3}$$

where $u \in \mathbb{R}$ denotes the control variable.

We write the control problems in the form

$$\min J(y,u) \quad \text{subject to } e(y,u) = 0. \tag{$\mathbf{P}$}$$

To solve **(P)** we use the augmented Lagrangian-SQP method. The infinite dimensional problems are analyzed so that the second-order convergence of Algorithm 1 on page 25 is guaranteed. The approximation of the infinite dimensional problems is done by using piecewise linear finite elements. We prove that Algorithm 2 on page 38 is well-defined for sufficiently small mesh-sizes. If in addition y_h^* is appropriate then the uniform discrete Babuška-Brezzi condition is satisfied for all h (Condition 6 on page 37). Both for the distributed and the boundary control problems numerical test examples are presented, which illustrate the second-order convergence of the Algorithm 2 and mesh-independence.

The chapter is organized as follows. The second section is devoted to prove that problem (4.2) has a unique weak solution if ν is sufficiently large. Distributed control problems for the stationary Burgers equation are considered in Section 3, where the existence of optimal solutions (Condition 1), the regular point condition (Condition 2) and the second-order sufficient optimality condition (Condition 3) are proved. Hence, Theorem 2.13 on page 26 guarantees the second-order convergence of Algorithm 1 if we choose the start-up values sufficiently close to the solution. If the viscosity parameter is sufficiently large, existence and uniqueness of a weak solution to (4.3) is ensured in Section 4. Analysis similar to the distributed control problem is done for the boundary control problem in Section 5. In the last section the approximation of the infinite dimensional problem is considered and numerical test examples are presented. The uniform discrete Babuška-Brezzi condition is verified if y_h^* is appropriate in the interval $[0,1]$.

2. The State Equation with Dirichlet Boundary Conditions

2.1. Boundary Value Problem. Let $f \in H^{-1}(0,1)$ and $\alpha, \beta \in \mathbb{R}$. We consider the *stationary Burgers equation*

$$-\nu y'' + yy' = f \qquad \text{in } (0,1) \tag{4.4a}$$

with Dirichlet boundary conditions

$$y(0) = \alpha\,, \qquad y(1) = \beta\,. \tag{4.4b}$$

$\nu > 0$ is called the *viscosity parameter.*

DEFINITION 2.1. A function $y \in H^1(0,1)$ is said to be a weak solution of (4.4) if

$$(\mathbf{B_1}) \qquad \begin{cases} \displaystyle\int_0^1 (\nu y'\varphi' + yy'\varphi)\,dx = \langle f, \varphi\rangle_{H^{-1},H_0^1} & \text{for all } \varphi \in H_0^1(0,1)\,, \\ y(0) = \alpha,\ y(1) = \beta \end{cases}$$

is valid.

By Lemma 3.7 on page 10 the weak solution y of (4.4) belongs to $C[0,1]$. Thus, the Dirichlet boundary conditions $y(0) = \alpha$ and $y(1) = \beta$ are meaningful.

LEMMA 2.2. *For every $\varepsilon > 0$ there exists a function $y_\circ = y_\circ(\varepsilon) \in H^1(0,1)$ such that $y_\circ(0) = \alpha$, $y_\circ(1) = \beta$ and $\|y_\circ\|_{L^2} \le \varepsilon$.*

PROOF. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. We define

$$y_\circ(x) = \begin{cases} (-2)^n \alpha \left(x - \frac{1}{2}\right)^n & \text{for all } x \in \left[0, \frac{1}{2}\right]\,, \\ 2^n \beta \left(x - \frac{1}{2}\right)^n & \text{for all } x \in \left(\frac{1}{2}, 1\right]\,. \end{cases} \tag{4.5}$$

Obviously, $y_\circ(0) = \alpha$, $y_\circ(1) = \beta$ and $y_\circ \in H^1(0,1)$, [**GR92**, p. 119]. Further,

$$\|y_\circ\|_{L^2}^2 = 4^n\alpha^2 \int_0^{\frac{1}{2}} \left(x - \frac{1}{2}\right)^{2n} dx + 4^n\beta^2 \int_{\frac{1}{2}}^1 \left(x - \frac{1}{2}\right)^{2n} dx = \frac{\alpha^2 + \beta^2}{4n+2}\,.$$

If we choose

$$n \ge \max\left(1, \frac{\alpha^2 + \beta^2 - 2\varepsilon^2}{4\varepsilon^2}\right)\,,$$

we derive that $\|y_\circ\|_{L^2} < \varepsilon$, and the claim is proved. □

For

$$n \ge \max\left(1, \frac{2(\alpha^2 + \beta^2) - \nu^2}{2\nu^2}\right)$$

we infer from the proof of Lemma 2.2 that $\|y_\circ\|_{L^2} \le \frac{\nu}{2}$ and we estimate

$$\|y_\circ\|_{H^1} = \sqrt{\frac{2n^2(\alpha^2 + \beta^2)}{2n-1} + \frac{\nu^2}{4}}\,.$$

Let $\tilde{f} \in H^{-1}(0,1)$ be defined by

$$\langle \tilde{f}, \varphi\rangle_{H^{-1},H_0^1} = \langle f, \varphi\rangle_{H^{-1},H_0^1} - \int_0^1 \nu y_\circ'\varphi' + y_\circ y_\circ'\varphi\,dx \quad \text{for } \varphi \in H_0^1(0,1)\,.$$

A weak solution satisfies the following a-priori estimates.

LEMMA 2.3. *Let y be a solution to* $(\mathbf{B_1})$. *We achieve*

$$\begin{aligned} \|y\|_{L^\infty} &\le \frac{2}{\nu}\,\|\tilde f\|_{H^{-1}} + \max(|\alpha|,|\beta|)\,,\\ \|y\|_{H^1} &\le \frac{\sqrt{8}}{\nu}\,\|\tilde f\|_{H^{-1}} + \|y_\circ\|_{H^1}\,. \end{aligned}$$

PROOF. Let y be a solution to $(\mathbf{B_1})$. Then $z = y - y_\circ \in H_0^1(0,1)$ satisfies

$$\int_0^1 \nu z'\varphi' + zz'\varphi + (y_\circ z)'\varphi\,dx = \langle \tilde f, \varphi\rangle_{H^{-1},H_0^1} \quad \text{for all } \varphi \in H_0^1(0,1)\,.$$

Setting $\varphi = z$ and integration by parts yields

$$\nu\,\|z\|_{H_0^1}^2 - \int_0^1 y_\circ z z'\,dx = \langle \tilde f, z\rangle_{H^{-1},H_0^1}\,, \tag{4.6}$$

where we use the identity

$$\int_0^1 zz'z\,dx = 0\,.$$

By duality inequality the right-hand side of equation (4.6) gives

$$\langle \tilde f, z\rangle_{H^{-1},H_0^1} \le \|\tilde f\|_{H^{-1}}\,\|z\|_{H_0^1}\,.$$

Moreover, we infer from the Hölder inequality and from Lemma 3.4 on page 9 that

$$\int_0^1 y_\circ z z'\,dx \le \|y_\circ\|_{L^2}\,\|z\|_{H_0^1}^2\,.$$

Hence, from (4.6) and $\|y_\circ\|_{L^2} \le \frac{\nu}{2}$ we conclude that

$$\|z\|_{H_0^1} \le \frac{2}{\nu}\,\|\tilde f\|_{H^{-1}}\,. \tag{4.7}$$

By triangle inequality and Lemma 3.4 on page 9 it follows that

$$\begin{aligned} \|y\|_{L^\infty} &\le \|z\|_{L^\infty} + \|y_\circ\|_{L^\infty}\\ &\le \|z\|_{H_0^1} + \max(|\alpha|,|\beta|)\\ &\overset{(4.7)}{\le} \frac{2}{\nu}\,\|\tilde f\|_{H^{-1}} + \max(|\alpha|,|\beta|) \end{aligned}$$

The H^1-estimate follows from $\|y\|_{H^1} \le \|y - y_\circ\|_{H^1} + \|y_\circ\|_{H^1}$ and Lemma 3.4 on page 9. $\square$

REMARK 2.4. If $f \in L^2(0,1)$ and $y_\circ \in H^2(0,1)$ hold we infer from the Hölder inequality and from Lemma 3.4 on page 9 that

$$\begin{aligned}\|\tilde f\|_{H^{-1}} &= \sup_{\|\varphi\|_{H_0^1}=1} \Big| \int_0^1 f\varphi - \nu y_\circ'\varphi' - y_\circ y_\circ'\varphi\,dx \Big| \\ &\le \sup_{\|\varphi\|_{H_0^1}=1} \left(\|f+\nu y_\circ'' - y_\circ y_\circ'\|_{L^2}\,\|\varphi\|_{L^2}\right) \\ &\le \sup_{\|\varphi\|_{H_0^1}=1} \left(\|f+\nu y_\circ'' - y_\circ y_\circ'\|_{L^2}\,\|\varphi\|_{H_0^1}\right) \\ &= \|f+\nu y_\circ'' - y_\circ y_\circ'\|_{L^2}\,.\end{aligned}$$

2.2. Existence and Uniqueness in Case of Homogeneous Boundary Conditions. Henceforth, we denote by $a : H^1(0,1)\times H^1(0,1)\to\mathbb{R}$ the symmetric bilinear form

$$a(\varphi,\psi) = \nu\,\langle\varphi,\psi\rangle_{H_0^1} \qquad \text{for } \varphi,\psi\in H^1(0,1)\,. \tag{4.8}$$

We also introduce the trilinear form $b(\cdot\,,\cdot\,,\cdot)$ on $(H^1(0,1))^3 = H^1(0,1)\times H^1(0,1)\times H^1(0,1)$ by

$$b(\phi,\varphi,\psi) = \frac{1}{3}\int_0^1 (\phi\varphi)'\psi + \phi\varphi'\psi\,dx\,. \tag{4.9}$$

The variational formulation for the stationary Burgers equation (4.4a) with homogeneous boundary conditions is given by

$$\begin{cases} a(y,\varphi) + b(y,y,\varphi) = \langle f,\varphi\rangle_{H^{-1},H_0^1} & \text{for all } \varphi\in H_0^1(0,1)\,,\\ y(0)=y(1)=0\,. \end{cases} \tag{4.10}$$

LEMMA 2.5. *$b(\cdot\,,\cdot\,,\cdot)$ is a continuous trilinear form on $(H^1(0,1))^3$.*

PROOF. Let $\phi,\varphi,\psi\in H^1(0,1)$. Applying (1.7) and Lemma 3.5 on page 10 implies that

$$\begin{aligned}|b(\phi,\varphi,\psi)| &\le \frac{1}{3}\|\phi'\|_{L^2}\,\|\varphi\|_{L^4}\|\psi\|_{L^4} + \frac{2}{3}\|\phi\|_{L^4}\,\|\varphi'\|_{L^2}\|\psi\|_{L^4} \\ &\le \|\phi\|_{H^1}\,\|\varphi\|_{H^1}\|\psi\|_{H^1}\,.\end{aligned}$$

Hence, b is continuous on $(H^1(0,1))^3$. It is clear, that b is linear in each component. □

LEMMA 2.6. *The trilinear form satisfies $b(\phi,\varphi,\psi)+b(\phi,\psi,\varphi)=0$ for all $\phi,\psi\in H^1(0,1)$ and $\varphi\in H_0^1(0,1)$.*

PROOF. Let $\phi,\psi\in H^1(0,1)$ and $\varphi\in H_0^1(0,1)$. Using (4.9) we calculate

$$b(\phi,\varphi,\psi)+b(\phi,\psi,\varphi) = \frac{1}{3}\int_0^1 -\phi\varphi\psi' + \phi\varphi'\psi - \phi\psi\varphi' + \phi\psi'\varphi\,dx = 0\,.$$

□

Lemma 2.6 implies

COROLLARY 2.7. *For all $\phi \in H^1(0,1)$ and $\varphi \in H_0^1(0,1)$ it follows that $b(\phi,\varphi,\varphi) = 0$.*

REMARK 2.8. With $\phi, \varphi, \psi \in \mathcal{H}$ holding we derive only

$$b(\phi,\varphi,\psi) + b(\phi,\psi,\varphi) = \frac{2}{3}\,\phi(1)\,\varphi(1)\,\psi(1)\,.$$

For problem $(\mathbf{B_1})$ with homogeneous boundary conditions existence can be proved by using the *Schauder fixed point theorem.*

THEOREM 2.9. *If M is a bounded, closed and convex set in a Banach space B and $T : M \to M$ is a compact mapping, then there exists a point $y \in M$ such that $Ty = y$.*

For the proof we refer the reader to [**GT77**, p. 222].

THEOREM 2.10. *For any $f \in H^{-1}(0,1)$ there exists at least one solution $y \in H_0^1(0,1)$ of (4.10).*

PROOF. We define $z = Tw$ via the mapping $T : H_0^1(0,1) \to H_0^1(0,1)$ as follows: $z \in H_0^1(0,1)$ is the unique solution to

$$a(z,\varphi) + b(w,z,\varphi) = \langle f,\varphi\rangle_{H^{-1},H_0^1} \qquad \text{for all } \varphi \in H_0^1(0,1)\,. \tag{4.11}$$

The bilinear form $a(\cdot\,,\cdot) + b(w,\cdot\,,\cdot)$ is continuous by (4.8) and Lemma 2.5 and coercive on $H_0^1(0,1)$ by Corollary 2.7. The existence and uniqueness of a solution for (4.11) can thus be shown by the *Lax-Milgram theorem* [**GT77**, p. 78]. The fixed points of T are the solutions of (4.10). Taking $\varphi = z$ in (4.11) above and using (4.8) and Lemma 2.6 we derive

$$\|z\|_{H_0^1} \le \frac{1}{\nu}\,\|f\|_{H^{-1}}\,. \tag{4.12}$$

Let us introduce the set

$$M = \left\{ w \in H_0^1(0,1) : \|w\|_{H_0^1} \le \frac{1}{\nu}\,\|f\|_{H^{-1}} \right\} \subset H_0^1(0,1)\,. \tag{4.13}$$

From (4.12) we infer that T maps M into itself. Since M is a closed ball in $H_0^1(0,1)$, M is bounded, closed and convex. Next we show that the mapping T is compact from M to M. Let $\{w_k\}_{k\in\mathbb{N}}$ be a bounded sequence in M. Then there exists a subsequence, which we also denote by $\{w_k\}_{k\in\mathbb{N}}$ for convenience, that is weakly convergent in $H_0^1(0,1)$, [**GT77**, p. 80]. Due to Remark 3.6 on page 10 this subsequence converges strongly in $L^4(0,1)$. Therefore, we obtain for any $w_n, w_m \in \{w_k\}_{k\in\mathbb{N}}$ and $z_n = Tw_n, z_m = Tw_m$

$$\begin{aligned} a(z_n,\varphi) + b(w_n,z_n,\varphi) &= \langle f,\varphi\rangle_{H^{-1},H_0^1} \qquad \text{for all } \varphi \in H_0^1(0,1)\,,\\ a(z_m,\varphi) + b(w_m,z_m,\varphi) &= \langle f,\varphi\rangle_{H^{-1},H_0^1} \qquad \text{for all } \varphi \in H_0^1(0,1) \end{aligned}$$

and subtracting the two equations above we derive that

$$a(z_n - z_m,\varphi) = b(w_m - w_n, z_m,\varphi) - b(w_n, z_n - z_m,\varphi)$$

for all $\varphi \in H_0^1(0,1)$. Taking $\varphi = z_n - z_m$ and using (4.8), Corollary 2.7 and Lemma 3.4 on page 9

$$\begin{aligned}
\nu \|z_n - z_m\|_{H_0^1}^2 &\le \frac{1}{3}\|w_m - w_n\|_{L^4}\|z_m\|_{L^4}\|z_n' - z_m'\|_{L^2} \\
&\quad + \frac{1}{3}\|w_m - w_n\|_{L^4}\|z_m'\|_{L^2}\|z_n - z_m\|_{L^4} \\
&\le \frac{2}{3}\|w_m - w_n\|_{L^4}\|z_m\|_{H_0^1}\|z_n - z_m\|_{H_0^1}.
\end{aligned}$$

Using the a-priori bound (4.12) we derive

$$\|z_n - z_m\|_{H_0^1} \le \frac{2}{3\nu^2}\|f\|_{H^{-1}}\|w_m - w_n\|_{L^4}.$$

Since $\{w_k\}_{k\in\mathbb{N}}$ converges in $L^4(0,1)$, the sequence $\{z_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $H_0^1(0,1)$. Therefore, $\{z_k\}_{k\in\mathbb{N}}$ converges in $H_0^1(0,1)$. Hence, T is compact [**Alt92**, p. 236]. The mapping T satisfies the assumptions of Theorem 2.9. Thus, there exists a fixed point $y \in M$. □

REMARK 2.11. From (4.12) above, we get the useful a-priori bound for the solution of (4.10)

$$\|y\|_{H_0^1} \le \frac{1}{\nu}\|f\|_{H^{-1}}.$$

Further assumptions are required to prove uniqueness of the solution of (4.10) whose existence we established in Theorem 2.10. Uniqueness can be achieved if the viscosity parameter is large enough with respect to the H^{-1}-norm of f.

THEOREM 2.12. *The solution of (4.10) is unique if* $\nu^2 > \|f\|_{H^{-1}}$.

PROOF. Assume that $y_1, y_2 \in H_0^1(0,1)$ are two solutions of (4.10). Then $z = y_1 - y_2 \in H_0^1(0,1)$ solves:

$$a(y_1 - y_2, \varphi) + b(y_1, y_1, z) - b(y_2, y_2, z) = 0 \qquad \text{for all } \varphi \in H_0^1(0,1).$$

Setting $\varphi = y_1 - y_2 \in H_0^1(0,1)$ we obtain from Corollary 2.7, Lemma 3.4 on page 9 and Remark 2.11

$$\begin{aligned}
\nu \|y_1 - y_2\|_{H_0^1}^2 &= b(y_2, y_2, y_1 - y_2) - b(y_1, y_1, y_1 - y_2) \\
&= b(y_2 - y_1, y_2, y_1 - y_2) + b(y_1, y_2, y_1 - y_2) \\
&\quad - b(y_1, y_1, y_1 - y_2) \\
&= b(y_2 - y_1, y_2, y_1 - y_2) + b(y_1, y_2 - y_1, y_1 - y_2) \\
&= b(y_2 - y_1, y_2, y_1 - y_2) \\
&\le \|y_1 - y_2\|_{H_0^1}^2 \|y_2\|_{H_0^1} \\
&\le \frac{1}{\nu}\|f\|_{H^{-1}}\|y_1 - y_2\|_{H_0^1}^2.
\end{aligned}$$

Hence, we have

$$\left(\nu - \frac{1}{\nu}\,\|f\|_{H^{-1}}\right)\|y_1 - y_2\|^2_{H^1_0} \le 0$$

and $y_1 = y_2$ by assumption of theorem. □

2.3. Non-Homogeneous Boundary Conditions. The case of non-homogeneous boundary conditions is a non-trivial extension of the homogeneous case.

If we choose

$$n > \max\left(0, \frac{\alpha^2 + \beta^2 - 2\varepsilon^2}{4\varepsilon^2}\right)$$

we derive $\|y_\circ\|_{L^2} < \varepsilon$ by Lemma 2.2. Applying Lemma 3.4 on page 9 leads to

$$\begin{aligned} |b(\varphi, y_\circ, \varphi)| &= |-\int_0^1 \varphi y_\circ \varphi'\,dx| \\ &\le \|\varphi\|_{L^\infty}\,\|y_\circ\|_{L^2}\,\|\varphi\|_{H^1_0} \\ &\le \|y_\circ\|_{L^2}\,\|\varphi\|^2_{H^1_0} \\ &< \varepsilon\,\|\varphi\|^2_{H^1_0} \qquad \text{for all } \varphi \in H^1_0(0,1)\,, \end{aligned}$$

Now the non-homogeneous problem $(\mathbf{B_1})$ can be converted to one with homogeneous boundary conditions by using the transformation $z = y - y_\circ$. To shorten notation we introduce

$$\langle \tilde{f}, \varphi\rangle_{H^{-1},H^1_0} = \langle f, \varphi\rangle_{H^{-1},H^1_0} - a(y_\circ, \varphi) - b(y_\circ, y_\circ, \varphi)$$

and

$$\tilde{a}(z,\varphi) = a(z,\varphi) + b(y_\circ, z, \varphi) + b(z, y_\circ, \varphi)\,.$$

Then we derive from $(\mathbf{B_1})$ that

$$\left\{\begin{array}{l} \tilde{a}(z,\varphi) + b(z,z,\varphi) = \langle \tilde{f}, \varphi\rangle_{H^{-1},H^1_0} \quad \text{for all } \varphi \in H^1_0(0,1)\,, \\ z(0) = z(1) = 0\,. \end{array}\right. \tag{4.14}$$

By (4.8), Corollary 2.7 and Lemma 2.2 we derive

$$\tilde{a}(z,z) \ge (\nu - \varepsilon)\,\|z\|^2_{H^1_0} \qquad \text{for all } \varphi \in H^1_0(0,1)\,.$$

Hence, if we choose $y_\circ$ as in Lemma 2.2 with $\varepsilon < \nu$ then $\tilde{a}(\cdot,\cdot)$ is coercive on $H^1_0(0,1) \times H^1_0(0,1)$. Also, $\tilde{f} \in H^{-1}(0,1)$. This is a direct consequence of the continuity of $a(\cdot\,,\cdot)$ on $H^1_0(0,1) \times H^1_0(0,1)$ and of $b(\cdot\,,\cdot\,,\cdot)$ on $(H^1(0,1))^3$. Then Theorem 2.9 yields existence of a solution of (4.14). Uniqueness is achieved by a "largeness" hypothesis on the viscosity parameter ν and this is analogously to the arguments leading to the proof of Theorem 2.12. We summarize this in the following theorem.

THEOREM 2.13. *Given $f \in H^{-1}(0,1)$ and $\alpha, \beta \in \mathbb{R}$, a weak solution $y \in H^1(0,1)$ of problem $(\mathbf{B_1})$ is unique if ν is sufficiently large.*

2.4. Regularity of the Solution. The regularity of the weak solution to $(\mathbf{B_1})$ depends on the regularity of the right-hand side f.

THEOREM 2.14. *Let ν satisfy the assumptions of Theorem 2.13 and f belong to $H^m(0,1)$, $m \geq 0$. Then the weak solution of $(\mathbf{B_1})$ is an element of $H^{m+2}(0,1)$.*

For the proof we refer the reader to [**GT77**, p. 173 and 176].

REMARK 2.15. Under the hypotheses of Theorem 2.14 we infer from Lemma 3.7 on page 10 that $y \in C^{m+1}[0,1]$ is valid.

2.5. Classical Existence Result. We consider the general Dirichlet problem

$$\left\{ \begin{array}{ll} y''(x) = g(x,y,y') & \text{in } (0,1)\,, \\ y(0) = \alpha,\; y(1) = \beta\,, \end{array} \right. \tag{4.15}$$

in which g is a continuous function on $[0,1] \times \mathbb{R}^2$. The differential inequality approach of Nagumo is based on the notion of a C^2-*subsolution* y_l and of a C^2-*supersolution* y_u, i.e. y_l and y_u possess the following properties

$$\left\{ \begin{array}{l} y_l \leq y_u \quad \text{in } [0,1]\,, \\ y_l'' \geq g(x,y_l,y_l'),\; y_u'' \leq g(x,y_u,y_u')\,, \\ y_l(0) \leq \alpha \leq y_u(0),\; y_l(1) \leq \beta \leq y_u(1)\,. \end{array} \right. \tag{4.16}$$

For the proof of the following result we refer the reader to [**CH84**, p. 7].

LEMMA 2.16. *Let $f \in C[0,1]$ and y_l and y_u be bounded functions satisfying (4.16). Assume moreover that the Nagumo-condition*

$$g(x,y,y') = O(|y'|^2) \quad \textit{as} \quad |y'| \to \infty \quad \textit{for all } (x,y) \in [0,1] \times [y_l, y_u]$$

holds. Then the Dirichlet problem (4.15) has a solution $y \in C^2[0,1]$ satisfying $y_l \leq y \leq y_u$ in $[0,1]$.

Now we investigate

$$\left\{ \begin{array}{l} y'' = \frac{1}{\nu}\,(y\,y' - f) = g(\cdot,y,y') \quad \text{in } (0,1)\,, \\ y(0) = \alpha,\; y(1) = \beta\,. \end{array} \right. \tag{4.17}$$

Setting

$$\begin{aligned} y_l(x) &= \min\Big(-\|f\|_{C[0,1]} - 1, \alpha, \beta - 1\Big) + x\,, \\ y_u(x) &= \max\Big(\|f\|_{C[0,1]}, \alpha, \beta - 1\Big) + x\,. \end{aligned}$$

we infer from Lemma 2.16 the following result.

LEMMA 2.17. *For $f \in C[0,1]$ the Dirichlet problem*

$$\left\{ \begin{array}{ll} -\nu\, y'' + y\, y' = f & \textit{in } (0,1)\,, \\ y(0) = \alpha,\; y(1) = \beta \end{array} \right.$$

has a (classical) solution $y \in C^2[0,1]$ *satisfying* $y_l \le y \le y_u$ *in* $[0,1]$.

REMARK 2.18. In case of $f = 0$ we can choose $y_l = \alpha$ and $y_u = \beta$ in $[0,1]$. Then, Lemma 2.17 implies that $\alpha \le y \le \beta$ in $[0,1]$.

2.6. Maximum Principle. If the right-hand side f is greater or less than zero we can prove a maximum principle for the stationary Burgers equation.

LEMMA 2.19. *Let* $f \in C[0,1]$ *with* $f < 0$ *(*$f > 0$*) in* $(0,1)$ *and* $y \in C^2(0,1) \cap C[0,1]$ *a solution of (4.4). Then we have:* $y \ge \min(\alpha,\beta)$ *(*$y \le \max(\alpha,\beta)$*) in the interval* $[0,1]$.

PROOF. If y has a minimum at $x_e \in (0,1)$ we infer $y''(x_e) \ge 0$ and $y'(x_e) = 0$ and therefore

$$0 \overset{(4.4a)}{=} -\underbrace{\nu y''(x_e)}_{\ge 0} + \underbrace{y(x_e)\, y'(x_e)}_{=0} - \underbrace{f(x_e)}_{>0} < 0\,,$$

which is a contradiction. Thus, $y \ge \min(\alpha,\beta)$ in $[0,1]$. The proof for $f < 0$ in $(0,1)$ follows analogously. ☐

3. The Distributed Control Problem

Now we shall formulate the distributed control problem for the stationary Burgers equation introduced in the last section. For this purpose let $\omega \subset (0,1)$ be an open set with positive measure, $f \in L^2(0,1)$ and $B \in L(L^2(\omega), L^2(0,1))$ given by

$$Bq = \begin{cases} q & \text{in } \omega\,, \\ 0 & \text{in } (0,1) \setminus \omega\,. \end{cases}$$

For a control $u \in L^2(\omega)$ the state $y \in H^1(0,1)$ is given by the solution of

$$\begin{cases} -\nu y'' + yy' = f + Bu & \text{in } (0,1)\,, \\ y(0) = \alpha,\ y(1) = \beta\,. \end{cases} \tag{4.18}$$

Due to Theorem 2.13 there exists a weak solution to (4.18) which is unique for sufficiently large viscosity parameter ν. We are also given an observation operator $C \in L(H^1(0,1), \mathcal{V})$, where $\mathcal{V}$ is a real Hilbert space. With every control $u \in L^2(\omega)$ we associate the cost of tracking type

$$J(y,u) = \frac{1}{2}\,\|Cy - z\|^2_{\mathcal{V}} + \frac{\sigma}{2}\,\|u\|^2_{L^2(\omega)}\,,$$

where $z \in \mathcal{V}$ is a given desired state and $\sigma > 0$ is fixed.

The control problem then is

$$\min J(y,u) \quad \text{such that } (y,u) \text{ solves (4.18).} \tag{4.19}$$

In the context of Chapter 2, Section 2, we set $\mathcal{Y} = H^1(0,1)$, $\mathcal{U} = L^2(\omega)$ and $Y = H_0^1(0,1) \times \mathbb{R}^2$ and identify with $H^{-1}(0,1) \times \mathbb{R}^2$ the dual of Y. We introduce the operator

$$e = (e_1, e_2) : H^1(0,1) \times L^2(\omega) \to H^{-1}(0,1) \times \mathbb{R}^2 \,,$$

whose action is defined by

$$\langle e_1(y,u), \varphi \rangle_{H^{-1},H_0^1} = \int_0^1 \nu y' \varphi' + y y' \varphi - (f + Bu)\varphi \, dx \quad \text{for } \varphi \in H_0^1(0,1) \,,$$

and $e_2(y,u) = (y(0) - \alpha, y(1) - \beta)$. Then we write (4.19) in the following form

$$(\mathbf{P_1}) \qquad \min J(y,u) \quad \text{subject to } e(y,u) = 0.$$

The Lagrangian associated with $(\mathbf{P_1})$ is expressed as

$$L(y,u,\lambda,\mu) = J(y,u) + \langle e_1(y,u), \lambda \rangle_{H^{-1},H_0^1} + \langle e_2(y,u), \mu \rangle_{\mathbb{R}^2} \,.$$

3.1. The Existence of an Optimal Solution.

THEOREM 3.1. *There exists a solution of* $(\mathbf{P_1})$.

PROOF. Let us first outline the proof: We begin by proving that J is unbounded for unbounded sequences in X and that is weakly lower semi-continuous. We shall denote by $\{(y^n, u^n)\}_{n \in \mathbb{N}}$ a minimizing sequence of J in X with $e(y^n, u^n) = 0$ for all n, the properties of J imply that there exists an element $(y^*, u^*) \in X$ such that $y^n \rightharpoonup y^*$ and $u^n \rightharpoonup u^*$ for $n \to \infty$. We shall proceed to show that $e(y^*, u^*) = 0$ holds. Since J is weakly lower semi-continuous and $\{(y^n, u^n)\}_{n \in \mathbb{N}}$ is a minimizing sequence of the cost functional, (y^*, u^*) minimizes J, and the proof will be complete.

Let $(y,u) \in X$ satisfy the equation $e(y,u) = 0$. Lemma 2.3 yields that $\|(y,u)\|_X \to \infty$ implies that $\|u\|_{L^2(\omega)} \to \infty$. Further we observe that

$$J(y,u) \geq \frac{\sigma}{2} \, \|u\|^2_{L^2(\omega)} \,.$$

Since ω has positive measure, we obtain that

$$J(y,u) \longrightarrow +\infty \quad \text{for} \quad \|(y,u)\|_X \to \infty \tag{4.20}$$

such that J is unbounded for unbounded sequences in X. The norm is weakly lower semi-continuous [**Wou79**, p. 351]. Further, C is continuous, and the inner product $\langle \cdot \,, z \rangle_{\mathcal{V}}$ is weakly continuous. Let $\{(y^n, u^n)\}_{n \in \mathbb{N}}$ be a sequence in X converging weakly to (y^*, u^*). It follows that

$$\liminf_{n \to \infty} J(y^n, u^n) \geq \frac{1}{2} \|Cy^*\|^2_{\mathcal{V}} - \langle Cy^*, z \rangle_{\mathcal{V}} + \frac{1}{2} \|z\|^2_{\mathcal{V}} + \frac{\sigma}{2} \|u^*\|^2_{L^2(\omega)} = J(y^*, u^*) \,,$$

and thus that J is weakly lower semi-continuous. As $J(y,u) \geq 0$ for all $(y,u) \in X$ holds, there is $\zeta \geq 0$ such that

$$\zeta = \inf_{e(y,u)=0} J(y,u) \,.$$

Let $\{(y^n, u^n)\}_{n\in\mathbb{N}}$ denote a minimizing sequence in the space X satisfying $\zeta = \lim_{n\to\infty} J(y^n, u^n)$. From property (4.20) we infer that $\{(y^n, u^n)\}_{n\in\mathbb{N}}$ is bounded. Then there exists $(y^*, u^*) \in X$ with

$$y^n \rightharpoonup y^* \quad \text{as } n\to\infty \quad \text{in } H^1(0,1)\,, \tag{4.21}$$

$$u^n \rightharpoonup u^* \quad \text{as } n\to\infty \quad \text{in } L^2(\omega)\,, \tag{4.22}$$

[**GT77**, p. 80]. We conclude from (4.21) that

$$\nu \int_0^1 (y^n - y^*)'\,\varphi'\,dx \overset{n\to\infty}{\longrightarrow} 0 \qquad \text{for all } \varphi \in H_0^1(0,1)\,,$$

[**Ada75**, p. 48]. Furthermore, (4.22) leads to

$$\int_0^1 (Bu^n - Bu^*)\varphi\,dx \overset{n\to\infty}{\longrightarrow} 0 \qquad \text{for all } \varphi \in H_0^1(0,1)\,.$$

Lemma 3.7 on page 10 yields $y^* \in C[0,1]$ and consequently $y^*\varphi \in L^2(0,1)$ for all $\varphi \in H_0^1(0,1)$. Moreover, $\|(y^n)'\|_{L^2}$ is bounded [**HS71**, p. 61]. Thus,

$$\int_0^1 (y^n - y^*)'\,y^*\varphi\,dx \overset{n\to\infty}{\longrightarrow} 0\,.$$

Applying Remark 3.6 on page 10 we deduce $y^n \to y^*$ in $L^2(0,1)$ as n tends to ∞ so that

$$\int_0^1 (y^n - y^*)\,(y^n)'\,\varphi\,dx \le \|y^n - y^*\|_{L^2}\,\|(y^n)'\|_{L^2}\,\|\varphi\|_{L^\infty} \overset{n\to\infty}{\longrightarrow} 0$$

for all $\varphi \in H_0^1(0,1)$. Hence

$$\begin{aligned}\int_0^1 \left(y^n(y^n)' - y^*(y^*)'\right)\varphi\,dx &= \int_0^1 (y^n - y^*)(y^n)'\,\varphi\,dx \\ &\quad + \int_0^1 (y^n - y^*)'\,y^*\varphi\,dx \overset{n\to\infty}{\longrightarrow} 0\end{aligned}$$

for all $\varphi \in H_0^1(0,1)$. Therefore, we have $e_1(y^*, u^*) = 0$. By Lemma 3.7 on page 10 we obtain that y^n converges to y^* in $C[0,1]$. Therefore,

$$y^*(0) = \lim_{n\to\infty} y^n(0) = \alpha\,, \quad y^*(1) = \lim_{n\to\infty} y^n(1) = \beta$$

and $e_2(y^*, u^*) = 0$ holds. Since J is weakly lower semi-continuous, it follows that

$$\zeta = \liminf_{n\to\infty} J(y^n, u^n) \ge J(y^*, u^*)\,,$$

i.e. (y^*, u^*) is a solution to $(\mathbf{P_1})$. $\square$

Remark 3.2. The optimal solution needs not to be unique. J is convex, but the set $\{(y,u) \in X : e(y,u) = 0\}$ is not convex in general.

3.2. The Fréchet-Derivatives of J and e. In Theorem 3.1 we have proved that $(\mathbf{P_1})$ has a solution. The requirements of Condition 1 on page 21 are met by the mapping J. Let $(y,u) \in X$ be arbitrary. Then the actions of the Fréchet-derivatives of J are given by

$$\begin{aligned}\langle J'(y,u),(v,q)\rangle_{X^*,X} &= \langle Cy-z, Cv\rangle_{\mathcal{V}} + \sigma\int_\omega uq\,dx\,,\\ \langle J''(y,u)(w,p),(v,q)\rangle_{X^*,X} &= \langle Cw, Cv\rangle_{\mathcal{V}} + \sigma\int_\omega pq\,dx\end{aligned}$$

for any directions (v,q), $(w,p) \in X$. Hence, the mapping $(y,u) \mapsto J''(y,u)$ is Lipschitz-continuous on the whole space X.

For arbitrary $(y,u) \in X$ we compute the following directional derivative of e_1

$$\langle e_1'(y,u)(v,q),\varphi\rangle_{H^{-1},H_0^1} = \int_0^1 \nu v'\varphi' + (yv)'\varphi - Bq\,\varphi\,dx \tag{4.23}$$

for $\varphi \in H_0^1(0,1)$. Using (1.7) we estimate

$$\begin{aligned}&\|e_1(y+v,u+q) - e_1(y,u) - e_1'(y,u)(v,q)\|_{H^{-1}}\\ &\quad\le \sup_{\|\varphi\|_{H_0^1}=1}\int_0^1 |vv'\varphi|\,dx \le \|v\|_{L^2}\,\|v'\|_{L^2} \le \|v\|_{H^1}^2\end{aligned}$$

and thus the directional derivative $e_1'(y,u)$ is the Fréchet-derivative of e_1 at (y,u). The second Fréchet-derivative of e_1 is given by

$$\langle e_1''(y,u)(w,p)(v,q),\varphi\rangle_{H^{-1},H_0^1} = \int_0^1 (wv)'\varphi\,dx \quad \text{for } \varphi \in H_0^1(0,1) \tag{4.24}$$

for any directions (v,q), $(w,p) \in X$.

For the operator e_2 we calculate the following Fréchet-derivatives

$$e_2'(y,u)(v,q) = (v(0),v(1))\,, \quad e_2''(y,u)(w,p)(v,q) = (0,0) \tag{4.25}$$

for any directions $(v,q),(w,p) \in X$ such that the mapping $(y,u) \mapsto e''(y,u)$ is also Lipschitz-continuous on X.

3.3. The Regular Point Condition. Now we are going to prove that the operator $e'(y,u)$ is surjective for all $(y,u) \in H^1(0,1) \times L^2(\omega)$. In particular, Condition 2 on page 22 is satisfied.

THEOREM 3.3. *The operator $e'(y,u)$ is surjective for all $(y,u) \in X$.*

PROOF. The operator $e'(y,u)$ is surjective if and only if for all $(g,a,b) \in H^{-1}(0,1) \times \mathbb{R}^2$ there exists $(v,q) \in H^1(0,1) \times L^2(\omega)$ such that

$$\begin{cases} -\nu v'' + (yv)' - Bq = g & \text{in } H^{-1}(0,1)\,,\\ v(0) = a,\ v(1) = b\,. \end{cases} \tag{4.26}$$

First we make problem (4.26) homogeneous. Setting $v_d(x) = a + (b-a)x$ and $w = v - v_d$ we conclude that $w \in H_0^1(0,1)$ solves

$$-\nu w'' + (yw)' - Bq = \tilde{g} \qquad \text{in } H^{-1}(0,1)\,, \tag{4.27}$$

where

$$\tilde{g} = g - (b-a)y - v_d y' \in H^{-1}(0,1)\,,$$

if and only if $v = w + v_d$ is a solution of (4.26).

We shall apply the Fredholm alternative [**Kre78**, p. 437] and rewrite (4.27) in operator notation. For this purpose we introduce the linear operator $K \in L(H_0^1(0,1), H^{-1}(0,1))$ by $Kw = (yw)'$. This operator K is compact. To see this we take a bounded sequence $\{w_n\}_{n\in\mathbb{N}}$ in $H_0^1(0,1)$. Since $H_0^1(0,1)$ is compactly embedded into $L^2(0,1)$ by Lemma 3.5 on page 10, there exists a subsequence $\{w_{n_k}\}_{k\in\mathbb{N}}$ and $w \in H_0^1(0,1)$ such that $w_{n_k} \to w$ in $L^2(0,1)$ as $k \to \infty$. Hence,

$$\|Kw_{n_k} - Kw\|_{H^{-1}} \le \|y\|_{L^\infty} \|w_{n_k} - w\|_{L^2} \xrightarrow{k\to\infty} 0$$

so that the sequence $\{Kw_n\}_{n\in\mathbb{N}}$ contains a strongly convergent subsequence in $H^{-1}(0,1)$. From this we obtain that the operator K is compact. Let $D \in L(H_0^1(0,1), H^{-1}(0,1))$ be given by $-\nu\, w''$. This implies

$$(D+K)w = \tilde{g} + Bq \qquad \text{in } H^{-1}(0,1)\,.$$

From the Lax-Milgram theorem [**GT77**, p. 78] we infer that D^{-1} is a bounded linear operator. Setting $G = D + K$ we have

$$-D^{-1}G = \underbrace{(-D^{-1}K)}_{=T} - I\,.$$

For a given $\bar{g} \in H^{-1}(0,1)$ the equation $Gw = \bar{g}$ is solvable if and only if $(T - I)w = -D^{-1}\bar{g}$ is solvable. Applying the Fredholm alternative yields: If the equation $T^*f = f$ has only the trivial solution, then $Gw = \bar{g}$ is solvable for every $\bar{g} \in H^{-1}(0,1)$. Setting $\phi = D^{-1}f \in H_0^1(0,1)$ the equation $T^*f = f$ leads to

$$-\nu\phi'' - y\phi' = 0 \qquad \text{in } H^{-1}(0,1)\,.$$

Since $y \in C[0,1]$ holds [**Ada75**, p. 98], the weak maximum principle [**GT77**, p. 168] implies $\phi = 0$ in $[0,1]$. Therefore the equation $T^*f = f$ has only the trivial solution. Hence, G is surjective, and is it obvious that there exists (w,q) solving (4.27) by setting $q = 0$ and $\bar{g} = \tilde{g}$. $\square$

3.4. The First-Order Necessary Optimality Conditions. If the point $(y^*, u^*) \in X$ is an optimal solution to $(\mathbf{P_1})$ we infer from Theorem 3.3 the existence of Lagrange multipliers $\lambda^* \in H_0^1(0,1)$ and $\mu^* = (\mu_1^*, \mu_2^*) \in \mathbb{R}^2$

satisfying the first-order necessary optimality conditions. In particular,

$$\frac{\partial L}{\partial y}(y^*, u^*, \lambda^*, \mu^*) = 0 \quad \text{in } H^1(0,1)^*, \tag{4.28a}$$

$$\frac{\partial L}{\partial u}(y^*, u^*, \lambda^*, \mu^*) = 0 \quad \text{in } L^2(\omega). \tag{4.28b}$$

Using (4.28a) we derive

$$\begin{aligned} &\langle C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z), v\rangle_{(H^1)^*,H^1} \\ &\quad + \int_0^1 \nu(\lambda^*)'v' - y^*(\lambda^*)'v\,dx + v(0)\mu_1^* + v(1)\mu_2^* = 0 \end{aligned} \tag{4.29}$$

for all $v \in H^1(0,1)$, specifically, for $v \in H_0^1(0,1)$. We find

$$\int_0^1 \nu(\lambda^*)'v' - y^*(\lambda^*)'v\,dx = -\langle C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z), v\rangle_{(H^1)^*,H^1}. \tag{4.30}$$

From (4.30) we derive the following uniqueness and regularity result for the Lagrange multiplier λ^*.

Lemma 3.4. *Let $C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z) \in L^2(0,1)$. Then the Lagrange multipliers λ^* and μ^* associated with an optimal solution (y^*, u^*) are unique. Further, $\lambda^* \in H^2(0,1) \cap H_0^1(0,1)$ and*

$$\|\lambda^*\|_{L^\infty} \le c_1 \|C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z)\|_{L^2},$$

where $c_1 = \frac{1}{\nu}\left[1 + \exp\left(\frac{2}{\nu}\|y^*\|_{L^1}\right)\right]\exp\left(\frac{1}{\nu}\|y^*\|_{L^1}\right)$.

Proof. Applying a well-known regularity result [**GT77**, p. 176] the Lagrange multiplier λ^* belongs to $H^2(0,1) \cap H_0^1(0,1)$. Hence,

$$\left\langle -\nu(\lambda^*)'' - y^*(\lambda^*)' + C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z), v\right\rangle_{L^2} = 0 \tag{4.31}$$

for all $v \in H_0^1(0,1)$. As $H_0^1(0,1)$ is dense in $L^2(0,1)$, equation (4.31) is valid for all $v \in L^2(0,1)$. Thus, Lemma 4.8 on page 16 implies the uniqueness of the solution λ^* of (4.30). If we set $\beta = 1$ and $\beta_1 = 0$, the estimate follows from Remark 4.9 on page 16.

Now we prove the uniqueness of μ_1^* and μ_2^*. Using (4.29) we find

$$\nu(\lambda^*)'(1)v(1) - \nu(\lambda^*)'(0)v(0) + \mu_1^*v(0) + \mu_2^*v(1) = 0$$

for all $v \in H^1(0,1)$. Choosing v appropriately we derive

$$\mu_1^* = -\nu(\lambda^*)'(0), \quad \mu_2^* = \nu(\lambda^*)'(1)$$

such that both μ_1^* and μ_2^* are unique determined by λ^*. □

Finally, we consider equation (4.28b). We derive that $\sigma u^* = \lambda^*$ in ω. Thus, $u^* \in H^2(\omega)$ holds.

3.5. The Second-Order Sufficient Optimality Condition. Now the second-order sufficient optimality condition – Condition 3 on page 22 – will be considered for the optimal control problem ($\mathbf{P_1}$). For this purpose we calculate

$$\langle L''(y^*, u^*, \lambda^*, \mu^*)(v, q), (v, q)\rangle_{X^*,X} = \|Cv\|_{\mathcal{V}}^2 + \sigma\,\|q\|_{L^2(\omega)}^2 + 2\int_0^1 \lambda^* v v'\,dx$$

for all $(v, q) \in X$.

THEOREM 3.5. a) *Let $\mathcal{V} = H^1(0,1)$ and $C = I$. If $\|y^* - z\|_{L^2}$ is sufficiently small, then the second-order sufficient optimality condition holds. Moreover, the operator $L''(y^*, u^*, \lambda^*, \mu^*)$ is coercive on the whole space X.*

b) *Let $\mathcal{V} = L^2(0,1)$ and C be the injection of $H^1(0,1)$ into $L^2(0,1)$. If $\|y^* - z\|_{L^2}$ is sufficiently small, the second-order sufficient optimality condition holds.*

PROOF. a) For all $(v, q) \in X$ it follows that

$$\begin{aligned}\langle L''(y^*, u^*, \lambda^*, \mu^*)(v, q), (v, q)\rangle_{X^*,X} &= \|v\|_{H^1}^2 + \sigma\,\|q\|_{L^2(\omega)}^2 + 2\int_0^1 \lambda^* v v'\,dx \\ &\geq (1 - 2\,\|\lambda^*\|_{L^\infty})\,\|v\|_{H^1}^2 + \sigma\,\|q\|_{L^2(\omega)}^2\,.\end{aligned}$$

We set $\kappa = \min(\sigma, 1 - 2c_1\|y^* - z\|_{L^2})$, where $c_1 > 0$ is given by Lemma 3.4. If $\|y^* - z\|_{L^2} < \frac{1}{2c_1}$ we derive that $\kappa > 0$ and

$$\langle L''(y^*, u^*, \lambda^*, \mu^*)(v, q), (v, q)\rangle_{X^*,X} \geq \kappa\,\|(v, q)\|_X^2 \quad \text{for all } (v, q) \in X\,,$$

in particular, for $(v, q) \in \ker e'(y^*, u^*)$.

b) Remark 4.13 on page 18 yields that $\|v\|_{H_0^1} \leq c_2\,\|q\|_{L^2(\omega)}$ for all $(v, q) \in \ker e'(y^*, u^*)$, where c_2 is an upper bound for $c_\circ$ given by

$$c_2 = \exp\left(\frac{6}{\nu}\,\|y^*\|_{L^1}\right)\left(1 + \frac{\|y^*\|_{L^\infty}}{\nu}\right)^2 .$$

Using Lemma 3.4 and Lemma 3.4 on page 9 we thus derive for L'' at $(y^*, u^*, \lambda^*, \mu^*)$ in directions $(v, q) \in \ker e'(y^*, u^*)$:

$$\begin{aligned}&\langle L''(y^*, u^*, \lambda^*, \mu^*)(v, q), (v, q)\rangle_{X^*,X} \\ &\geq \|v\|_{L^2}^2 + \sigma\,\|q\|_{L^2(\omega)}^2 - 2\,\|\lambda^*\|_{L^\infty}\,\|v\|_{H^1}^2 \\ &\geq \sigma\,\|q\|_{L^2(\omega)}^2 - 2c_1\,\|y^* - z\|_{L^2}\,\|v\|_{H^1}^2 \\ &\geq \frac{\sigma}{2}\,\|q\|_{L^2(\omega)}^2 + \frac{\sigma}{2c_2}\,\|v\|_{H_0^1}^2 - 2c_1\,\|y^* - z\|_{L^2}\,\|v\|_{H^1}^2 \\ &\geq \frac{\sigma}{2}\,\|q\|_{L^2(\omega)}^2 + \left(\frac{\sigma}{4c_2} - 2c_1\|y^* - z\|_{L^2}\right)\|v\|_{H^1}^2\,.\end{aligned}$$

If $\|y^* - z\|_{L^2} < \frac{\sigma}{8c_1c_2}$, we obtain

$$\kappa = \min\left(\frac{\sigma}{2}, \frac{\sigma}{4c_2} - 2c_1\|y^* - z\|_{L^2}\right) > 0\,,$$

and the claim follows.

□

REMARK 3.6. The second-order sufficient optimality condition holds, if the residuum $y^* - z$ is small enough.

3.6. The Scheme of the Augmented Lagrangian-SQP Method. In this section we shall formulate Algorithm 1 on page 25 for our distributed control problem ($\mathbf{P_1}$).

a) *Choose* $(y^0, u^0, (\lambda^0, \mu_1^0, \mu_2^0)) \in \mathcal{Y} \times \mathcal{U} \times Y$, $c \geq 0$ *and put* $n = 0$.

b) *Calculate* $e^n \in H_0^1(0,1)$ *solving*

$$-(e^n)'' = e_1(y^n, u^n) \qquad in\ H^{-1}(0,1)$$

and define

$$\tilde{\lambda}^n = \lambda^n + c\,e^n,\ \tilde{\mu}_1^n = \mu_1^n + c\,(y^n(0) - \alpha)\,,\ \tilde{\mu}_2^n = \mu_2^n + c\,(y^n(1) - \beta)\ .$$

c) *Solve for* $(\bar{y}, \bar{u}, (\bar{\lambda}, \bar{\mu}_1, \bar{\mu}_2)) \in \mathcal{Y} \times \mathcal{U} \times Y$ *the following linear system:*

$$\begin{aligned}&\langle L''(y^n, u^n, \tilde{\lambda}^n, \tilde{\mu}_1^n, \tilde{\mu}_2^n)(\bar{y}, \bar{u}) + e'(y^n, u^n)^*(\bar{\lambda}, \bar{\mu}_1, \bar{\mu}_2), (v, q)\rangle_{X^*,X}\\ &\quad = \langle L''(y^n, u^n, \tilde{\lambda}^n, \tilde{\mu}_1^n, \tilde{\mu}_2^n)(y^n, u^n) + e'(y^n, u^n)^*(\tilde{\lambda}^n, \tilde{\mu}_1^n, \tilde{\mu}_2^n), (v, q)\rangle_{X^*,X}\\ &\qquad -\langle L'(y^n, u^n, \tilde{\lambda}^n, \tilde{\mu}_1^n, \tilde{\mu}_2^n), (v, q)\rangle_{X^*,X}\end{aligned}$$

$$\langle e'(y^n, u^n)(\bar{y}, \bar{u}), \varphi\rangle_{Y^*,Y} = \langle e'(y^n, u^n)(y^n, u^n) - e(y^n, u^n), \varphi\rangle_{Y^*,Y}$$

for all $(v, q, \varphi) \in \mathcal{Y} \times \mathcal{U} \times Y$.

d) *Set* $(y^{n+1}, u^{n+1}, \lambda^{n+1}, \mu_1^{n+1}, \mu_2^{n+1}) = (\bar{y}, \bar{u}, \bar{\lambda}, \bar{\mu}_1, \bar{\mu}_2)$, $n = n+1$ *and goto step* b).

4. The State Equation with Mixed Boundary Conditions

4.1. Boundary Value Problem. We have introduced the Hilbert space $\mathcal{H}$ in (1.8), which is a Hilbert space endowed with the inner product (1.9) and the associated norm (1.10), see Lemma 3.3 on page 9. Let $f \in \mathcal{H}^*$ and $\alpha, \beta \in \mathbb{R}$. We consider the *stationary Burgers equation*

$$-\nu y'' + yy' = f \qquad \text{in } (0,1) \tag{4.32a}$$

with mixed boundary conditions

$$y(0) = \alpha\,, \qquad y'(1) = \beta\,. \tag{4.32b}$$

$\nu > 0$ is called the *viscosity parameter.*

DEFINITION 4.1. A function $y \in H^1(0,1)$ is called a weak solution of (4.32) if

$$
(\mathbf{B_2}) \qquad \begin{cases} \int_0^1 (\nu y'\varphi' + yy'\varphi)\,dx = \nu\beta\varphi(1) + \langle f,\varphi\rangle_{\mathcal{H}^*,\mathcal{H}} & \text{for all } \varphi \in \mathcal{H}\,, \\ y(0) = \alpha \end{cases}
$$

is valid.

4.2. Existence and Uniqueness in Case of Homogeneous Boundary Conditions. Problem $(\mathbf{B_2})$ with homogeneous boundary conditions reads

$$
\begin{cases} \int_0^1 (\nu y'\varphi' + yy'\varphi)\,dx = \langle f,\varphi\rangle_{\mathcal{H}^*,\mathcal{H}} \quad \text{for all } \varphi \in \mathcal{H}\,, \\ y(0) = 0\,. \end{cases} \tag{4.33}
$$

We define the set

$$
M = \left\{ w \in \mathcal{H} : \|w\|_{\mathcal{H}} \le \sqrt{\|f\|_{\mathcal{H}^*}} \right\} \subset \mathcal{H}\,. \tag{4.34}
$$

Since M is a closed ball in $\mathcal{H}$, it follows that M is bounded, convex and closed in $\mathcal{H}$. We define $z = Tw$ via the mapping $T : M \to \mathcal{H}$ as follows: $z \in \mathcal{H}$ is the unique solution to

$$
\int_0^1 (\nu z'\varphi' + wz'\varphi)\,dx = \langle f,\varphi\rangle_{\mathcal{H}^*,\mathcal{H}} \qquad \text{for all } \varphi \in \mathcal{H}\,. \tag{4.35}
$$

The fixed points of T are the solutions of (4.33).

LEMMA 4.2. *If $\nu \ge 2\sqrt{\|f\|_{\mathcal{H}^*}}$, the mapping T is well-defined.*

PROOF. Since $w \in M$ holds, we have $\|w\|_{L^2} \le \sqrt{\|f\|_{\mathcal{H}^*}}$ by Lemma 3.4 on page 9. Setting

$$
a(z,\varphi) = \int_0^1 (\nu z'\varphi' + wz'\varphi)\,dx \qquad \text{for } z,\varphi \in \mathcal{H}
$$

we derive that $a(\cdot\,,\cdot)$ is continuous and coercive on $\mathcal{H} \times \mathcal{H}$:

$$
\begin{aligned}
|a(z,\varphi)| &\le \left(\nu + \sqrt{\|f\|_{\mathcal{H}^*}}\right) \|z\|_{\mathcal{H}}\, \|\varphi\|_{\mathcal{H}}\,, \\
a(z,z) &\ge \nu\,\|z\|_{\mathcal{H}}^2 - \|w\|_{L^2}\, \|z\|_{\mathcal{H}}\, \|z\|_{L^\infty} \ge \frac{\nu}{2}\,\|z\|_{\mathcal{H}}^2\,.
\end{aligned}
$$

By Lax-Milgram theorem [**GT77**, p. 78] the claim follows. ☐

THEOREM 4.3. *If $\nu \ge 2\sqrt{\|f\|_{\mathcal{H}^*}}$, then there exists at least one weak solution of (4.33) for any $f \in \mathcal{H}^*$.*

PROOF. To apply Theorem 2.9 we shall prove the requirements for the fixed point operator T. The operator T is well-defined by Lemma 4.2. Now we shall prove that $z := Tw \in M$ for $w \in M$. Setting $\varphi = z$ in (4.35) yields:

$$\nu \|z\|_{\mathcal{H}}^2 - \|w\|_{L^2} \|z\|_{\mathcal{H}} \|z\|_{L^\infty} \leq \|f\|_{\mathcal{H}^*} \|z\|_{\mathcal{H}}$$

and by Lemma 3.4 on page 9 and assumption $\nu \geq 2\sqrt{\|f\|_{\mathcal{H}^*}}$:

$$\|z\|_{\mathcal{H}} \leq \frac{\|f\|_{\mathcal{H}^*}}{\nu - \sqrt{\|f\|_{\mathcal{H}^*}}} \leq \sqrt{\|f\|_{\mathcal{H}^*}}\,. \tag{4.36}$$

Thus, $z \in M$. Next we show that the mapping T is compact. Let $\{w_k\}_{k\in\mathbb{N}}$ be a bounded sequence in M. By Remark 3.6 on page 10 and $\mathcal{H} \subset H^1(0,1)$ we have a subsequence, which we also denote by $\{w_k\}_{k\in\mathbb{N}}$ for convenience, that converges in $L^4(0,1)$. Therefore, we obtain for any $w_n, w_m \in \{w_k\}_{k\in\mathbb{N}}$ and $z_n = Tw_n$, $z_m = Tw_m$

$$\int_0^1 (\nu z_n' \varphi' + w_n z_n' \varphi)\, dx = \langle f, \varphi\rangle_{\mathcal{H}^*,\mathcal{H}} \quad \text{for all } \varphi \in \mathcal{H}\,,$$
$$\int_0^1 (\nu z_m' \varphi' + w_m z_m' \varphi)\, dx = \langle f, \varphi\rangle_{\mathcal{H}^*,\mathcal{H}} \quad \text{for all } \varphi \in \mathcal{H}$$

and subtracting the two equations above we derive

$$\int_0^1 (\nu(z_n - z_m)'\varphi' + w_n(z_n - z_m)'\,\varphi)\, dx = \int_0^1 (w_m - w_n) z_m' \varphi\, dx$$

for all $\varphi \in \mathcal{H}$. Taking $\varphi = z_n - z_m \in \mathcal{H}$ we obtain

$$\nu \|z_n - z_m\|_{\mathcal{H}}^2 - \|w\|_{\mathcal{H}} \|z_n - z_m\|_{\mathcal{H}}^2 \leq \|w_m - w_n\|_{L^4} \|z_m\|_{\mathcal{H}} \|z_n - z_m\|_{\mathcal{H}}\,.$$

As $z_m \in M$ and $\nu > \sqrt{\|f\|_{\mathcal{H}^*}}$ we find

$$\|z_n - z_m\|_{\mathcal{H}} \leq \frac{\|f\|_{\mathcal{H}^*}}{\nu - \sqrt{\|f\|_{\mathcal{H}^*}}} \|w_m - w_n\|_{L^4}\,.$$

Since $\{w_k\}_{k\in\mathbb{N}}$ converges in $L^4(0,1)$, it follows that $\{z_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$. Hence, $\{z_k\}_{k\in\mathbb{N}}$ converges in $\mathcal{H}$ and T is compact.

Thus, T satisfies the requirements of Theorem 2.9 such that there exists a fixed point $y \in M$. $\square$

REMARK 4.4. In contrast to Theorem 2.10 we need a restriction on the viscosity parameter to prove existence. From $y \in M$ we get the following useful a-priori bound for the solution of (4.33)

$$\|y\|_{\mathcal{H}} \leq \sqrt{\|f\|_{\mathcal{H}^*}}$$

for $\nu \geq 2\sqrt{\|f\|_{\mathcal{H}^*}}$.

THEOREM 4.5. *The solution of (4.33) is unique if* $\nu \geq 2\sqrt{\|f\|_{\mathcal{H}^*}}$.

PROOF. Assume that $y_1, y_2 \in \mathcal{H}$ are two solutions of (4.33). Then $z = y_1 - y_2$ solves:

$$\begin{cases} \displaystyle\int_0^1 \nu (y_1 - y_2)'\varphi' + \frac{1}{2}(y_1^2 - y_2^2)'\varphi \, dx = 0 \qquad \text{for all } \varphi \in \mathcal{H}, \\ z(0) = 0 . \end{cases}$$

Setting $\varphi = y_1 - y_2$ we obtain

$$\nu \, \|y_1 - y_2\|_{\mathcal{H}}^2 - \frac{1}{2}\int_0^1 (y_1^2 - y_2^2)(y_1 - y_2)' \, dx = 0 . \tag{4.37}$$

By Remark 4.4 we derive $\|y_1 + y_2\|_{\mathcal{H}} \le 2\sqrt{\|f\|_{\mathcal{H}^*}}$. Then we obtain

$$\begin{aligned} \int_0^1 (y_1^2 - y_2^2)(y_1 - y_2)' \, dx &= \int_0^1 (y_1 + y_2)(y_1 - y_2)(y_1 - y_2)' \, dx \\ &\le \|y_1 + y_2\|_{L^\infty} \|y_1 - y_2\|_{L^2} \|y_1 - y_2\|_{\mathcal{H}} \\ &\le \|y_1 + y_2\|_{\mathcal{H}} \|y_1 - y_2\|_{\mathcal{H}}^2 \\ &\le 2\sqrt{\|f\|_{\mathcal{H}^*}} \, \|y_1 - y_2\|_{\mathcal{H}}^2 . \end{aligned}$$

Hence, we have

$$\left(\nu - \sqrt{\|f\|_{\mathcal{H}^*}}\right) \|y_1 - y_2\|_{\mathcal{H}}^2 \le 0 .$$

From $\nu > \sqrt{\|f\|_{\mathcal{H}^*}}$ we infer that

$$\|y_1 - y_2\|_{\mathcal{H}} = 0 .$$

Since $y_1(0) = y_2(0) = 0$, we conclude that $y_1 = y_2$. ☐

4.3. Non-Homogeneous Mixed Boundary Conditions.

THEOREM 4.6. *Let $f \in \mathcal{H}^*$ and $\alpha, \beta \in \mathbb{R}$. If ν is sufficiently large, then there exists a unique weak solution $y \in H^1(0,1)$ of problem* ($\mathbf{B_2}$)*.*

PROOF. We begin by transforming ($\mathbf{B_2}$) into an equation with a homogeneous Dirichlet boundary condition at $x = 0$. For this purpose let

$$y_\circ(x) = \begin{cases} (-2)^n \alpha \left(x - \frac{1}{2}\right)^n & \text{for all } x \in \left[0, \frac{1}{2}\right) , \\ 0 & \text{for all } x \in \left[\frac{1}{2}, 1\right] , \end{cases} \tag{4.38}$$

$n \in \mathbb{N}$. It follows that $y_\circ \in H^1(0,1)$ and $y_\circ(0) = \alpha$. We introduce $\tilde{f} \in \mathcal{H}^*$ by

$$\langle \tilde{f}, \varphi \rangle_{\mathcal{H}^*,\mathcal{H}} = \nu\beta\varphi(1) + \langle f, \varphi \rangle_{\mathcal{H}^*,\mathcal{H}} - \int_0^1 y_\circ y_\circ' \varphi + \nu y_\circ' \varphi' \, dx \quad \text{for all } \varphi \in \mathcal{H} .$$

Let $\nu > 2\sqrt{\|\tilde{f}\|_{\mathcal{H}^*}}$. Consequently, $\varepsilon = \nu - 2\sqrt{\|\tilde{f}\|_{\mathcal{H}^*}} > 0$. We choose

$$n \ge \max\left(1, \frac{9\alpha^2 - 2\varepsilon^2}{4\varepsilon^2}\right)$$

such that $3\,\|y_\circ\|_{L^2} < \varepsilon$ from definition (4.38). Further, $z = y - y_\circ \in \mathcal{H}$ solves problem ($\mathbf{B_2}$) with homogeneous Dirichlet boundary condition at $x = 0$

$$(4.39)\qquad \begin{cases} \displaystyle\int_0^1 \nu z'\varphi' + (z + y_\circ)z' + y_\circ' z\varphi\,dx = \langle \tilde{f}, \varphi\rangle_{\mathcal{H}^*,\mathcal{H}} \quad \text{for all } \varphi \in \mathcal{H}\,, \\ z(0) = 0\,. \end{cases}$$

To apply the Schauder fixed point theorem (Theorem 2.9) we define

$$M = \left\{ w \in \mathcal{H} : \|w\|_{\mathcal{H}} \le \sqrt{\|\tilde{f}\|_{\mathcal{H}^*}} \right\} \subset \mathcal{H}\,.$$

Clearly, M is bounded, convex and closed in $\mathcal{H}$. We define $z = Tw$ via the mapping $T : M \to \mathcal{H}$ as follows: $z \in \mathcal{H}$ is the unique solution to

$$(4.40)\qquad \begin{cases} \displaystyle\int_0^1 \nu z'\varphi' + (w + y_\circ)z'\,\varphi + y_\circ' z\varphi\,dx = \langle \tilde{f}, \varphi\rangle_{\mathcal{H}^*,\mathcal{H}} \quad \text{for all } \varphi \in \mathcal{H}\,, \\ y(0) = 0\,. \end{cases}$$

The fixed points of T are the solutions of (4.39). Now we proceed to prove that the fixed point operator T satisfies the requirements of the Schauder fixed point theorem. Therefore, we define the bilinear form $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by

$$a(z, \varphi) = \int_0^1 (\nu z'\varphi' + (w + y_\circ)z'\,\varphi + y_\circ' z\varphi)\,dx\,.$$

$a(\cdot\,,\cdot)$ is continuous on $\mathcal{H} \times \mathcal{H}$:

$$\begin{aligned} |a(z, \varphi)| &\le (\nu + 3\,\|y_\circ\|_{L^2} + \|w\|_{\mathcal{H}})\,\|z\|_{\mathcal{H}}\,\|\varphi\|_{\mathcal{H}} \\ &< \left(\nu + \varepsilon + \sqrt{\|\tilde{f}\|_{\mathcal{H}^*}}\right)\|z\|_{\mathcal{H}}\,\|\varphi\|_{\mathcal{H}}\,. \end{aligned}$$

Setting $\varphi = z$ we derive

$$a(z, z) \ge \Big(\underbrace{\nu - \varepsilon - \sqrt{\|\tilde{f}\|_{\mathcal{H}^*}}}_{=\sqrt{\|\tilde{f}\|_{\mathcal{H}^*}}}\Big)\|z\|_{\mathcal{H}}\,,$$

i.e. $a(\cdot\,,\cdot)$ is coercive. By Lax-Milgram theorem [**GT77**, p. 78] there exists a unique solution for (4.40) for any $\tilde{f} \in \mathcal{H}^*$. Hence, T is well-defined. From (4.40) we conclude that

$$\|z\|_{\mathcal{H}} \le \frac{1}{\nu - \varepsilon - \sqrt{\|\tilde{f}\|_{\mathcal{H}^*}}}\,\|\tilde{f}\|_{\mathcal{H}^*} \le \frac{1}{\sqrt{\|\tilde{f}\|_{\mathcal{H}^*}}}\,\|\tilde{f}\|_{\mathcal{H}^*} = \sqrt{\|\tilde{f}\|_{\mathcal{H}^*}}\,.$$

Consequently, $z = Tw \in M$. If $z_1 = Tw_1$ and $z_2 = Tw_2$ we obtain that

$$\|z_1 - z_2\|_{\mathcal{H}} \le \frac{\|f\|_{\mathcal{H}^*}}{\nu - \sqrt{\|f\|_{\mathcal{H}^*}} - \varepsilon}\,\|w_2 - w_1\|_{L^4}\,.$$

This estimate shows that T is compact. By Theorem 2.9 there exists an element $y \in M$ solving $(\mathbf{B_2})$. $\square$

REMARK 4.7. The uniqueness follows again if the viscosity parameter is large with respect to the $\mathcal{H}^*$-norm of f.

4.4. Regularity of the Solution. The regularity of the weak solution to $(\mathbf{P_1})$ depends on the regularity of the right-hand side f.

THEOREM 4.8. *Let ν satisfy the assumptions of Theorem 4.3 and let f belong to $H^m(0,1)$, $m \geq 0$. Then the solution of* $(\mathbf{B_2})$ *is an element of $H^{m+2}(0,1)$.*

For the proof we refer the reader to [**GT77**, p. 176].

REMARK 4.9. Under the hypotheses of Theorem 4.8 we infer from Lemma 3.7 on page 10 that $y \in C^{m+1}[0,1]$ is valid.

4.5. Classical Existence Result. We consider the general mixed boundary value problem

$$\left\{ \begin{array}{ll} y''(x) = g(x, y, y') & \text{in } (0,1)\,, \\ y(0) = \alpha,\, y'(1) = \beta\,, & \end{array} \right. \tag{4.41}$$

in which g is a continuous function on $[0,1] \times \mathbb{R}^2$. Let there exist a C^2-subsolution y_l and a C^2-supersolution y_u, i.e. y_l and y_u possess the following properties

$$\left\{ \begin{array}{l} y_l \leq y_u \quad \text{in } [0,1]\,, \\ y_l'' \geq g(x, y_l, y_l'),\, y_u'' \leq g(x, y_u, y_u')\,, \\ y_l(0) \leq \alpha \leq y_u(0),\, y_l'(1) \leq \beta \leq y_u'(1)\,. \end{array} \right.$$

For the proof of the following lemma we refer the reader to [**Hei74**, p. 7].

LEMMA 4.10. *Let $f \in C[0,1]$ and y_l and y_u be a C^2-sub- and a C^2-supersolution of (4.41), respectively. Moreover, the function $g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is assumed to satisfy the Nagumo-condition*

$$g(x, y, y') = O(|y'|^2) \quad \textit{as} \quad |y'| \to \infty \quad \textit{for all } (x, y) \in [0,1] \times [y_l, y_u]\,.$$

Then the mixed boundary value problem (4.41) *has a solution $y \in C^2[0,1]$ satisfying $y_l \leq y \leq y_u$ in $[0,1]$.*

Now we investigate

$$\left\{ \begin{array}{l} y'' = \frac{1}{\nu}\,(y\,y' - f) = g(\cdot, y, y') \quad \text{in } (0,1)\,, \\ y(0) = \alpha,\, y'(1) = \beta\,. \end{array} \right. \tag{4.42}$$

We choose

$$\begin{aligned} y_l(x) &= -\min(\alpha, -1 - \|f\|_{C[0,1]}) - x\,, \\ y_u(x) &= \max\Big(\alpha, \|f\|_{C[0,1]}\Big) + x \end{aligned}$$

if $\beta = 0$ and

$$\begin{aligned} y_l(x) &= \min\left(-\frac{1}{\beta}\,\|f\|_{C[0,1]} - \beta, \alpha\right) + \beta x\,, \\ y_u(x) &= \max\left(\frac{1}{\beta}\,\|f\|_{C[0,1]}, \alpha\right) + \beta x\,, \end{aligned}$$

otherwise. Then, from Lemma 4.10 we derive existence of a C^2-solution of (4.42).

LEMMA 4.11. *For $f \in C[0,1]$ the mixed boundary value problem*

$$\begin{cases} -\nu\, y'' + y\, y' = f & \text{in } (0,1)\,, \\ y(0) = \alpha,\ y'(1) = \beta \end{cases}$$

has a (classical) solution $y \in C^2[0,1]$ satisfying $y_l \le y \le y_u$ in $[0,1]$.

5. The Boundary Control Problem

Now we shall formulate the boundary control problem for the stationary Burgers equation with mixed boundary conditions. For this purpose let $f \in L^2(0,1)$. For a control $u \in \mathbb{R}$ the state $y \in H^1(0,1)$ is given by the solution of

$$\begin{cases} -\nu y'' + y y' = f & \text{in } (0,1)\,, \\ y(0) = \alpha,\ y'(1) = u\,. \end{cases} \tag{4.43}$$

We make the following assumption throughout this section.

CONDITION 14. Let the viscosity parameter ν be sufficiently large such that there exists a weak solution to (4.43) by Theorem 4.6.

With every control $u \in \mathbb{R}$ we associate the cost of tracking type

$$J(y,u) = \frac{1}{2}\,\|Cy - z\|_{\mathcal{V}}^2 + \frac{\sigma}{2}\,u^2\,,$$

where C, $z \in \mathcal{V}$ and $\sigma > 0$ are introduced in Section 3.

The control problem then is

$$\min J(y,u) \quad \text{such that } (y,u) \text{ solves (4.43).} \tag{4.44}$$

In the context of Chapter 2, Section 2, we set $\mathcal{Y} = H^1(0,1)$, $\mathcal{U} = \mathbb{R}$ and $Y = \mathcal{H} \times \mathbb{R}$ and identify with $\mathcal{H}^* \times \mathbb{R}$ the dual of Y. We introduce the operator

$$e = (e_1, e_2) : H^1(0,1) \times \mathbb{R} \to \mathcal{H}^* \times \mathbb{R},$$

whose action is defined by

$$\langle e_1(y,u), \varphi\rangle_{\mathcal{H}^*,\mathcal{H}} = \int_0^1 (\nu y'\varphi' + yy'\,\varphi - f\varphi)\,dx - \nu u\varphi(1) \qquad \text{for } \varphi \in \mathcal{H}\,,$$

and $e_2(y,u) = y(0) - \alpha$. Then we write (4.44) in the following form

$$\text{minimize } J(y,u) \quad \text{subject to } e(y,u) = 0. \tag{$\mathbf{P_2}$}$$

The Lagrangian associated with $(\mathbf{P_2})$ is expressed as

$$L(y,u,\lambda,\mu) = J(y,u) + \langle e_1(y,u),\lambda\rangle_{\mathcal{H}^*,\mathcal{H}} + e_2(y,u)\,\mu\,.$$

5.1. The Existence of an Optimal Solution.

THEOREM 5.1. *There exists a solution of* $(\mathbf{P_2})$.

PROOF. Let $(y,u) \in X$ satisfy the equation $e(y,u) = 0$. From the proof of Theorem 4.6 we derive that $\|(y,u)\|_X \to \infty$ implies that $|u| \to \infty$. Hence,

$$J(y,u) \geq \frac{\sigma}{2}\,u^2\,,$$

leads to

$$J(y,u) \longrightarrow +\infty \quad \text{for} \quad \|(y,u)\|_X \to \infty\,, \tag{4.45}$$

such that J is unbounded for unbounded sequences in X. Since weak convergence in $\mathbb{R}$ is equal to strong convergence, we analogously prove as in the proof of Theorem 3.1 that J is weakly lower semi-continuous and that

$$\lim_{n\to\infty} \nu u^n \varphi(1) = \nu u^* \varphi(1) \qquad \text{for all } \varphi \in \mathcal{H}$$

and for any sequence $\{u^n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ satisfying $u^n \rightharpoonup u^*$ as $n \to \infty$.

The rest of the proof is analogous to that of Theorem 3.1. □

5.2. The Fréchet-Derivatives of J and e. In Section 5.1 we have proved that $(\mathbf{P_2})$ has a solution. The requirements of Condition 1 on page 21 are clearly met by the mapping J. Let $(y,u) \in X$ be arbitrary. Then the actions of the Fréchet-derivatives of J are given by

$$\begin{aligned} \langle J'(y,u),(v,q)\rangle_{X^*,X} &= \langle Cy - z, Cv\rangle_{\mathcal{V}} + \sigma u q\,, \\ \langle J''(y,u)(w,p),(v,q)\rangle_{X^*,X} &= \langle Cw, Cv\rangle_{\mathcal{V}} + \sigma p q \end{aligned}$$

for any directions (v,q), $(w,p) \in X$. Thus, the mapping $(y,u) \mapsto J''(y,u)$ is Lipschitz-continuous on the whole space X.

As we discuss in Section 3.2 we analogously conclude that for all $(y,u) \in X$ the operator $e(y,u)$ is twice Fréchet-differentiable. The actions of the Fréchet-derivatives of e are given by

(4.46a)

$$\begin{aligned} &\langle e_1'(y,u)(v,q),\varphi\rangle_{\mathcal{H}^*,\mathcal{H}} = \int_0^1 \nu v'\varphi' + (vy)'\varphi\,dx - \nu q\varphi(1) \quad \text{for } \varphi \in \mathcal{H}\,, \\ &e_2'(y,u)(v,q) = v(0) \end{aligned}$$

and

$$\begin{aligned} &\langle e_1''(y,u)(w,p)(v,q),\varphi\rangle_{\mathcal{H}^*,\mathcal{H}} = \int_0^1 (vw)'\varphi\,dx \quad \text{for } \varphi \in \mathcal{H}\,, \\ &e_2''(y,u)(w,p)(v,q) = 0 \end{aligned} \tag{4.46b}$$

for any directions $(v,q),(w,p) \in X$.

5.3. The Regular Point Condition. Now we are going to prove that the operator $e'(y,u)$ is surjective for all $(y,u) \in \mathcal{H} \times \mathbb{R}$, in particular, Condition 2 on page 22 is satisfied.

THEOREM 5.2. *The operator $e'(y,u)$ is surjective for all $(y,u) \in \mathcal{H} \times \mathbb{R}$.*

PROOF. The operator $e'(y,u)$ is surjective if and only if for all $(g,a) \in \mathcal{H} \times \mathbb{R}$ there exists $(v,q) \in X$ such that

$$\begin{cases} \displaystyle\int_0^1 \nu v'\varphi' + (yv)'\varphi\,dx - \nu q\varphi(1) = \langle g,\varphi\rangle_{\mathcal{H}^*,\mathcal{H}} \text{ for all } \varphi \in \mathcal{H}\,, \\ v(0) = a\,. \end{cases} \tag{4.47}$$

First we make problem (4.47) homogeneous. Setting $v_d(x) = a(1-x)$ and $w = v - v_d$ we conclude that $w \in \mathcal{H}$ solves

$$\int_0^1 \nu w'\varphi' + (yw)'\varphi\,dx - \nu q\varphi(1) = \langle \tilde{g},\varphi\rangle_{\mathcal{H}^*,\mathcal{H}} \quad \text{for all } \varphi \in \mathcal{H} \tag{4.48}$$

where

$$\tilde{g} = g - ay - v_d y' \in \mathcal{H}^*\,,$$

if and only if $v = w + v_d$ is a solution of (4.47). Setting $q = \frac{1}{\nu}\,y(1)w(1)$ equation (4.48) reads

$$\int_0^1 \nu w'\varphi' + (yw)'\varphi\,dx - y(1)w(1)\varphi(1) = \langle \tilde{g},\varphi\rangle_{\mathcal{H}^*,\mathcal{H}} \text{ for all } \varphi \in \mathcal{H}\,. \tag{4.49}$$

We shall apply the Fredholm alternative [**Kre78**, p. 437] and rewrite (4.49) in operator notation. For this purpose we introduce the operator $K \in L(\mathcal{H},\mathcal{H}^*)$, whose action is given by

$$\langle Kw,\varphi\rangle_{\mathcal{H}^*,\mathcal{H}} = \int_0^1 (yw)'\varphi\,dx - y(1)w(1)\varphi(1) \qquad \text{for } \varphi \in \mathcal{H}\,.$$

The operator K is compact. To see this we take a bounded sequence $\{w_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$. Since $\mathcal{H} \subset H^1(0,1) \hookrightarrow\hookrightarrow L^2(0,1)$ by Lemma 3.5 on page 10 there exists a subsequence $\{w_{n_k}\}_{k\in\mathbb{N}}$ and $w \in \mathcal{H}$ such that $w_{n_k} \to w$ in $L^2(0,1)$ as $k \to \infty$. Hence,

$$\|Kw_{n_k} - Kw\|_{\mathcal{H}^*} \le \|y\|_{L^\infty}\,\|w_{n_k} - w\|_{L^2} \overset{k\to\infty}{\longrightarrow} 0$$

so that the sequence $\{Kw_n\}_{n\in\mathbb{N}}$ contains a strongly convergent subsequence in $\mathcal{H}^*$. From this we obtain that the operator K and thus K^* is compact [**Kre78**, p. 407 and 416]. Let $D \in L(\mathcal{H},\mathcal{H}^*)$ be given by $-\nu\, w''$ with a homogeneous Neumann boundary condition at $x = 1$. Then we are able to write (4.49) in the form

$$(D + K)w = \tilde{g} \qquad \text{in } \mathcal{H}^*\,.$$

From the Lax-Milgram theorem [**GT77**, p. 78] we infer that D^{-1} is a bounded linear operator. Therefore, we have for $G = D + K$

$$-D^{-1}G = \underbrace{(-D^{-1}K)}_{=T} - I\,.$$

For a given $\bar{g}$ the equation $Gw = \bar{g}$ is solvable if and only if $(T-I)w = -D^{-1}\bar{g}$ is solvable. Applying the Fredholm alternative yields: If the equation $T^*f = f$ has only the trivial solution, then $Gw = \bar{g}$ is solvable for every $\bar{g} \in \mathcal{H}^*$. Setting $\phi = D^{-1}f \in \mathcal{H}$ the equation $T^*f = f$ leads to

$$\begin{cases} -\nu\phi'' - y\phi' = 0 & 0 < x < 1\,, \\ \phi(0) = \phi'(1) = 0\,. \end{cases}$$

Applying Remark 4.7 on page 15 with $\beta = 0$ and $\beta_1 = 1$ implies that $\phi = 0$ in $[0,1]$. Therefore, the equation $T^*f = f$ has only the trivial solution. Hence, G is surjective, and is it obvious that there exists w solving (4.49) by setting $\bar{g} = \tilde{g}$. ☐

5.4. The First-Order Necessary Optimality Conditions. If the point $(y^*, u^*) \in X$ is an optimal solution to $(\mathbf{P_2})$ we infer from Theorem 5.2 the existence of Lagrange multipliers $\lambda^* \in \mathcal{H}$ and $\mu^* \in \mathbb{R}$ satisfying the first-order necessary optimality conditions. Specifically,

$$\frac{\partial L}{\partial y}(y^*, u^*, \lambda^*, \mu^*) = 0 \quad \text{in } H^1(0,1)^*\,, \tag{4.50a}$$

$$\frac{\partial L}{\partial u}(y^*, u^*, \lambda^*, \mu^*) = 0\,. \tag{4.50b}$$

Using (4.50a) we derive

(4.51)

$$\langle C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z), v\rangle_{(H^1)^*,H^1} + \int_0^1 \nu(\lambda^*)'v' + (y^*v)'\lambda^*\,dx + v(0)\mu^* = 0$$

for all $v \in H^1(0,1)$, in particular, for $v \in H_0^1(0,1)$. For this choice of v we find

$$\int_0^1 \nu(\lambda^*)'v' - y^*(\lambda^*)'v\,dx = -\langle C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z), v\rangle_{(H^1)^*,H^1}\,. \tag{4.52}$$

LEMMA 5.3. *Let $C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z) \in L^2(0,1)$. Then $\lambda^* \in H^2(0,1) \cap \mathcal{H}$. If*

$$\nu \neq \frac{y^*(1)\exp(-\frac{1}{\nu}\int_0^1 y^*(s)\,ds)}{\int_0^1 \exp(-\frac{1}{\nu}\int_0^t y^*(s)\,ds)dt}\,,$$

the Lagrange multipliers λ^* and μ^* associated with an optimal solution (y^*, u^*) are unique. Further, there exists a constant $c_\lambda > 0$ such that

$$\|\lambda^*\|_{L^\infty} \leq c_\lambda \|C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z)\|_{L^2}\,.$$

PROOF. Since $y^* \in C[0,1]$ and $C^*\mathfrak{J}_\mathcal{V}^{-1}(Cy^*-z) \in L^2(0,1)$ hold, $\lambda^* \in \mathcal{H}$ satisfies

$$-\nu(\lambda^*)'' = y^*(\lambda^*)' - C^*\mathfrak{J}_\mathcal{V}^{-1}(Cy^*-z) \in L^2(0,1)\,.$$

Applying the regularity result in [**GT77**, p. 176] the Lagrange-multiplier λ^* belongs to $H^2(0,1) \cap \mathcal{H}$. Thus, we derive from (4.52)

$$\big\langle -\nu(\lambda^*)'' - y^*(\lambda^*)' + C^*\mathfrak{J}_\mathcal{V}^{-1}(Cy^*-z), v\big\rangle_{L^2} = 0 \tag{4.53}$$

for all $v \in H_0^1(0,1)$. As $H_0^1(0,1)$ is dense in $L^2(0,1)$, equation (4.53) is valid for all $v \in L^2(0,1)$. Using (4.51) we find

$$\nu(\lambda^*)'(1)v(1) - \nu(\lambda^*)'(0)v(0) + y^*(1)\lambda^*(1)v(1) + \mu^* v(0) = 0$$

for all $v \in H^1(0,1)$. Choosing v appropriately we derive

$$\mu^* = -\nu(\lambda^*)'(0)\,, \quad y^*(1)\lambda^*(1) = \nu(\lambda^*)'(1)$$

such that μ^* is unique determined by λ^*. If we set

$$c_* = \nu \int_0^1 \exp\left(-\frac{1}{\nu}\int_0^t y^*(s)\,ds\right)dt - y^*(1)\exp\left(-\frac{1}{\nu}\int_0^1 y^*(s)\,ds\right) \neq 0$$

and

$$c_\lambda = \frac{1}{\nu}\exp\left(\frac{1}{\nu}\,\|y^*\|_{L^1}\right)\left(1 + \frac{\nu}{|c_*|}\exp\left(\frac{1}{\nu}\,\|y^*\|_{L^1}\right)\right)\,,$$

the estimate follows from Remark 4.9 on page 16. ☐

Finally, we consider equation (4.50b). We derive that

$$\lambda^*(1) = \frac{\sigma u^*}{\nu}\,.$$

holds.

5.5. The Second-Order Sufficient Optimality Condition. Now the second-order sufficient optimality condition – Condition 3 on page 22 – will be considered for the optimal control problem ($\mathbf{P_2}$). For this purpose we calculate

$$\langle L''(y^*,u^*,\lambda^*,\mu^*)(v,q),(v,q)\rangle_{X^*,X} = \|Cv\|_\mathcal{V}^2 + \sigma\,|q|^2 + 2\int_0^1 \lambda^* v v'\,dx$$

for all $(v,q) \in X$. To prove Theorem 5.5-b) we need the following estimate.

LEMMA 5.4. *If*

$$\nu \neq -y^*(1)\int_0^1 \exp\left(-\frac{1}{\nu}\int_0^1 y^*(s)\,ds\right)dt \tag{4.54}$$

holds, then there exists a constant $c_{\ker} > 0$ *such that*

$$\|v\|_{H^1} \leq c_{\ker}\,|q| \qquad \textit{for all } (v,q) \in \ker e'(y^*,u^*)\,.$$

PROOF. Let $(v, q) \in \ker e'(y^*, u^*)$. Hence, v solves

$$\begin{cases} -\nu v'' + (y^* v)' = 0 \quad \text{in } (0,1)\,, \\ v(0) = 0,\ v'(1) = q\,. \end{cases} \tag{4.55}$$

First we make problem (4.55) homogeneous. For this purpose we introduce $v_\circ(x) = qx \in \mathcal{H}$. Then we infer from (4.55) that $w = v - v_\circ$ is a solution of

$$\begin{cases} -\nu w'' + (y^* w)' = -(y^* v_\circ)' \quad \text{in } (0,1)\,, \\ w(0) = 0,\ w'(1) = 0\,. \end{cases} \tag{4.56}$$

Relation (4.54) leads to

$$c_* = \frac{y^*(1)}{\nu} \int_0^1 \exp\left(\frac{1}{\nu}\int_t^1 y^*(s)\,ds\right) dt + 1 \neq 0$$

such that (4.56) is uniquely solvable by Remark 4.11 on page 17. Applying Remark 4.13 on page 18 we have

$$\|w\|_{\mathcal{H}} \leq 2c_\circ \|y^*\|_{H^1} |q|\,,$$

where

$$c_\circ = \exp\left(\frac{3}{\nu}\|y^*\|_{L^1}\right)\left(1 + \frac{\|y^*\|_{L^\infty}}{\nu}\right)^2 \left[1 + \frac{|y^*(1)|}{\nu |c_*|} \exp\left(\frac{1}{\nu}\|y^*\|_{L^1}\right)\right].$$

By Lemma 3.4 on page 9 and by triangle inequality we thus conclude that

$$\|v\|_{H^1} \leq \sqrt{2}\,(\|v - v_\circ\|_{\mathcal{H}} + \|v_\circ\|_{\mathcal{H}}) \leq \underbrace{\sqrt{2}\,(1 + c_\circ \|y^*\|_{H^1})}_{=c_{\ker}} |q|\,,$$

which gives the claim. $\square$

THEOREM 5.5. *Assume that*

$$\nu \neq \frac{y^*(1)\exp(-\frac{1}{\nu}\int_0^1 y^*(s)\,ds)}{\int_0^1 \exp(-\frac{1}{\nu}\int_0^t y^*(s)\,ds)dt}$$

holds.

a) *Let $\mathcal{V} = H^1(0,1)$, $C = I$. If the residue $\|y^* - z\|_{L^2}$ is sufficiently small, then Condition 3 on page 22 is satisfied. Moreover, the operator $L''(y^*, u^*, \lambda^*, \mu^*)$ is coercive on the whole space X.*

b) *Let $\mathcal{V} = L^2(0,1)$, C is the injection of $H^1(0,1)$ into $L^2(0,1)$. If*

$$\nu \neq -y^*(1) \int_0^1 \exp\left(-\frac{1}{\nu}\int_0^1 y^*(s)\,ds\right) dt$$

is valid and the residue $\|y^* - z\|_{L^2}$ is sufficiently small, then Condition 3 on page 22 is satisfied.

PROOF. a) For all $(v,q) \in X$ it follows

$$\begin{aligned}\langle L''(y^*,u^*,\lambda^*,\mu^*)(v,q),(v,q)\rangle_{X^*,X} &= \|v\|^2_{H^1} + \sigma\,|q|^2 + 2\int_0^1 \lambda^* v v'\,dx\,,\\ &\geq (1-2\,\|\lambda^*\|_{L^\infty})\,\|v\|^2_{H^1} + \sigma\,|q|^2\,.\end{aligned}$$

We set $\kappa = \min\left(\sigma, 1-2c_\lambda\|y^*-z\|_{L^2}\right)$, where $c_\lambda > 0$ is given by Lemma 5.3. If $\|y^*-z\|_{L^2} < \frac{1}{2c_\lambda}$ we derive $\kappa > 0$ and

$$\langle L''(y^*,u^*,\lambda^*,\mu^*)(v,q),(v,q)\rangle_{X^*,X} \geq \kappa\,\|(v,q)\|^2_X \quad \text{for all } (v,q)\in X\,,$$

in particular, for $(v,q)\in\ker e'(y^*,u^*)$.

b) Lemma 5.4 yields that $\|v\|_{H^1} \leq c_{\text{ker}}\,|q|$ for all $(v,q) \in \ker e'(y^*,u^*)$. Using Lemma 3.4 on page 9 we thus derive for $(v,q)\in\ker e'(y^*,u^*)$:

$$\begin{aligned}\langle L''(y^*,u^*,\lambda^*,\mu^*)(v,q),(v,q)\rangle_{X^*,X} &\geq \|v\|^2_{L^2} + \sigma\,|q|^2 - 2\,\|\lambda^*\|_{L^\infty}\,\|v\|^2_{H^1}\\ &\geq \sigma\,|q|^2 - 2c_1\,\|y^*-z\|_{L^2}\,\|v\|^2_{H^1}\\ &\geq \frac{\sigma}{2}\,|q|^2 + \frac{\sigma}{2c_{\text{ker}}}\,\|v\|^2_{H^1}\\ &\quad -2c_\lambda\,\|y^*-z\|_{L^2}\,\|v\|^2_{H^1}\\ &\geq \frac{\sigma}{2}\,|q|^2 + \left(\frac{\sigma}{2c_{\text{ker}}} - 2c_\lambda\|y^*-z\|_{L^2}\right)\|v\|^2_{H^1}\,.\end{aligned}$$

If $\|y^*-z\|_{L^2} < \frac{\sigma}{4c_{\text{ker}}c_\lambda}$, we derive that

$$\kappa = \min\left(\frac{\sigma}{2}, \frac{\sigma}{2c_{\text{ker}}} - 2c_\lambda\|y^*-z\|_{L^2}\right) > 0\,,$$

and the claim follows.

□

5.6. The Scheme of the Augmented Lagrangian-SQP Method. This section is devoted to formulate Algorithm 1 on page 25 for our boundary control problem ($\mathbf{P_2}$).

a) *Choose* $(y^0,u^0,(\lambda^0,\mu^0))\in\mathcal{Y}\times\mathcal{U}\times Y$, $c\geq 0$ *and put* $n=0$.
b) *Calculate* $e^n\in\mathcal{H}$ *solving*

$$\langle e^n,\varphi\rangle_{\mathcal{H}} = \langle e_1(y^n,u^n),\varphi\rangle_{\mathcal{H}^*,\mathcal{H}} \qquad \text{for all } \varphi\in\mathcal{H}$$

and define $\tilde\lambda^n = \lambda^n + c\,e^n$ *and* $\tilde\mu^n = \mu^n + c(y^n(0)-\alpha)$.

c) *Solve for* $(\bar{y}, \bar{u}, (\bar{\lambda}, \bar{\mu})) \in \mathcal{Y} \times \mathcal{U} \times Y$ *the following linear system:*

$$\begin{aligned}
&\langle L''(y^n, u^n, \tilde{\lambda}^n, \tilde{\mu}^n)(\bar{y}, \bar{u}) + e'(y^n, u^n)^*(\bar{\lambda}, \bar{\mu}), (v, q)\rangle_{X^*,X} \\
&\quad = \langle L''(y^n, u^n, \tilde{\lambda}^n, \tilde{\mu}^n)(y^n, u^n) + e'(y^n, u^n)^*(\tilde{\lambda}^n, \tilde{\mu}^n), (v, q)\rangle_{X^*,X} \\
&\qquad - \langle L'(y^n, u^n, \tilde{\lambda}^n, \tilde{\mu}^n), (v, q)\rangle_{X^*,X}
\end{aligned}$$

$$\begin{aligned}
&\langle e'(y^n, u^n)(\bar{y}, \bar{u}), \varphi\rangle_{Y^*,Y} \\
&\quad = \langle e'(y^n, u^n)(y^n, u^n) - e(y^n, u^n), \varphi\rangle_{Y^*,Y}
\end{aligned}$$

for all $(v, q, \varphi) \in \mathcal{Y} \times \mathcal{U} \times Y$.

d) *Set* $(y^{n+1}, u^{n+1}, \lambda^{n+1}, \mu^{n+1}) = (\bar{y}, \bar{u}, \bar{\lambda}, \bar{\mu})$, $n = n + 1$ *and goto step* b).

6. The Numerical Treatment of the Control Problems

In Sections 3 and 5 we have posed two types of control problems for the stationary Burgers equation, $(\mathbf{P_1})$ and $(\mathbf{P_2})$, respectively. To solve these control problems we shall use the augmented Lagrangian-SQP method. But both $(\mathbf{P_1})$ and $(\mathbf{P_2})$ are infinite dimensional problems. Hence, for the numerical treatment we have to discretize our problems. To apply the results of Chapter 2, Section 3, we shall present approximations of the control problems satisfying the requirements of Theorem 3.38 on page 39, which guarantees the convergence of the internal approximation of the augmented Lagrangian-SQP method (Algorithm 2 on page 38).

6.1. Approximation of the Hilbert Spaces. For $m \in \mathbb{N}$ the mesh-size $h = h(m)$ and the associated grid-points $x_i \in [0, 1]$ are given by $h = \frac{1}{m}$ and $x_i = ih$ for $i = 0, \dots, m$, respectively.

In Example 3.1 on page 27 we have introduced the piecewise linear functions $\varphi_0, \dots, \varphi_m \in H^1(0, 1)$.

We will restrict our discussions to the case that the set $\omega = (a, b) \subseteq (0, 1)$ is an open interval. The numbers $i_a, i_b \in \{1, \dots, m-1\}$ are defined by

$$a < x_{i_a} < x_{i_a+1} < \ldots < x_{i_b} < b\,.$$

For the approximation of the Hilbert spaces in case of $(\mathbf{P_1})$ we set

$$\begin{array}{rclrcl}
\mathcal{Y}_h &=& \operatorname{Span}\{\varphi_0, \dots, \varphi_m\}\,, && & \\
\mathcal{U}_h &=& \operatorname{Span}\{\varphi_{i_a}, \dots, \varphi_{i_b}\}\,, & X_h &=& \mathcal{Y}_h \times \mathcal{U}_h\,, \\
Y_h &=& \operatorname{Span}\{\varphi_1, \dots, \varphi_{m-1}\} \times \mathbb{R}^2\,, & Z_h &=& X_h \times Y_h
\end{array}$$

and in case of $(\mathbf{P_2})$ we put

$$\begin{array}{rclrcl}
\mathcal{Y}_h &=& \operatorname{Span}\{\varphi_0, \dots, \varphi_m\}\,, && & \\
\mathcal{U}_h &=& \mathcal{U} = \mathbb{R}, & X_h &=& \mathcal{Y}_h \times \mathcal{U}\,, \\
Y_h &=& \operatorname{Span}\{\varphi_1, \dots, \varphi_m\} \times \mathbb{R}, & Z_h &=& X_h \times Y_h\,.
\end{array}$$

Now we are going to define the prolongations and restrictions. We set

$$p_h(y_h, u_h, (\lambda_h, \mu)) = (y_h, u_h, (\lambda_h, \mu)) \quad \text{for } (y_h, u_h) \in X_h \text{ and } (\lambda_h, \mu) \in Y_h\,,$$

if we handle the approximation of $(\mathbf{P_1})$, and

$$p_h(y_h, u, (\lambda_h, \mu)) = (y_h, u, (\lambda_h, \mu)) \quad \text{for } (y_h, u) \in X_h \text{ and } (\lambda_h, \mu) \in Y_h\,.$$

in case of $(\mathbf{P_2})$. Hence, the finite dimensional Hilbert spaces X_h and Y_h are endowed with the inner products and norms introduced in X and Y, respectively. It follows directly that in both cases p_h is linear and injective operator satisfying $\|p_h\|_{L(Z_h,Z)} = 1$.

For $(\mathbf{P_1})$ we introduce the restriction

$$r_h(y, u, (\lambda, \mu)) = \Big(\sum_{i=0}^{m} y(x_i)\varphi_i, \frac{1}{h} \sum_{i=i_a}^{i_b} \int_{x_i-\frac{h}{2}}^{x_i+\frac{h}{2}} u\, dx\, \varphi_i, \Big(\sum_{i=1}^{m-1} \lambda(x_i)\varphi_i, \mu \Big)\Big)$$

for $(y, u) \in X$ and $(\lambda, \mu) \in Y$. Note that $u \in L^2(\omega)$ holds so that $u(x_i)$ need not to be meaningful.

In case of $(\mathbf{P_2})$ let the restriction be given by

$$r_h(y, u, (\lambda, \mu)) = \Big(\sum_{i=0}^{m} y(x_i)\varphi_i, u, \Big(\sum_{i=1}^{m-1} \lambda(x_i)\varphi_i, \mu \Big)\Big)$$

for $(y, u) \in X$ and $(\lambda, \mu) \in Y$. Obviously, r_h is linear and surjective for both problems.

Remark 6.1. We point out that the use of optimal Galerkin approximations is limited in many cases, because in this case the matrices of the approximate problems often are not sparse [**Aub72**, p. 69-74].

6.2. Approximation of the Dual Spaces. To calculate the internal approximations of the constraint e and the internal approximations of the Fréchet-derivatives of the constraint e and of the Lagrangian L – see Chapter 2, Section 3.4 – we need the restriction $s_h = (s_h^X, s_h^Y) : Z^* \to X_h^* \times Y_h^*$. From (2.10) we derive that s_h^X is defined by

$$\langle s_h^X f, (y_h, u_h) \rangle_{X_h^*, X_h} = \langle f, (y_h, u_h) \rangle_{X^*, X} \text{ for all } f \in X^*, (y_h, u_h) \in X_h\,,$$

and s_h^Y is given by

$$\langle s_h^Y g, \lambda_h \rangle_{Y_h^*, Y_h} = \langle g, \lambda_h \rangle_{Y^*, Y} \quad \text{for all } g \in Y^*, \lambda_h \in Y_h\,. \tag{4.57}$$

6.3. Convergent Approximations. The next result ensures that Condition 5 on page 35 is satisfied.

Lemma 6.2. *The approximations (Z_h, p_h, r_h) of Z are convergent both for $(\mathbf{P_1})$ and for $(\mathbf{P_2})$.*

For the proof we refer the reader to [**Aub72**, p. 42].

Remark 6.3. Remark 3.27 on page 35 leads to the existence of a constant $c_\circ > 0$ such that $\|r_h\|_{L(Z,Z_h)} \le c_\circ$ for all h. Hence, $\{r_h\}_h$ is a family of uniformly bounded linear operator in case of $(\mathbf{P_1})$ and of $(\mathbf{P_2})$.

6.4. Internal Approximation of the Operators. By (2.12) the internal approximations of the cost functional and of the constraint yield $J_h = J|_{X_h}$ and $e_h = s_h^Y e|_{X_h}$, respectively. It follows that

$$\langle e_h(y_h, u_h), \phi_h \rangle_{Y_h^*, Y_h} \overset{(4.57)}{=} \langle e(y_h, u_h), \phi_h \rangle_{Y^*, Y} \qquad \text{for all } \phi_h \in Y_h\,.$$

LEMMA 6.4. *For every h the equation*

$$\langle e(y_h, u_h), \phi_h \rangle_{Y^*, Y} = 0 \qquad \textit{for all } \phi_h \in Y_h \tag{4.58}$$

possesses a solution $(y_h, u_h) \in X_h$ for $(\mathbf{P_1})$.

PROOF. From $Y_h = \text{Span}\,\{\varphi_1, \dots, \varphi_{m-1}\}$ and (4.58) we infer that (y_h, u_h) satisfies

$$\begin{cases} \displaystyle\int_0^1 \nu y_h' \varphi_i' + y_h y_h' \varphi_i - (f + Bu_h)\varphi_i \, dx \quad \text{for } 1 \le i \le m-1\,, \\ y_h(0) = \alpha,\ y_h(1) = \beta\,. \end{cases} \tag{4.59}$$

Instead of (4.13) we introduce the set

$$M_h = \left\{ w_h \in \text{Span}\{\varphi_1, \dots, \varphi_{m-1}\} : \|w_h\|_{H_0^1} \le \frac{1}{\nu}\, \|f + Bu_h\|_{L^2} \right\} \subset M$$

for every h. Then existence of a solution to (4.59) with $\alpha = \beta = 0$ follows analogously as in the proof of Theorem 2.10. The inhomogeneous case can be treated as in the proof of Theorem 2.13. □

To guarantee existence of a solution to (4.58) for problem $(\mathbf{P_2})$ we have to assume that the viscosity parameter is sufficiently large with respect to the $\mathcal{H}^*$-norm of f and the values of α and u. The proof of this result is similar to that of Theorems 4.3 and 4.6.

LEMMA 6.5. *For every h the equation*

$$\langle e(y_h, u_h), \phi_h \rangle_{Y^*, Y} = 0 \qquad \textit{for all } \phi_h \in Y_h \tag{4.60}$$

has a solution $(y_h, u_h) \in X_h$ for $(\mathbf{P_2})$ if the viscosity parameter is sufficiently large.

6.5. Approximation of the Control Problems. Together with $(\mathbf{P_1})$ and $(\mathbf{P_2})$ we investigate the finite dimensional discrete problems

$(\mathbf{P_{i,h}})$ $\quad \min J(y_h, u_h)$ subject to $\langle e(y_h, u_h), \phi_h \rangle_{Y^*, Y} = 0$ for all $\phi_h \in Y_h$

for $\mathbf{i} = 1, 2$.

The next result ensures that both $(\mathbf{P_{1,h}})$ and $(\mathbf{P_{2,h}})$ have local solutions for every h (Condition 4 on page 34).

THEOREM 6.6. *If C is the canonical injection of $H^1(0,1)$ into $L^2(0,1)$ or $C = I$, the finite dimensional problems $(\mathbf{P_{1,h}})$ and $(\mathbf{P_{2,h}})$ have a local solution for every h.*

PROOF. Analysis similar to that in the infinite dimensional case shows that from equations (4.58) and (4.60) we can derive the same a-priori bounds for y_h as in the proofs of Theorem 3.1 and Theorem 5.1. Since $J_h = J|_{X_h}$ and $X_h \subset X$ hold, J_h has the same properties as J, i.e. J_h is radically unbounded in X_h and weakly lower semi-continuous. Thus, there exists a minimizing sequence $\{(y_h^n, u_h^n)\}_{n\in\mathbb{N}}$ in X_h which converges weakly to a limit point $(y_h^*, u_h^*) \in X$. Since X_h is finite dimensional, this sequence converges strongly and (y_h^*, u_h^*) belongs to X_h. Further, the continuity of e implies the continuity of e_h. Therefore, from $e_h(y_h^n, u_h^n) = 0$ for all n we derive $e_h(y_h^*, u_h^*) = \lim_{n\to\infty} e_h(y_h^n, u_h^n) = 0$. □

In $(\mathbf{OS_h})$ on page 37 we gave the first-order necessary optimality condition of the finite dimensional discrete problems. The variational formulation of $(\mathbf{OS_h})$ is expressed as

$$\text{(4.61)} \quad \begin{cases} \langle L'(y_h, u_h, \lambda_h + c\,\mathfrak{J}_h e_h(y_h, u_h)), \psi_h\rangle_{X^*,X} = 0 & \text{for all } \psi_h \in X_h\,, \\ \langle e(y_h, u_h), \phi_h\rangle_{Y^*,Y} = 0 & \text{for all } \phi_h \in Y_h\,. \end{cases}$$

To solve (4.61) we apply the discretized augmented Lagrangian-SQP method (Algorithm 2).

6.6. The Babuška-Brezzi Condition. Right away we remark that in general for $(y_h, u_h) \in X_h$ we have $\ker e_h(y_h, u_h) \not\subset \ker e'(y_h, u_h)$, because X_h and Y_h are proper subspaces of X and Y, respectively. Hence, the uniform discrete Babuška-Brezzi condition in the form of Condition 6 on page 37 does not follow from Condition 2 on page 22.

LEMMA 6.7. *Let $(y_h^*, u_h^*) \in X_h$ denote a local solution of $(\mathbf{P_{i,h}})$, $\mathbf{i} = 1, 2$. If the mesh-size is sufficiently small, then for every h there exists a constant $\beta_h > 0$ such that for the above choices of approximations we have*

$$\text{(4.62)} \qquad \inf_{\lambda_h \in Y_h} \sup_{(y_h,u_h)\in X_h} \frac{\langle e'(y_h^*, u_h^*)^*\lambda_h, (y_h, u_h)\rangle_{X^*,X}}{\|(y_h, u_h)\|_X \, \|\lambda_h\|_Y} \geq \beta_h\,.$$

PROOF. We shall prove the claim for control problem $(\mathbf{P_{1,h}})$. In case of problem $(\mathbf{P_{2,h}})$ the proof is similar.

Let $(y_h^*, u_h^*) \in X_h$ denote a local solution of $(\mathbf{P_{1,h}})$. Then there are coefficients $y_1^*, \ldots, y_{m-1}^* \in \mathbb{R}$ such that

$$y_h^*(x) = \alpha\varphi_0(x) + \sum_{j=1}^{m-1} y_j^*\varphi_j(x) + \beta\varphi_m(x)\,.$$

We shall prove that the operator $e_h'(y_h^*, u_h^*)$ is surjective for sufficiently small mesh-size. Then, by Lemma 2.4 on page 7 the claim follows.

The operator $e_h'(y_h^*, u_h^*)$ is surjective if and only if for all $(g_h, a, b) \in Y_h^*$ there exists $(y_h, u_h) \in X_h$ satisfying

$$\langle e_1'(y_h^*, u_h^*)(y_h, u_h), \varphi_i\rangle_{H^{-1},H_0^1} = \langle g_h, \varphi_i\rangle_{H^{-1},H_0^1} \text{ for } 1 \le i \le m-1\,, \tag{4.63}$$

$$e_2'(y_h^*, u_h^*)(y_h, u_h) = (a, b)\,. \tag{4.64}$$

By (4.25) equation (4.64) yields $y_h(0) = a$ and $y_h(1) = b$. Since the operator $e_1'(y_h^*, u_h^*)$ is linear, we derive that

$$\begin{aligned}
&\langle e_1'(y_h^*, u_h^*)(y_h, u_h), \varphi_i\rangle_{H^{-1},H_0^1} \\
&\quad = \sum_{j=0}^{m} y_h(x_j)\langle e_1'(y_h^*, u_h^*)(\varphi_j, 0), \varphi_i\rangle_{H^{-1},H_0^1} \\
&\qquad + \sum_{j=i_a}^{i_b} \frac{1}{h}\int_{x_i-\frac{h}{2}}^{x_i+\frac{h}{2}} u\,dx\,\langle e_1'(y_h^*, u_h^*)(0, \varphi_j), \varphi_i\rangle_{H^{-1},H_0^1}
\end{aligned}$$

for $1 \le i \le m-1$. These equations can be written in matrix notation. For this purpose we introduce the two matrices $\mathbf{A}_h \in \mathbb{R}^{(m-1)\times(m-1)}$ and $\mathbf{B}_h \in \mathbb{R}^{(m-1)\times(i_b-i_a)}$ by

$$((\mathbf{A}_h))_{ij} = \langle e_1'(y_h^*, u_h^*)(\varphi_j, 0), \varphi_i\rangle_{H^{-1},H_0^1} \tag{4.65}$$

for $1 \le i, j \le m-1$ and

$$((\mathbf{B}_h))_{ij} = \langle e_1'(y_h^*, u_h^*)(0, \varphi_j), \varphi_i\rangle_{H^{-1},H_0^1} \tag{4.66}$$

for $1 \le i \le m-1$ and $1 \le j \le i_b - i_a$. In addition we define the vectors $\mathbf{y}_h = (y(x_1), \dots, y(x_{m-1}))^T \in \mathbb{R}^{m-1}$ and

$$\mathbf{u}_h = \frac{1}{h}\Big(\int_{x_{i_a}-\frac{h}{2}}^{x_{i_a}+\frac{h}{2}} u\,dx, \dots, \int_{x_{i_b}-\frac{h}{2}}^{x_{i_b}+\frac{h}{2}} u\,dx\Big)^T \in \mathbb{R}^{i_b-i_a}\,.$$

Then equation (4.63) leads to

$$(\mathbf{A}_h \mid \mathbf{B}_h)\cdot\begin{pmatrix}\mathbf{y}_h\\ \mathbf{u}_h\end{pmatrix} = \mathbf{g}_h\,, \tag{4.67}$$

where

$$(\mathbf{g}_h)_i = \langle g_h, \varphi_i\rangle_{H^{-1},H_0^1} - \langle e_1'(y_h^*, u_h^*)(a\varphi_0 + b\varphi_m, 0), \varphi_i\rangle_{H^{-1},H_0^1}$$

for $i = 1, \dots, m$.

To prove the theorem we have to show that the matrix

$$\mathbf{E}_h = (\mathbf{A}_h \mid \mathbf{B}_h) \in \mathbb{R}^{(m-1)\times(m-1+i_b-i_a)} \tag{4.68}$$

is surjective. The matrix $\mathbf{A}_h$ is equal to

$$\begin{pmatrix} \frac{2\nu}{h}+\frac{y_2^*-\alpha}{6} & -\frac{\nu}{h}+\frac{2y_2^*+y_1^*}{6} & & 0 \\ -\frac{\nu}{h}-\frac{y_2^*+2y_1^*}{6} & \frac{2\nu}{h}+\frac{y_3^*-y_1^*}{6} & -\frac{\nu}{h}+\frac{2y_3^*+y_2^*}{6} & \\ \ddots & \ddots & \ddots & \\ -\frac{\nu}{h}-\frac{y_{m-2}^*+2y_{m-3}^*}{6} & \frac{2\nu}{h}+\frac{y_{m-1}^*-y_{m-3}^*}{6} & -\frac{\nu}{h}+\frac{2y_{m-1}^*+y_{m-2}^*}{6} & \\ 0 & -\frac{\nu}{h}-\frac{y_{m-1}^*+2y_{m-2}^*}{6} & \frac{2\nu}{h}+\frac{\beta-y_{m-2}^*}{6} & \end{pmatrix}$$

Moreover, we derive

$$\mathbf{B}_h = \begin{pmatrix} \mathbf{0} \\ \hline \mathbf{B}_h^\circ \\ \hline \mathbf{0} \end{pmatrix} \begin{matrix} \} \in \mathbb{R}^{(i_a-1)\times(i_b-i_a)} \\ \} \in \mathbb{R}^{(i_b-i_a)\times(i_b-i_a)} \\ \} \in \mathbb{R}^{(m-i_b+1)\times(i_b-i_a)} \end{matrix}$$

where either $\mathbf{B}_h^\circ = -\frac{2h}{3}$ if $i_b - i_a = 1$ or

$$\mathbf{B}_h^\circ = -\frac{h}{6}\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$$

if $i_b - i_a = 2$ or

$$\mathbf{B}_h^\circ = -\frac{h}{6}\begin{pmatrix} 4 & 1 & & 0 \\ 1 & 4 & 1 & \\ \ddots & \ddots & \ddots & \\ & 1 & 4 & 1 \\ 0 & & 1 & 4 \end{pmatrix},$$

otherwise. Let $y_{\max}^* = \max_{0\le i\le m} y_i^*$ and $y_{\min}^* = \min_{0\le i\le m} y_i^*$. We distinguish two cases:

a) If $y_{\min}^* \geq 0$ holds, then it follows that $y_i^* \geq 0$ and therefore for all mesh-sizes h

$$-\frac{\nu}{h} - \frac{2y_{i-1}^* + y_i^*}{6} < 0 \quad \text{for } i = 2, \dots, m-1\,. \tag{4.69}$$

By interchanging the rows of $\mathbf{A}_h$ we obtain the matrix

$$\begin{pmatrix} -\frac{\nu}{h}-\frac{y_2^*+2y_1^*}{6} & \frac{2\nu}{h}+\frac{y_3^*-y_1^*}{6} & -\frac{\nu}{h}+\frac{2y_3^*+y_2^*}{6} & & 0 \\ & \ddots & \ddots & \ddots & \\ & -\frac{\nu}{h}-\frac{y_{m-2}^*+2y_{m-3}^*}{6} & \frac{2\nu}{h}+\frac{y_{m-1}^*-y_{m-3}^*}{6} & & -\frac{\nu}{h}+\frac{2y_{m-1}^*+y_{m-2}^*}{6} \\ & & -\frac{\nu}{h}-\frac{y_{m-1}^*+2y_{m-2}^*}{6} & & \frac{2\nu}{h}+\frac{\beta-y_{m-2}^*}{6} \\ \frac{2\nu}{h}+\frac{y_2^*-\alpha}{6} & -\frac{\nu}{h}+\frac{2y_2^*+y_1^*}{6} & 0 & \cdots & 0 \end{pmatrix}$$

If we apply the Gauß-procedure we derive the matrix

$$\begin{pmatrix} -\frac{\nu}{h}-\frac{y_2^*+2y_1^*}{6} & \frac{2\nu}{h}+\frac{y_3^*-y_1^*}{6} & -\frac{\nu}{h}+\frac{2y_3^*+y_2^*}{6} & 0 \\ & \ddots & \ddots & \ddots \\ & -\frac{\nu}{h}-\frac{y_{m-2}^*+2y_{m-3}^*}{6} & \frac{2\nu}{h}+\frac{y_{m-1}^*-y_{m-3}^*}{6} & -\frac{\nu}{h}+\frac{2y_{m-1}^*+y_{m-2}^*}{6} \\ & & -\frac{\nu}{h}-\frac{y_{m-1}^*+2y_{m-2}^*}{6} & \frac{2\nu}{h}+\frac{\beta-y_{m-2}^*}{6} \\ 0 & & & \tilde{a}_{m-1,m-1} \end{pmatrix}$$

which has the same rank as the matrix $\mathbf{A}_h$. By (4.69) we conclude that the first $m-2$ columns are linear independent, i.e. $\operatorname{rank}\mathbf{A}_h \geq m-2$ for all mesh-sizes. We point out that we do not know if $\tilde{a}_{m-1,m-1} \neq 0$. But the same procedure applied to $\mathbf{B}_h$ leads to a matrix, which has at least one nonzero element in the last row for all h. Hence, $\operatorname{rank}\mathbf{E}_h = m-1$ for all mesh-sizes.

b) In case of $y^*_{\min} < 0$ there can arise two cases:

(1) Let $y^*_{\max} \leq 0$. Then we conclude that $y^*_i \leq 0$ and thus for all mesh-sizes h

$$-\frac{\nu}{h}+\frac{2y^*_{i+1}+y^*_i}{6} < 0 \quad \text{for } i = 1,\dots,m-2\,. \tag{4.70}$$

If we define the matrix $\tilde{\mathbf{E}}_h$ by $((\tilde{\mathbf{E}}_h))_{ij} = ((\tilde{\mathbf{E}}_h))_{m-i,j}$ for $1 \leq i \leq m-1$ and $1 \leq j \leq m-1+i_b-i_a$ and apply an analogous procedure as in case a), we derive that $\operatorname{rank}\tilde{\mathbf{E}}_h = m-1$ for all h, which implies that $\operatorname{rank}\mathbf{E}_h = m-1$ for all mesh-sizes.

(2) For $y^*_{\max} > 0$ we assume that

$$0 < h < \frac{2\nu}{\max\left(-y^*_{\min}, y^*_{\max}\right)}\,. \tag{4.71}$$

Then we obtain either (4.69) or (4.70) so that we can apply case a) or case b)-(1). This gives $\operatorname{rank}\mathbf{E}_h = m-1$ for all mesh-sizes h satisfying (4.71).

□

REMARK 6.8. Note that Lemma 6.7 ensures already the well-posed of Algorithm 2, if there exists a constant $\kappa_h > 0$ for every h so that

$$\begin{cases} \langle L''(y_h^*, u_h^*, \lambda_h^*)(y_h, u_h), (y_h, u_h)\rangle_{X^*,X} \geq \kappa_h \,\|(y_h, u_h)\|_{X_h}^2 \\ \text{for all } (y_h, u_h) \in \ker e_h'(y_h^*, u_h^*) \end{cases}$$

holds. To guarantee the convergence for $h \to 0$ we have to require that there are constants $\beta^* > 0$ and $\kappa^* > 0$ with $\beta_h \geq \beta^* > 0$ and $\kappa_h \geq \kappa^* > 0$ for all h sufficiently small.

To check Condition 6 we make use of the following lemma. For the proof we refer the reader to [**BF91**, p. 58].

LEMMA 6.9. *Let $(y_h^*, u_h^*) \in X_h$ denote a solution to $(\mathbf{P_{i,h}})$, $\mathbf{i = 1, 2}$, and let $\beta > 0$ such that*

$$\sup_{(y,u)\in X} \frac{\langle e'(y_h^*, u_h^*)(y,u), \phi_h\rangle_{Y^*,Y}}{\|(y,u)\|_X} \geq \beta \,\|\phi_h\|_Y \quad \textit{for all } \phi_h \in Y_h\,. \tag{4.72}$$

Suppose that there exists a family of uniformly continuous restrictions $\Pi_h : X \to X_h$ satisfying

$$\begin{cases} \langle e_h'(y_h^*, u_h^*)[\Pi_h(y,u) - (y,u)], \phi_h\rangle_{Y^*,Y} = 0 & \textit{for all } \phi_h \in Y_h\,, \\ \|\Pi(y,u)\|_X \leq \gamma \,\|(y,u)\|_X\,. \end{cases}$$

Then the uniform discrete Babuška-Brezzi condition holds at (y_h^*, u_h^*), i.e.

$$\sup_{(y_h,u_h)\in X_h} \frac{\langle e'(y_h^*, u_h^*)(y_h,u_h), \phi_h\rangle_{Y^*,Y}}{\|(y_h,u_h)\|_X} \geq \frac{\beta}{\gamma} \,\|\phi_h\|_Y \quad \textit{for all } \phi_h \in Y_h\,.$$

First we shall discuss the uniform discrete Babuška-Brezzi condition for control problem $(\mathbf{P_{1,h}})$. Henceforth we set

$$Y_{1,h} = \mathrm{Span}\,\{\varphi_1, \dots, \varphi_{m-1}\} \subset H_0^1(0,1)\,.$$

LEMMA 6.10. *For each h let $(y_h^*, u_h^*) \in X_h$ denote a local solution of $(\mathbf{P_{1,h}})$ satisfying*

$$0 \leq \nu \,\|\varphi_h\|_{H_0^1}^2 + \int_0^1 (y_h^*)'\varphi_h^2\,dx \qquad \textit{for all } \varphi_h \in Y_{1,h}\,.$$

Then for every $(g_h, a, b) \in Y_h^*$ there exists a unique $y_h \in \mathcal{Y}_h$ such that

$$\begin{aligned} \langle e_1'(y_h^*, u_h^*)(y_h, 0), \varphi_i\rangle_{H^{-1},H_0^1} &= \langle g_h, \varphi_i\rangle_{H^{-1},H_0^1} \quad \textit{for } 1 \leq i \leq m-1\,, \\ e_2'(y_h^*, u_h^*)(y_h, 0) &= (a,b)\,. \end{aligned}$$

PROOF. We shall prove that for all $(g_h, a, b) \in Y_h^*$ there exists $y_h \in Y_{1,h}$ such that

$$(4.73) \qquad \begin{cases} \displaystyle\int_0^1 \nu y_h' \varphi_i' + (y_h^* y_h)' \varphi_i \, dx = \langle g_h, \varphi_i \rangle_{H^{-1}, H_0^1} \quad \text{for } 1 \le i \le m-1\,, \\ y_h(0) = a,\; y_h(1) = b\,. \end{cases}$$

First we make problem (4.73) homogeneous. Setting $v_d(x) = a + (b-a)x$ and $w_h = y_h - v_d$ we conclude that $w_h \in Y_{1,h}$ solves

$$(4.74) \qquad \int_0^1 \nu w_h' \varphi_i' + (y_h^* w_h)' \varphi_i \, dx = \langle \tilde{g}_h, \varphi_i \rangle_{H^{-1}, H_0^1} \quad \text{for } 1 \le i \le m-1\,,$$

where

$$\langle \tilde{g}_h, \varphi_i \rangle_{H^{-1}, H_0^1} = \langle g_h, \varphi_i \rangle_{H^{-1}, H_0^1} - \int_0^1 (b-a) y_h^* \varphi_i + v_d (y_h^*)' \varphi_i \, dx\,,$$

if and only if $v_h = w_h + v_d$ is a solution of (4.73).

We shall apply the Fredholm alternative [**Kre78**, p. 437] and rewrite (4.74) in operator notation. For this purpose we introduce the linear operator $K_h \in L(Y_{1,h}, Y_{1,h}^*)$ by $K_h w_h = (y_h^* w_h)'$. Since $Y_{1,h}$ is finite dimensional, the operator K_h is compact for every h. Let $D_h \in L(Y_{1,h}, Y_{1,h}^*)$ be given by $-\nu\, w_h''$. This implies

$$(D_h + K_h) w_h = \tilde{g}_h \qquad \text{in } Y_{1,h}^*\,.$$

For every h the operator D_h possesses the matrix representation

$$\mathbf{D}_h = \frac{\nu}{h} \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & -1 & 2 & -1 \\ 0 & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{(m-1)\times(m-1)}\,,$$

which is positive definite and thus invertible. Hence, D_h^{-1} exists and is linear, and we can express $G_h = D_h + K_h$ as

$$-D_h^{-1} G_h = \underbrace{(-D_h^{-1} K_h)}_{=T_h} - I_h\,.$$

For a given $\bar{g}_h \in Y_{1,h}^*$ the equation $G_h w_h = \bar{g}_h$ is solvable if and only if $(T_h - I_h) w_h = -D_h^{-1} \bar{g}_h$ is solvable. Applying the Fredholm alternative yields: If the equation $T_h^* f_h = f_h$ has only the trivial solution, then $G_h w_h = \bar{g}_h$ is solvable for every $\bar{g}_h \in Y_{1,h}^*$. Setting $\phi_h = D_h^{-1} f_h \in Y_{1,h}$ the equation $T_h^* f_h = f_h$ leads to

$$(4.75) \qquad -\nu \phi_h'' - y_h^* \phi_h' = 0 \qquad \text{in } Y_{1,h}^*\,.$$

Multiplying (4.75) by ϕ_h and integrating over the interval $(0,1)$ we obtain

$$\nu \, \|\phi_h\|^2_{H^1_0} + \frac{1}{2} \int_0^1 (y_h^*)' \phi_h^2 \, dx = 0 \, ,$$

which implies by assumption that $\|\phi_h\|_{H^1_0} = 0$. Since $\phi_h \in H^1_0(0,1)$ holds, we infer that $\phi_h = 0$ in $[0,1]$. Therefore the equation $T_h^* f_h = f_h$ has only the trivial solution. Hence, G_h is surjective, i.e. there exists y_h solving (4.74) by setting $\bar{g}_h = \tilde{g}_h$.

As the operator $e_h'(y_h^*, u_h^*) : \mathcal{Y}_h \times \{0\} \to Y_h$ is surjective and $\dim \mathcal{Y}_h \times \{0\} = \dim Y_h = m+1$ holds, the operator $e_h'(y_h^*, u_h^*)|_{\mathcal{Y}_h \times \{0\}}$ is also injective. Hence, y_h is unique determined. □

Now we define the restriction operator, which is needed in Lemma 6.9. Let $(y,u) \in X$ and

$$0 \le \nu \, \|\varphi_h\|^2_{H^1_0} + \int_0^1 (y_h^*)' \varphi_h^2 \, dx \qquad \text{for all } \varphi_h \in Y_{1,h} \, . \tag{4.76}$$

Further, we assume that there exists a constant $c_1 > 0$, which does not depend on h and satisfies $\|y_h^*\|_{H^1} \le c_1 \, \|y^*\|_{H^1}$ for all h. By Lemma 6.10 there is a unique element $y_h \in \mathcal{Y}_h$ satisfying

$$e_h'(y_h^*, u_h^*)(y_h, 0) = s_h^Y e'(y_h^*, u_h^*)(y,u) \qquad \text{in } Y_h^* \, . \tag{4.77}$$

We introduce the linear restriction operator $\Pi_h : X \to X_h$ by $\Pi_h(y,u) = (y_h, 0)$ for all $(y,u) \in X$. From (4.76) and (4.77) we conclude that

$$\frac{\nu}{2} \, \|y_h'\|_{L^2} \le \underbrace{(\nu + c_1 \, \|y^*\|_{H^1})}_{=c_2} \, \|(y,u)\|_X \, . \tag{4.78}$$

Let $v_d(x) = y(0) + [y(1) - y(0)]x$. Then equation (4.77) implies that $y_h - v_d \in H^1_0(0,1)$. Moreover, $\|v_d'\|_{L^2} \le \sqrt{2} \, \|y\|_{H^1}$ and $\|v_d\|_{H^1} \le \sqrt{3} \, \|y\|_{H^1}$. From the triangle inequality and Lemma 3.4 on page 9 we infer that

$$\begin{aligned} \|y_h\|_{H^1} &\le \|y_h - v_d\|_{H^1} + \|v_d\|_{H^1} \\ &\le \sqrt{2} \, \|y_h - v_d\|_{H^1_0} + \sqrt{3} \, \|y\|_{H^1} \\ &\le \sqrt{2} \, \|y_h'\|_{L^2} + \sqrt{2} \, \|v_d'\|_{L^2} + \sqrt{3} \, \|y\|_{H^1} \\ &\overset{(4.78)}{\le} \underbrace{\Big(\frac{2c_2}{\nu} \sqrt{2} + 2 + \sqrt{3} \Big)}_{=c_3} \|(y,u)\|_X \end{aligned}$$

Hence, we have

$$\langle e'(y_h^*, u_h^*)[\Pi_h(y,u) - (y,u)], \phi_h \rangle_{Y^*,Y} = 0 \qquad \text{for all } \phi_h \in Y_h$$

and

$$\|\Pi_h(y,u)\|_X \le c_3 \, \|(y,u)\|_X \, .$$

By Theorem 3.3 assumption (4.72) follows from Lemma 2.4 on page 7. By Lemma 6.9 we thus derive the following theorem.

THEOREM 6.11. *Let $(y_h^*, u_h^*) \in X_h$ denote a solution to $(\mathbf{P_{1,h}})$ with*

$$0 \le \nu \, \|\varphi_h\|_{H_0^1}^2 + \int_0^1 (y_h^*)' \varphi_h^2 \, dx \qquad \text{for all } \varphi_h \in \operatorname{Span}\{\varphi_1, \dots, \varphi_{m-1}\}.$$

Moreover, assume the existence of a constant $c_1 > 0$ such that $\|y_h^*\|_{H^1} \le c_1 \, \|y^*\|_{H^1}$ for all h. Then Condition 6 is satisfied at (y_h^*, u_h^*).

Now we shall consider Condition 6 for control problem $(\mathbf{P_{2,h}})$. Henceforth we set

$$Y_{2,h} = \operatorname{Span}\{\varphi_1, \dots, \varphi_m\} \in \mathcal{H}.$$

LEMMA 6.12. *For each h let $(y_h^*, u_h^*) \in X_h$ denote a local solution of $(\mathbf{P_{2,h}})$ satisfying*

$$0 \le \nu \, \|\varphi_h\|_{\mathcal{H}}^2 + \int_0^1 (y_h^* \varphi_h)' \varphi_h \, dx \qquad \text{for all } \varphi_h \in Y_{2,h}.$$

Then for every $(g_h, a) \in Y_h^*$ there exists a unique $y_h \in \mathcal{Y}_h$ such that

$$\begin{aligned} \langle e_1'(y_h^*, u_h^*)(y_h, 0), \varphi_i \rangle_{\mathcal{H}^*, \mathcal{H}} &= \langle g_h, \varphi_i \rangle_{\mathcal{H}^*, \mathcal{H}} \quad \text{for } 1 \le i \le m, \\ e_2'(y_h^*, u_h^*)(y_h, 0) &= a. \end{aligned}$$

PROOF. We shall prove that for all $(g_h, a) \in Y_h^*$ there exists $y_h \in Y_{2,h}$ such that

$$\left\{ \begin{array}{l} \displaystyle\int_0^1 \nu y_h' \varphi_i' + (y_h^* y_h)' \varphi_i \, dx = \langle g_h, \varphi_i \rangle_{\mathcal{H}^*, \mathcal{H}} \quad \text{for } 1 \le i \le m, \\ y_h(0) = a. \end{array} \right. \tag{4.79}$$

First we make problem (4.79) homogeneous. Setting $v_d(x) = a(1-x)$ and $w_h = y_h - v_d$ we conclude that $w_h \in Y_{2,h}$ solves

$$\int_0^1 \nu w_h' \varphi_i' + (y_h^* w_h)' \varphi_i \, dx = \langle \tilde{g}_h, \varphi_i \rangle_{\mathcal{H}^*, \mathcal{H}} \quad \text{for } 1 \le i \le m, \tag{4.80}$$

where

$$\langle \tilde{g}_h, \varphi_i \rangle_{\mathcal{H}^*, \mathcal{H}} = \langle g_h, \varphi_i \rangle_{\mathcal{H}^*, \mathcal{H}} - \int_0^1 a y_h^* \varphi_i + v_d (y_h^*)' \varphi_i \, dx,$$

if and only if $v_h = w_h + v_d$ is a solution of (4.79).

We shall apply the Fredholm alternative [**Kre78**, p. 437] and rewrite (4.80) in operator notation. For this purpose we introduce the linear operator $K_h \in L(Y_{2,h}, Y_{2,h}^*)$ by $K_h w_h = (y_h^* w_h)'$. The action of the dual operator is given by

$$\langle K_h^* w_h, \varphi_h \rangle_{\mathcal{H}^*, \mathcal{H}} = -\int_0^1 (y_h^*)' w_h \varphi_h \, dx + y_h^*(1) w_h(1) \varphi_h(1) \quad \text{for } \varphi_h \in Y_{2,h}.$$

Since $Y_{2,h}$ is finite dimensional, the operator K_h is compact for every h. Let $D_h \in L(Y_{2,h}, Y_{2,h}^*)$ be given by $-\nu\, w_h''$. This implies

$$(D_h + K_h)w_h = \tilde{g}_h \qquad \text{in } Y_{2,h}^* .$$

For every h the operator D_h possesses the matrix representation

$$\mathbf{D}_h = \frac{\nu}{h} \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & -1 & 2 & -1 \\ 0 & & -1 & 1 \end{pmatrix} \in \mathbb{R}^{m\times m} ,$$

which is positive definite and thus invertible. Hence, D_h^{-1} exists and is linear, and we have for $G_h = D_h + K_h$

$$-D_h^{-1}G_h = \underbrace{(-D_h^{-1}K_h)}_{=T_h} - I_h .$$

For a given $\bar{g}_h \in Y_{2,h}^*$ the equation $G_h w_h = \bar{g}_h$ is solvable if and only if $(T_h - I_h)w_h = -D_h^{-1}\bar{g}_h$ is solvable. Applying the Fredholm alternative yields: If the equation $T_h^* f_h = f_h$ has only the trivial solution, then $G_h w_h = \bar{g}_h$ is solvable for every $\bar{g}_h \in Y_{2,h}^*$. Setting $\phi_h = D_h^{-1} f_h \in Y_{2,h}$ the equation $T_h^* f_h = f_h$ leads to

$$\int_0^1 \nu \phi_h' \varphi_h' - y_h^* \phi_h' \varphi_h \, dx + y_h^*(1)\phi_h(1)\varphi_h(1) = 0 \qquad \text{for } \varphi_h \in Y_{2,h} . \tag{4.81}$$

Setting $\varphi_h = \phi_h$ we derive from (4.81) that

$$\nu \, \|\phi_h\|_{\mathcal{H}}^2 + \int_0^1 (y_h^* \phi_h)' \phi_h \, dx = 0 ,$$

which implies by assumption that $\|\phi_h\|_{\mathcal{H}} = 0$. Since $\phi_h \in \mathcal{H}$ holds, we infer that $\phi_h = 0$ in $[0,1]$. Therefore the equation $T_h^* f_h = f_h$ has only the trivial solution. Hence, G_h is surjective, i.e. there exists y_h solving (4.80) by setting $\bar{g}_h = \tilde{g}_h$.

As the operator $e_h'(y_h^*, u_h^*) : \mathcal{Y}_h \times \{0\} \to Y_h$ is surjective and $\dim \mathcal{Y}_h \times \{0\} = \dim Y_h = m+1$ holds, the operator $e_h'(y_h^*, u_h^*)|_{\mathcal{Y}_h \times \{0\}}$ is also injective. Hence, y_h is unique determined. □

Now we define the restriction operator, which is needed in Lemma 6.9. Let $(y,u) \in X$ and

$$0 \le \nu \, \|\varphi_h\|_{\mathcal{H}}^2 + \int_0^1 (y_h^* \varphi_h)' \varphi_h \, dx \qquad \text{for all } \varphi_h \in Y_{2,h} . \tag{4.82}$$

In addition, we suppose that there is a constant $c_2 > 0$ independent of h such that $\|y_h^*\|_{H^1} \le c_2 \, \|y^*\|_{H^1}$ for all h. By Lemma 6.12 there exists a unique element $y_h \in \mathcal{Y}_h$ satisfying

$$e_h'(y_h^*, u_h^*)(y_h, 0) = s_h^Y e'(y_h^*, u_h^*)(y, u) \qquad \text{in } Y_h^* . \tag{4.83}$$

We introduce the linear restriction operator $\Pi_h : X \to X_h$ by $\Pi_h(y,u) = (y_h, 0)$ for all $(y,u) \in X$. From (4.82) and (4.83) we conclude that

$$\frac{\nu}{2} \|y_h'\|_{L^2} \leq \underbrace{(\nu + 2c_4 [\|y^*\|_{H^1} + \|y^*\|_{L^2}])}_{=c_3} \|(y,u)\|_X . \tag{4.84}$$

Let $v_d(x) = y(0)(1-x)$. Then equation (4.83) implies that $y_h - v_d \in \mathcal{H}$. Moreover, $\|v_d'\|_{L^2} \leq \|y\|_{H^1}$ and $\|v_d\|_{H^1} \leq \sqrt{2}\, \|y\|_{H^1}$. From the triangle inequality and Lemma 3.4 on page 9 we infer that

$$\begin{aligned} \|y_h\|_{H^1} &\leq \|y_h - v_d\|_{\mathcal{H}} + \|v_d\|_{H^1} \\ &\leq \sqrt{2}\, \|y_h - v_d\|_{H_0^1} + \sqrt{2}\, \|y\|_{H^1} \\ &\leq \sqrt{2}\, \|y_h'\|_{L^2} + \sqrt{2}\, \|v_d'\|_{L^2} + \sqrt{2}\, \|y\|_{H^1} \\ &\overset{(4.84)}{\leq} \Big(\underbrace{\sqrt{2}(\frac{2c_3}{\nu} + 2)}_{=c_4}\Big) \|(y,u)\|_X \end{aligned}$$

Hence, we have

$$\langle e'(y_h^*, u_h^*)[\Pi_h(y,u) - (y,u)], \phi_h \rangle_{Y^*,Y} = 0 \qquad \text{for all } \phi_h \in Y_h$$

and

$$\|\Pi_h(y,u)\|_X \leq c_4 \, \|(y,u)\|_X .$$

By Theorem 5.2 assumption (4.72) follows from Lemma 2.4 on page 7. By Lemma 6.9 we derive the following theorem.

THEOREM 6.13. *Let $(y_h^*, u_h^*) \in X_h$ denote a solution to $(\mathbf{P_{2,h}})$ with*

$$0 \leq \nu \, \|\varphi_h\|_{\mathcal{H}}^2 + \int_0^1 (y_h^* \varphi_h)' \varphi_h \, dx \qquad \text{for all } \varphi_h \in \operatorname{Span}\{\varphi_1, \dots, \varphi_m\} .$$

Further, we assume that there exists a constant $c_2 > 0$, which does not depend on h and satisfies $\|y_h^*\|_{H^1} \leq c_4 \, \|y^*\|_{H^1}$ for all h. Then Condition 6 is satisfied at (y_h^*, u_h^*).

6.7. The Second-Order Sufficient Optimality Condition. Now we proceed with the study of the second-order sufficient optimality condition (Condition 7 on page 37). Observe that $\langle L''(y,u,(\lambda,\mu)) \cdot , \cdot \rangle_{X^*,X}$ is coercive on X_h as soon as

$$\langle L''(y,u,\lambda,\mu)(\bar{y}_h, \bar{u}_h), (\bar{y}_h, \bar{u}_h) \rangle_{X^*,X} > 0 \qquad \text{for all } (\bar{y}_h, \bar{u}_h) \neq 0 .$$

LEMMA 6.14. *Let $\mathcal{V} = H^1(0,1)$, $C = I$. If $\|y^* - z\|_{L^2}$ is sufficiently small and $(y_h^*, u_h^*, \lambda_h^*, \mu_h^*)$ is sufficiently close to $(y^*, u^*, \lambda^*, \mu^*)$, then Condition 7 on page 37 is satisfied for both control problems.*

PROOF. By Theorem 3.5 and Theorem 5.5 the operator $L''(y^*, u^*, \lambda^*, \mu^*)$ is coercive on the whole space X. Since L'' is continuous the claim follows. □

6.8. Condition 8. Using (2.11) and Remark 3.27 on page 35 we estimate

$$
\begin{aligned}
&\|r_h^Y \mathfrak{J}_Y e(y,u) - \mathfrak{J}_h e_h(r^X(y,u))\|_{Y_h} \\
&\quad \le \|p_h^Y r_h^Y \mathfrak{J}_Y e(y,u) - \mathfrak{J}_Y e(y,u)\|_Y + \|\mathfrak{J}_Y e(y,u) - p_h^Y \mathfrak{J}_h s_h^Y e(y,u)\|_Y \\
&\qquad + \|\mathfrak{J}_h s_h^Y e(y,u) - \mathfrak{J}_h s_h^Y e(p_h^X r_h^X(y,u))\|_{Y_h} \\
&\quad \le \|p_h^Y r_h^Y \mathfrak{J}_Y e(y,u) - \mathfrak{J}_Y e(y,u)\|_Y + \|\mathfrak{J}_Y e(y,u) - p_h^Y \mathfrak{J}_h s_h^Y e(y,u)\|_Y \\
&\qquad + c_\circ^2 \gamma_e \, \|(y,u) - p_h^X r^X(y,u)\|_X \,.
\end{aligned}
$$

Since the internal approximation of Z is convergent by Lemma 6.2 we conclude that

$$\lim_{h\to 0} \|p_h^Y r_h^Y \mathfrak{J}_Y e(y,u) - \mathfrak{J}_Y e(y,u)\|_Y = 0$$

and

$$\lim_{h\to 0} \|(y,u) - p_h^X r_h^X(y,u)\|_X = 0$$

for all $(y,u) \in X$ hold. Setting $w = \mathfrak{J}_Y e(y,u)$ and $w_h = \mathfrak{J}_h s_h^Y e(y,u)$ we derive that

$$
\begin{aligned}
\langle w, \phi\rangle_Y &= \langle e(y,u), \phi\rangle_{Y^*,Y} \quad \text{for all } \phi \in Y\,, \\
\langle w_h, \phi_h\rangle_Y &= \langle e(y,u), \phi_h\rangle_{Y^*,Y} \quad \text{for all } \phi_h \in Y_h\,.
\end{aligned} \tag{4.85}
$$

If we prove that

$$\lim_{h\to 0} \|w - p_h^Y w_h\|_Y = 0\,, \tag{4.86}$$

then Condition 8 is satisfied.

For $(\mathbf{P_1})$ and $(\mathbf{P_2})$ we have $Y = H_0^1(0,1)\times\mathbb{R}^2$ and $Y = \mathcal{H}\times\mathbb{R}$, respectively. Hence, (4.86) follows from error estimates for piecewise linear finite elements [**Ha86**, p. 170].

REMARK 6.15. Analogously, we can show Condition 13 on page 65.

6.9. Numerical Examples. We carried out numerous tests for Examples 6.16-6.18 and focused out interest on the following aspects

- mesh-independence,
- rate of convergence.

The programs were written in MATLAB, and we use an IBM RS/6000 590-computer.

EXAMPLE 6.16 ($\mathbf{P_1}$). Let $\omega = (0, 0.9)$, $\sigma = 10$, $\nu = 0.08$, $\mathcal{V} = L^2(0,1)$, C is the injection of $H^1(0,1)$ into $L^2(0,1)$ and $z(x) = -x^2$. We consider the problem

$$\text{(4.87)} \qquad \begin{cases} \text{minimize} & \dfrac{1}{2}\|y - z\|^2_{L^2} + \dfrac{\sigma}{2}\|u\|^2_{L^2(\omega)} \\ \text{subject to} & \begin{cases} -\nu y'' + yy' = 2(\nu + x^3) + Bu & \text{in } (0,1)\,, \\ y(0) = 0,\ y(1) = -1\,. \end{cases} \end{cases}$$

For the augmented Lagrangian-SQP method we choose $c = 40$. The solution to (4.87) is given by $y^*(x) = -x^2$ and $u^* = 0$. In Figure 1 and Figure 2 the graphs of the iterates y_h^i and u_h^i, $i = 0, \dots, 5$ are compared with the solution (y_h^*, u_h^*).

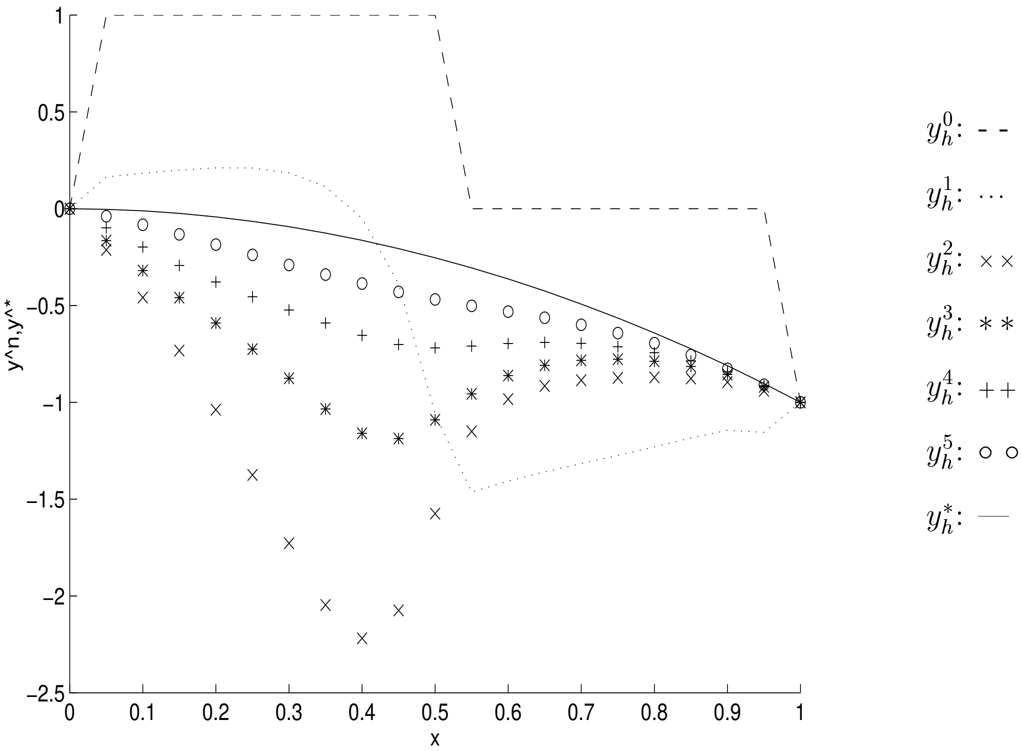

FIGURE 1. Iterates y_h^n and solution y_h^* for $h = 0.05$

To show that the asymptotic mesh-independence can be observed numerically, we specify the smallest iteration number $n = n_h(\varepsilon)$ such that

$$\text{(4.88)} \qquad \|F_h(y_h^n, u_h^n, \tilde{\lambda}_h^n, \tilde{\mu}_{1,h}^n, \tilde{\mu}_{2,h}^n)\|_{Z_h^*} < \varepsilon$$

holds, where F_h is defined in (2.29). Table 1 yields that there is at most a difference of one between the number of steps required to converge within a given tolerance $\varepsilon > 0$. Hence, the asymptotic mesh-independence result

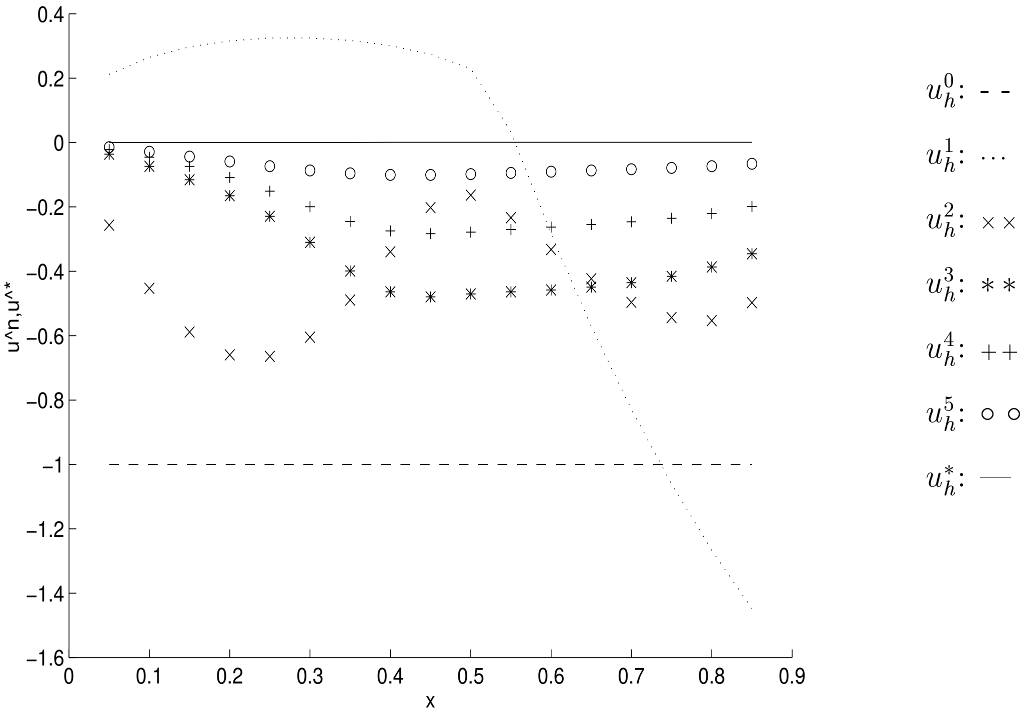

FIGURE 2. Iterates u_h^n and solution u_h^* for $h = 0.05$

of Theorem 3.4 on page 66 can be observed numerically, i.e. Algorithm 2 behaves essentially the same for all sufficiently fine discretizations. We emphasize that the mesh-independence principle is already satisfied for $h \leq 0.05$.

m	20	25	50	100	200	300	400	500	600	700	800	900
$\varepsilon = 10^{-1}$	5	5	5	5	5	5	5	5	5	5	5	6
$\varepsilon = 10^{-2}$	6	6	6	6	6	6	6	6	6	6	6	7
$\varepsilon = 10^{-3}$	6	6	7	7	7	7	7	7	7	7	7	7
$\varepsilon = 10^{-4}$	7	7	7	8	8	8	8	8	8	8	8	8
$\varepsilon = 10^{-5}$	7	7	8	8	8	8	8	8	8	8	8	8
$\varepsilon = 10^{-6}$	7	7	8	8	8	8	8	8	8	8	8	8
$\varepsilon = 10^{-7}$	8	8	8	8	9	9	9	9	9	9	9	9
$\varepsilon = 10^{-8}$	8	8	8	9	9	9	9	9	9	9	9	9
$\varepsilon = 10^{-9}$	8	8	9	9	9	9	9	9	9	9	9	9
$\varepsilon = 10^{-10}$	8	8	9	9	9	9	9	9	9	9	9	9

TABLE 1.

Another aspect is the fast convergence of the algorithm after nine steps. Let

$$e_h(n) = \|y_h^n - y_h^{n-1}\|_{L^2} + \|u_h^n - u_h^{n-1}\|_{L^2} \qquad \text{for } n = 2, \dots .$$

We found the results shown in Table 2.

n	$e_h(n)$	$\frac{e_h(n)}{e_h(n-1)}$	$10^{-2} \cdot \frac{e_h(n)}{e_h(n-1)^2}$
2	6.01e-01	0.3021645	0.002
3	3.32e-01	0.5515550	0.009
4	1.30e-01	0.3911646	0.012
5	4.44e-02	0.3419812	0.026
6	4.49e-03	0.1012156	0.023
7	3.44e-05	0.0076506	0.017
8	1.79e-09	0.0000520	0.018
9	7.30e-16	0.0000004	2.282

TABLE 2.

The second column of Table 2 shows the superlinear rate of convergence of the algorithm, i.e.

$$\lim_{n\to\infty} \frac{\|y_h^n - y_h^{n-1}\|_{L^2} + \|u_h^n - u_h^{n-1}\|_{L^2}}{\|y_h^{n-1} - y_h^{n-2}\|_{L^2} + \|u_h^{n-1} - u_h^{n-2}\|_{L^2}} = 0\,.$$

Moreover, we observe from the third column that $\frac{e_h(n)}{e_h(n-1)^2}$ is bounded, which yields the quadratic rate of convergence. We mention that the floating point relative accuracy in MATLAB is given by $\varepsilon = 2.2204 \cdot 10^{-16}$. Since $e_h(9) \approx 7.30 \cdot 10^{-16}$ holds, the value of $\frac{e_h(9)}{e_h(8)^2}$ is much greater than for the other iteration levels.

Note that $(y_h^*)' \leq 0$ such that the uniform discrete Babuška-Brezzi condition need not to be satisfied. However, the asymptotic mesh-independence holds numerically.

EXAMPLE 6.17 ($\mathbf{P_1}$). Let $\omega = (0.25, 0.75)$, $\sigma = 10$, $\nu = 0.05$, $c = 50$, $\mathcal{V} = L^2(0,1)$, C is the injection of $H^1(0,1)$ into $L^2(0,1)$ and $z(x) = x$. The distributed control problem is given by

$$\text{(4.89)} \qquad \begin{cases} \text{minimize} & \dfrac{1}{2}\,\|y - z\|_{L^2}^2 + \dfrac{\sigma}{2}\,\|u\|_{L^2(\omega)}^2 \\ \text{subject to} & \begin{cases} -\nu y'' + yy' = x + Bu & \text{in } (0,1)\,, \\ y(0) = 0,\; y(1) = 1\,. \end{cases} \end{cases}$$

It is easy to calculate the solution of (4.89): $y^*(x) = x$ and $u^* = 0$. Let us define $(y_h^*, u_h^*) = (x, 0)$ and $(\lambda_h^*, \mu_h^*) = (0, (0, 0))$. Then, $(y_h^*, u_h^*, \lambda_h^*, \mu_h^*)$ solves

$$\begin{aligned} \langle L'(y_h^*, u_h^*, \tilde{\lambda}_h^*, \tilde{\mu}_h^*), \psi_h \rangle_{X^*,X} &= 0 \qquad \text{for all } \psi_h \in X_h\,, \\ \langle e(y_h^*, u_h^*), \phi_h \rangle_{Y^*,Y} &= 0 \qquad \text{for all } \phi_h \in Y_h\,. \end{aligned}$$

In Figure 3 and Figure 4 we give the graphs of the iterates y_h^i and u_h^i, $i = 0, \ldots, 5$, and compare them with the solution (y_h^*, u_h^*).

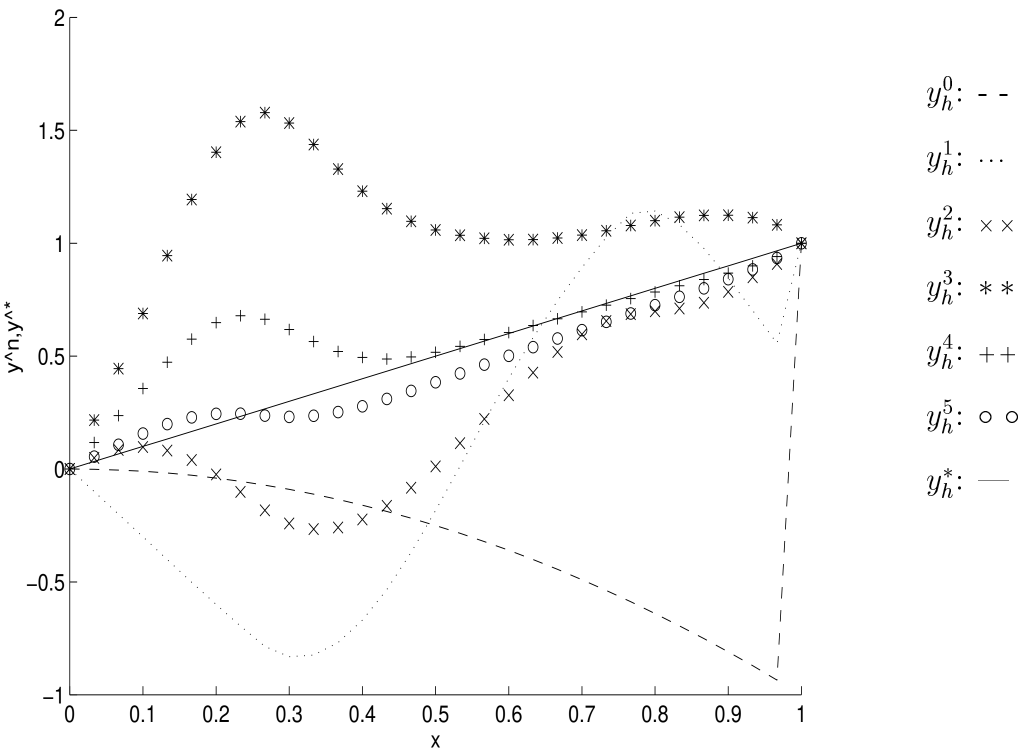

FIGURE 3. Iterates y_h^n and solution y_h^* for $h = \frac{1}{90}$

We again observe that Algorithm 2 behaves essentially the same for all sufficiently fine discretizations, and the asymptotic mesh-independence can numerically be observed for $h \leq \frac{1}{90}$, see Table 3.

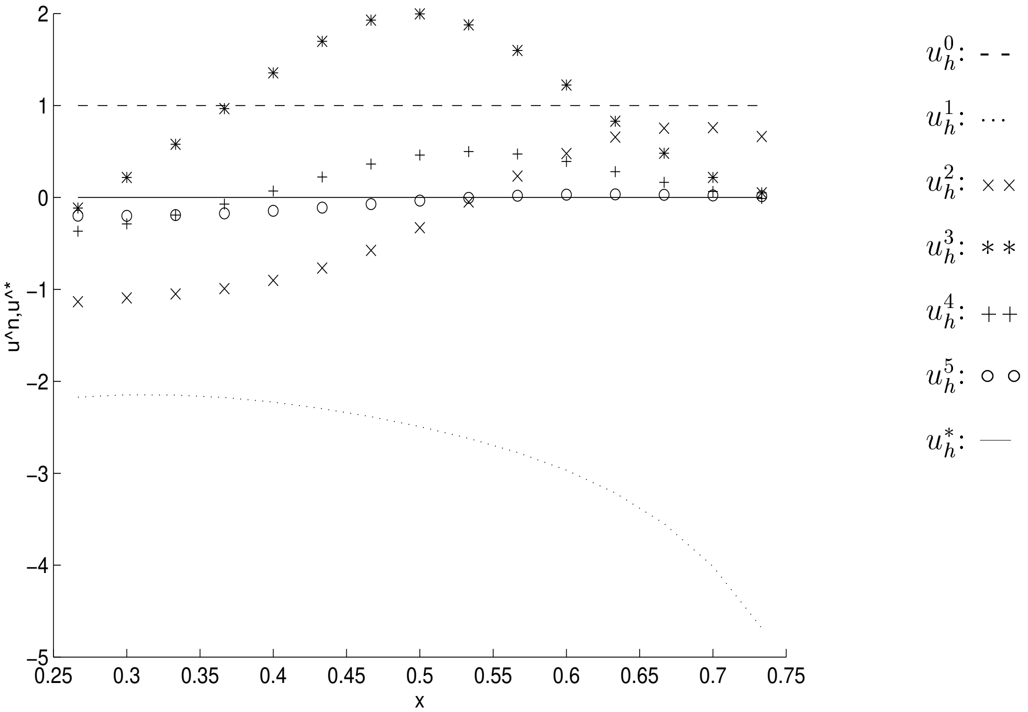

FIGURE 4. Iterates u_h^n and solution u_h^* for $h = \frac{1}{90}$

m	90	100	200	300	400	500	600	700	800	900	1200
$\varepsilon = 10^{-1}$	6	6	6	6	6	6	6	5	5	5	5
$\varepsilon = 10^{-2}$	7	7	7	7	7	7	7	7	7	7	6
$\varepsilon = 10^{-3}$	8	8	7	7	7	7	7	7	7	7	7
$\varepsilon = 10^{-4}$	8	8	8	8	8	8	8	8	8	8	7
$\varepsilon = 10^{-5}$	8	8	8	8	8	8	8	8	8	8	7
$\varepsilon = 10^{-6}$	9	8	8	8	8	8	8	8	8	8	8
$\varepsilon = 10^{-7}$	9	9	8	9	9	9	9	9	9	9	8
$\varepsilon = 10^{-8}$	9	9	9	9	9	9	9	9	9	9	8

TABLE 3.

The rates for $e_h(n)$ are given in Table 4. They have the same behavior as in Table 2 of Example 6.16.

n	$e_h(n)$	$\frac{e(n)}{e(n-1)}$	$10^{-2} \cdot \frac{e(n)}{e(n-1)^2}$
2	2.97e+00	0.670464	0.002
3	3.10e+00	1.045712	0.004
4	1.78e+00	0.573721	0.002
5	6.33e-01	0.355421	0.002
6	1.96e-01	0.309407	0.005
7	1.83e-02	0.093557	0.005
8	1.62e-04	0.008843	0.005
9	1.14e-08	0.000070	0.004
10	1.30e-14	0.000001	1.002

TABLE 4.

Since $(y_h^*)' = 1$ and $\|y_h^*\|_{H^1} = \|y^*\|_{H^1}$ hold, the uniform discrete Babuška-Brezzi condition is satisfied by Theorem 6.11.

EXAMPLE 6.18 ($\mathbf{P_2}$). Let $\sigma = 0.001$, $\nu = 0.05$, $c = 150$, $u^0 = 1$, $\mathcal{V} = L^2(0,1)$, C is the injection of $H^1(0,1)$ into $L^2(0,1)$ and $z = -1$. We consider the problem

$$
\left\{
\begin{array}{ll}
\text{minimize} & \dfrac{1}{2}\,\|y - z\|_{L^2}^2 + \dfrac{\sigma}{2}\,u^2 \\
\text{subject to} & \left\{ \begin{array}{ll} -\nu y'' + yy' = x & \text{in } (0,1)\,, \\ y(0) = 0,\ y'(1) = u\,. & \end{array} \right.
\end{array}
\right. \tag{4.90}
$$

The graphs of (y_h^*, u_h^*) and of the iterates y_h^i, $i \in \{0,2,4,7,9,11,13\}$, and u_h^i are presented in Figure 5 and Figure 6.

Note that Algorithm 2 takes more than fifteen iterations. We denote by v_h the numerical solution of the stationary Burgers equation with homogeneous boundary conditions, i.e. v_h solves

$$
\left\{
\begin{array}{l}
\displaystyle\int_0^1 \nu v_h' \varphi_j' + v_h v_h' \varphi_j \, dx = \int_0^1 x \varphi_j \, dx \quad \text{for } 1 \le j \le m\,, \\
v_h(0) = 0\,.
\end{array}
\right.
$$

Let z_h be the discretization of the desired state z. In Figure 7 both v_h and z_h are compared with the start-up value y_h^0 and with the solution y_h^*.

The control of the differential equation only takes place at the point $x = 1$. Thus, we observe that the iterates y_h^i, $i = 1, \ldots, 6$ are hardly influenced by the control in the interval $[0, 0.4]$. Figure 8 shows that the difference between y_h^6 and v_h is very small in the interval $[0, 0.4]$.

The asymptotic mesh-independence can be seen in Table 5, where we specify the smallest iteration number $n = n_h(\varepsilon)$ such that

$$
\|F_h(y_h^n, u_h^n, \tilde{\lambda}_h^n, \tilde{\mu}_h^n)\|_{Z_h^*} < \varepsilon\,.
$$

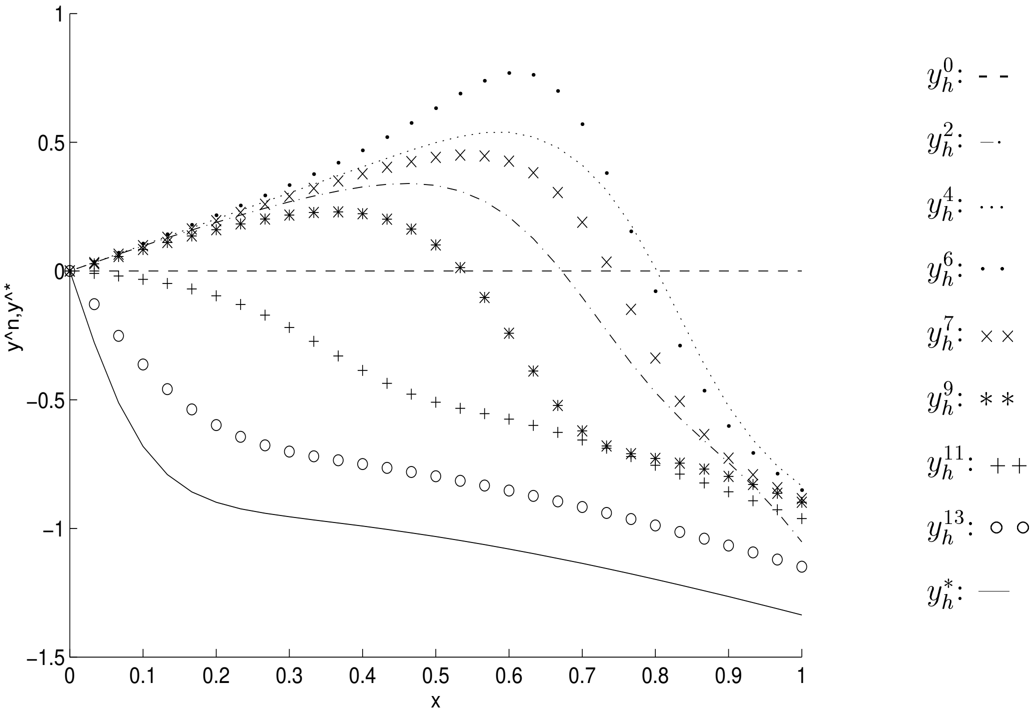

FIGURE 5. Iterates y_h^n and solution y_h^* for $h = 0.01$

Algorithm 2 behaves essentially the same for $h \leq 0.01$.

m	100	200	300	400	500	600	700	800	900	1000	1400
$\varepsilon = 10^{-3}$	16	16	15	15	15	15	15	15	15	15	15
$\varepsilon = 10^{-4}$	17	17	17	17	17	17	17	17	17	17	17
$\varepsilon = 10^{-5}$	17	17	17	17	17	17	17	17	17	17	17
$\varepsilon = 10^{-6}$	17	17	17	17	17	17	17	17	17	17	17
$\varepsilon = 10^{-7}$	17	17	17	17	17	17	17	17	17	17	17
$\varepsilon = 10^{-8}$	18	17	17	17	17	17	17	17	17	17	17
$\varepsilon = 10^{-9}$	18	18	18	18	18	18	18	18	18	18	18

TABLE 5.

Let

$$e_h(n) = \|y_h^n - y_h^{n-1}\|_{L^2} + |u^n - u^{n-1}| \qquad \text{for } n = 2, \ldots .$$

The convergence behavior is shown in Table 6. We note that it takes more iterations before quadratic convergence is obtained. Only when the sixteenth iteration level is reached, the terms $e_h(n)$ strictly decrease and superlinear convergence can be observed from the second column of Table 6.

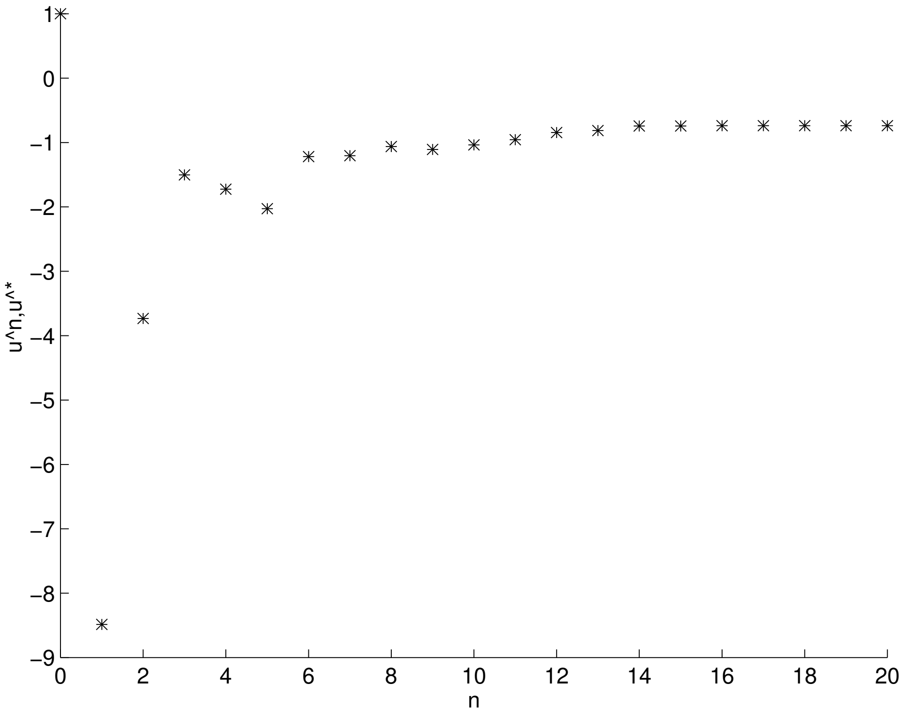

FIGURE 6. Iterates u_h^n and solution u_h^* for $h = 0.01$

n	$e(n)$	$\frac{e(n)}{e(n-1)}$	$10^{-2} \cdot \frac{e(n)}{e(n-1)^2}$
2	4.94e+00	0.4911164	0.0005
3	2.26e+00	0.4580805	0.0009
4	5.31e-01	0.2346843	0.0010
5	4.09e-01	0.7699888	0.0014
6	9.64e-01	2.3564779	0.0576
7	1.58e-01	0.1635922	0.0017
8	3.86e-01	2.4514159	0.1555
9	2.09e-01	0.5411460	0.0140
10	2.53e-01	1.2080849	0.0578
11	2.05e-01	0.8119142	0.0321
12	3.07e-01	1.4982540	0.0730
13	1.02e-01	0.3307544	0.0107
14	2.17e-01	2.1308668	0.2960
15	1.03e-02	0.004.758	0.0022
16	1.07e-02	1.0397945	1.8834
17	4.20e-05	0.0039194	0.0037
18	7.70e-08	0.0018337	0.4365
19	2.58e-14	0.0000003	4.3516

TABLE 6.

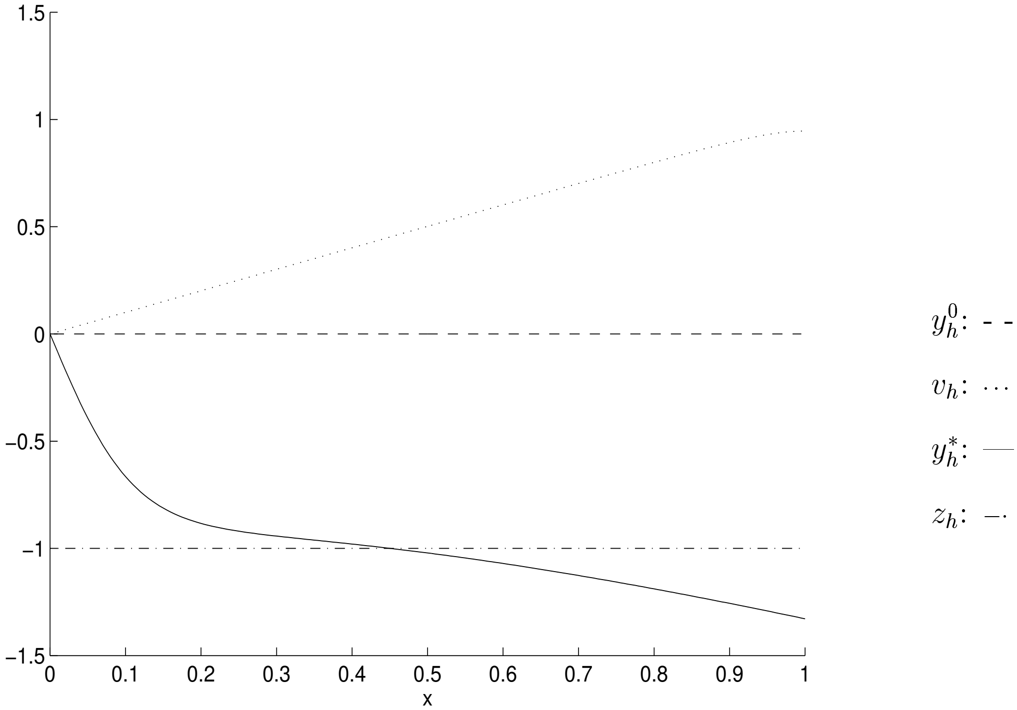

FIGURE 7. y_h^0, v_h, y_h^* and z_h

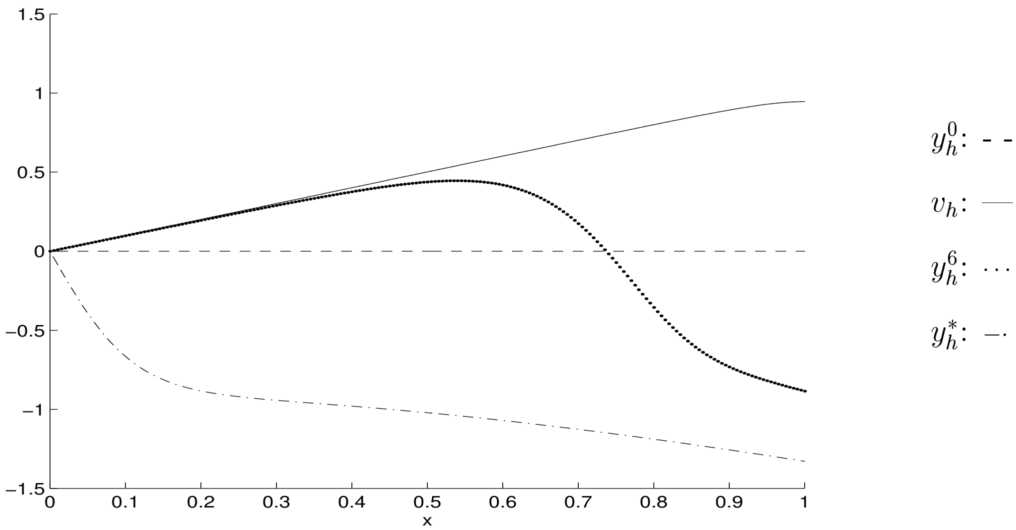

FIGURE 8. y_h^0, v_h, y_h^6 and y_h^*

Let us emphasize that divergence is observed for $c = 0$.

EXAMPLE 6.19 ($\mathbf{P_2}$). Choosing $\sigma = 10$, $\nu = 0.1$, $c = 10$, $u^0 = 0$, $\mathcal{V} = L^2(0,1)$, C is the injection of $H^1(0,1)$ into $L^2(0,1)$ and $z = -1$ we consider the problem

$$
\text{(4.91)} \qquad \begin{cases} \text{minimize} & \dfrac{1}{2}\,\|y-z\|_{L^2}^2 + \dfrac{\sigma}{2}\,u^2 \\ \text{subject to} & \begin{cases} -\nu y'' + yy' = 0 & \text{in } (0,1)\,, \\ y(0) = -1,\ y'(1) = u\,. \end{cases} \end{cases}
$$

The solution to (4.91) is given by $y^* = -1$ and $u^* = 0$, and the graphs of the iterates y_h^i, $i \in \{0,1,2,4,6\}$, and u_h^i are presented in Figure 9 and Figure 10.

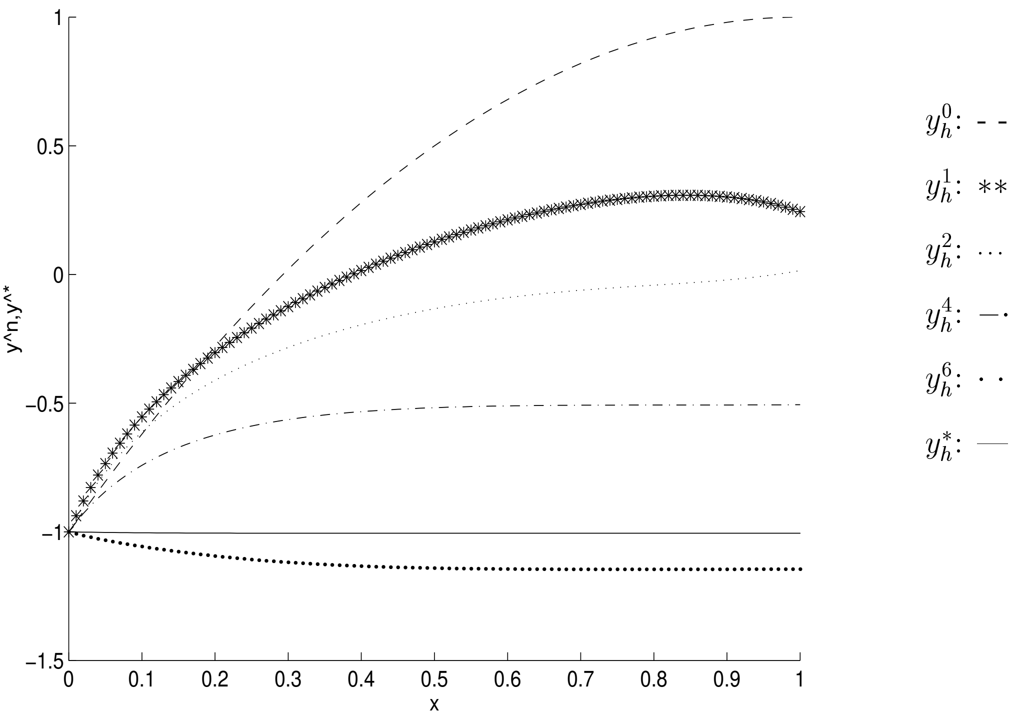

FIGURE 9. Iterates y_h^n and solution y_h^* for $h = 0.01$

Table 7 illustrates again that the asymptotic mesh-independence of the augmented Lagrangian-SQP method can be observed numerically.

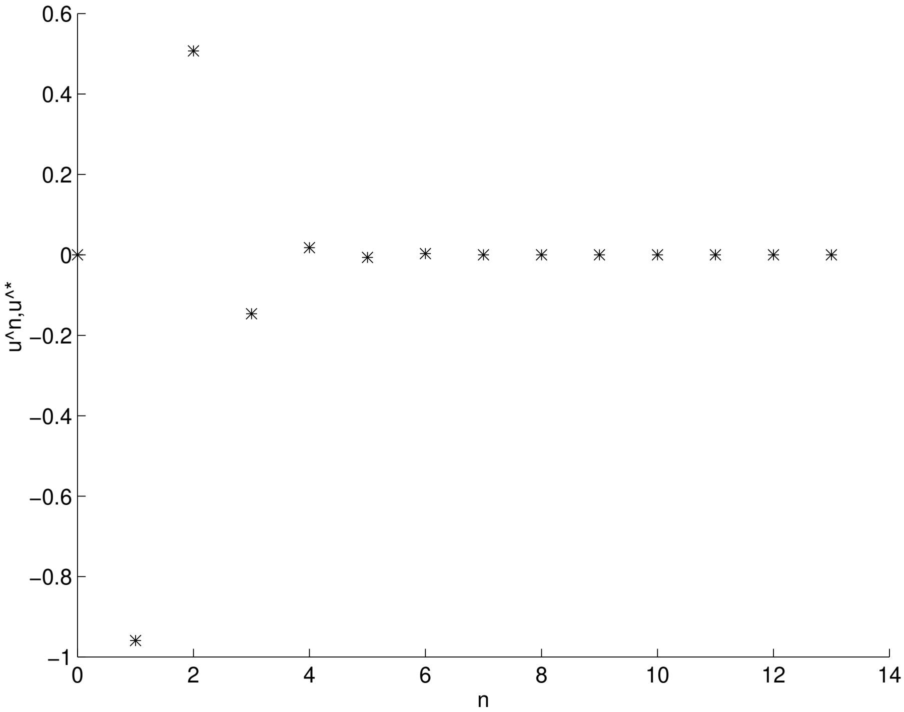

FIGURE 10. Iterates u_h^n and solution u_h^* for $h = 0.01$

m	100	200	300	400	500	600	700	800	900	1000	1500
$\varepsilon = 10^{-2}$	8	8	7	7	7	7	7	7	7	7	7
$\varepsilon = 10^{-3}$	8	8	8	8	8	8	8	8	8	8	8
$\varepsilon = 10^{-4}$	9	9	8	8	8	8	8	8	8	8	8
$\varepsilon = 10^{-5}$	9	9	9	9	9	9	9	9	9	9	9
$\varepsilon = 10^{-6}$	10	9	9	9	9	9	9	9	9	9	9
$\varepsilon = 10^{-7}$	10	10	10	10	10	10	10	10	10	10	10
$\varepsilon = 10^{-8}$	10	10	10	10	10	10	10	10	10	10	10
$\varepsilon = 10^{-9}$	10	10	10	10	10	10	10	10	10	10	10

TABLE 7.

The rates for $e(n)$ are shown in Table 8. The superlinear convergence can just be observed for iteration level $n \geq 6$. The term $\frac{e(n)}{e(n-1)^2}$ fluctuates very much for $n = 2, \dots, 8$. Therefore, the algorithm just reaches the second-order convergence area at iteration level eight. However, the method converges, which illustrates the global behavior of augmented Lagrangian-SQP methods.

n	$e(n)$	$\frac{e(n)}{e(n-1)}$	$\frac{e(n)}{e(n-1)^2}$
2	1.70e+00	1.229141	0.89
3	8.48e-01	0.499861	0.29
4	3.31e-01	0.390875	0.46
5	2.36e-01	0.712062	2.14
6	3.60e-01	1.526040	6.47
7	1.34e-01	0.371884	1.03
8	1.25e-02	0.093432	0.70
9	1.01e-03	0.080570	6.43
10	1.45e-06	0.001442	1.43
11	5.13e-12	0.000004	2.43

TABLE 8.

CHAPTER 5

Control of the Unsteady Burgers Equation

1. Introduction

Since Burgers introduced the equation

$$y_t + yy_x = \nu y_{xx} \tag{5.1}$$

as a simple model for turbulence [**Bur40, Bur48, Bur72**], where $\nu > 0$ denotes a viscosity parameter, the conservation law

$$y_t + yy_x = 0$$

and the so-called viscosity solution

$$y = \lim_{\nu \to 0} y^\nu ,$$

where y^ν satisfies equation (5.1), has been considered, see for example in [**Hop50**]. A complete discussion of results concerned with the conservation law and the viscosity solution may be found in [**Smo83**]. Further, in [**Paz83**] existence and uniqueness of a solution to (5.1) with homogeneous Dirichlet boundary conditions is proved by using the theory of semigroups. We refer the book [**KL89**, Chapter 4], where the Cauchy problem for the viscous and inviscid ($\nu = 0$ in (5.1)) Burgers' equation is investigated.

Let us mention some papers concerning control problems for Burgers' equation. Chen, Wang and Weerakoon handle an initial value control problem for the Burgers equation and show numerical results in [**Wee84, CWW85**]. Burns, Ito and Kang investigate several control problems for Burgers' equation [**BK90, Kan90, BK91, KIB91**]. Using a linearized model feedback control laws are computed, which are applied to the full nonlinear model. The problems, which are under consideration, contain both unbounded inputs and observations. Numerical examples illustrates the theoretical results. Dean and Gubernatis consider the Burgers equation with homogeneous Dirichlet-Neumann boundary conditions on the time boundary [**DG91**]. To obtain a given output function they establish a time control $v(t)$ at a fixed point $x_\circ \in (0, 1)$. In the article [**CT92**] the authors describe the role of incremental unknowns for approximating attractors and inertial manifolds arising in fluid dynamics. Especially, they concern with the Burgers equation with stochastic forces. In [**CTMK93**] a feedback control method of minimizing a cost function is developed and applied to the Burgers equation as a first step towards application to fluid mechanics problems. Ito and Kang construct a dissipative feedback control to regulate the systems arising in fluid dynamics [**IK94**]. For the numerical experiments they use the Burgers equation, but also the Navier-Stokes equations. We mention the paper of Ikeda [**Ike94**], where a method for active fluid flow control based on control theory is analyzed and applied to stochastic Burgers' equation. Numerical results are also presented. Diaz concerns with approximate controllability for a final observation [**DI94**]. He considers the Burgers equation over the space interval $(0, L)$, where the control takes place at $x = L$. In [**BGS95**] a boundary control problem for a forced Burgers equation in a Hilbert space is considered. For the unforced problem, the stability of equilibria is investigated, and for the forced problem the authors concern with steady state behavior of solutions. In the paper [**LMT96**] the results of Burns et al. and Byrnes at al. are generalized. Further, the Burgers equation is treated with other nonlinear boundary conditions.

This chapter is organized as follows: In Section 2 we present an existence and a uniqueness result for the unsteady Burgers equation with homogeneous Dirichlet boundary conditions. This equation is used as the state equation of the distributed control problem in Section 3. We prove that Conditions 1-3 are satisfied such that Algorithm 1 is well-defined and has quadratic rate of convergence if we start sufficiently close to the solution of the first-order necessary optimality system. Section 4 is devoted to the unsteady Burgers equation with inhomogeneous Neumann boundary conditions. Applying the Galerkin procedure we prove the existence of a unique solution for a short time interval. In the fifth section we discuss a boundary control problem for the unsteady Burgers equation, where the solution of the Burgers equation is controlled by the Neumann boundary conditions. To guarantee the well-posedness and quadratic rate of convergence we analyze the regular-point condition — Condition 2 — and the second-order sufficient optimality condition — Condition 3 — and give sufficient assumptions. Some numerical experiments are presented in Section 6, which illustrate that asymptotic mesh-independence can be observed numerically.

2. The Dirichlet Boundary Value Problem

2.1. Existence and Uniqueness. Let $T > 0$, $Q = (0,T) \times (0,1)$, $V = H_0^1(0,1)$ and $H = L^2(0,1)$. We recall the definition of the continuous trilinear form

$$b(\varphi, \psi, \phi) = \frac{1}{3} \int_0^1 (\varphi\psi)'\phi + \varphi\psi'\phi \, dx \qquad \text{for } \varphi, \psi, \phi \in H^1(0,1)$$

and denote by $y(t)$ and $f(t)$ the functions $y(t, \cdot)$ and $f(t, \cdot)$, respectively, considered as functions of x only when t is fixed.

Let us consider the *unsteady Burgers equation*

$$y_t - \nu y_{xx} + y y_x = f \qquad \text{in } L^2(V^*) \tag{5.2a}$$

with the initial condition

$$y(0) = \phi \qquad \text{in } H \tag{5.2b}$$

and with homogeneous Dirichlet boundary conditions

$$y(t,0) = y(t,1) = 0 \qquad \text{for } t \in [0,T] \text{ a.e.}\,, \tag{5.2c}$$

where $f \in L^2(V^*)$ and $\phi \in H$ are given and $\nu > 0$ is the viscosity parameter.

DEFINITION 2.1. A function $y \in W(V)$ is called a weak solution to (5.2) if

$$\langle y_t(t), \varphi \rangle_{V^*,V} + \nu \langle y(t), \varphi \rangle_V + b(y(t), y(t), \varphi) = \langle f(t), \varphi \rangle_{V^*,V} \tag{5.3a}$$

for all $\varphi \in V$ and a.e. $t \in [0,T]$ and

$$y(0) = \phi \tag{5.3b}$$

are valid.

The next theorem ensures the existence of a unique $y \in W(V)$ satisfying (5.3).

THEOREM 2.2. *With $f \in L^2(V^*)$ and $\phi \in H$ holding there exists a unique weak solution $y \in W(V)$ to (5.2).*

PROOF. As in the proof for the unsteady Navier-Stokes equations in [**Tem79**, p. 283-289 and p. 294-295] we derive that there exists a unique $y \in W(V)$ satisfying (5.3). □

REMARK 2.3. From Lemma 3.12 on page 12 we conclude that $y \in C(H)$.

LEMMA 2.4. *For all $\varphi \in V$ the estimate*

$$\|\varphi\|_{L^4} \le \sqrt{2\,\|\varphi\|_H\,\|\varphi\|_V}$$

holds.

PROOF. Since $C_0^\infty(0,1)$ is dense in V, [**Ada75**, p. 31], it is sufficient to prove the claim for all $\varphi \in C_0^\infty(0,1)$. Writing

$$|\varphi(x)|^2 = 2\int_0^x \varphi(s)\varphi'(s)\,ds$$

we derive

$$|\varphi(x)|^2 \le 2\int_0^x |\varphi(s)\varphi'(s)|\,ds\,. \tag{5.4}$$

By Hölder's inequality it follows that

$$\begin{aligned}\int_0^1 |\varphi(x)|^4\,dx &= \int_0^1 |\varphi(x)|^2\,|\varphi(x)|^2\,dx\\ &\overset{(5.4)}{\le} 4\int_0^1 \Big(\int_0^x |\varphi(s)\varphi'(s)|\,ds\Big)^2\,dx\\ &\le 4\int_0^1 \|\varphi\|_H^2\,\|\varphi\|_V^2\,dx\\ &= 4\,\|\varphi\|_H^2\,\|\varphi\|_V^2\,,\end{aligned}$$

which is our assertion. □

REMARK 2.5. Due to Theorem 2.2, Remark 2.3 and Lemma 2.4 we estimate

$$\begin{aligned}\|y\|^4_{L^4(L^4)} &\le 4\int_0^T \|y(t)\|_H^2\,\|y(t)\|_V^2\,dt\\ &\le 4\,\|y\|_{C(H)}\int_0^T \|y(t)\|_V^2\,dt\\ &= 4\,\|y\|^2_{C(H)}\,\|y\|^2_{L^2(V)}\,.\end{aligned}$$

Consequently, $y \in L^4(L^4)$ holds.

The following theorem asserts the existence of a more regular weak solution of problem (5.2).

THEOREM 2.6. *Let $f \in L^2(H)$ and $\phi \in V$. Then the unique weak solution y of problem (5.2) satisfies*

$$y \in L^2(H^2 \cap V)\,, \qquad y_t \in L^2(H)\,.$$

PROOF. The assertion follows directly from [**Tem79**, p. 314]. □

2.2. Estimates for the Solution. The following estimate will be needed in Section 3.

LEMMA 2.7. *Let $f \in L^2(V^*)$ and $\phi \in H$. Then there exists a constant $c_\circ > 0$ such that*

$$\|y\|_{W(V)} \le c_\circ\left(1 + [\|\phi\|_H + \|f\|_{L^2(V^*)}]^2\right)\,.$$

PROOF. Let us first prove that

$$\|y\|^2_{L^2(V)} \le c_1\left(\|\phi\|_H + \|f\|_{L^2(V^*)}\right)^2 \tag{5.5}$$

holds for a constant $c_1 > 0$. For this purpose we multiply (5.2a) by y and integrate the equation over the interval $(0, 1)$. This leads to

$$\frac{1}{2}\frac{d}{dt}\,\|y(t)\|_H^2 + \nu\,\|y(t)\|_V^2 = \langle f(t), y(t)\rangle_{V^*,V}\,. \tag{5.6}$$

Applying the Hölder inequality we obtain

$$\int_0^T \langle f(t), y(t)\rangle_{V^*,V}\, dt \le \int_0^T \|f(t)\|_{V^*} \|y(t)\|_V \, dt \le \|f\|_{L^2(V^*)} \|y\|_{L^2(V)} .$$

Integrating (5.6) over the interval $(0, T)$ we thus derive

$$\frac{1}{2}\|y(T)\|_H^2 + \nu \|y\|_{L^2(V)}^2 \le \frac{1}{2}\|\phi\|_H^2 + \|f\|_{L^2(V^*)} \|y\|_{L^2(V)} . \tag{5.7}$$

Hence, Young's inequality gives

$$2\nu \|y\|_{L^2(V)}^2 \le \|\phi\|_H^2 + \frac{1}{\nu}\|f\|_{L^2(V^*)}^2 + \nu \|y\|_{L^2(V)}^2 ,$$

and therefore

$$\begin{aligned} \|y\|_{L^2(V)}^2 &\le \frac{1}{\nu}\|\phi\|_H^2 + \frac{1}{\nu^2}\|f\|_{L^2(V^*)}^2 \\ &\le \underbrace{\max\left(\frac{1}{\nu}, \frac{1}{\nu^2}\right)}_{=c_1} \left(\|\phi\|_H + \|f\|_{L^2(V^*)}\right)^2 \end{aligned} \tag{5.8}$$

such that estimate (5.5) holds.

To prove that $\|y_t\|_{L^2(V^*)}^2$ is bounded in terms of $\|\phi\|_H$ and $\|f\|_{L^2(V^*)}$ we show that

$$\|y\|_{C(H)} \le \max(1, \sqrt{c_1}) \left(\|\phi\|_H + \|f\|_{L^2(V^*)}\right)^2 . \tag{5.9}$$

Due to (5.6)

$$\begin{aligned} \|y(t)\|_H^2 &\le \|\phi\|_H^2 + 2\|f\|_{L^2(V^*)} \|y\|_{L^2(V)} \\ &\overset{(5.5)}{\le} \|\phi\|_H^2 + 2\|f\|_{L^2(V^*)}\sqrt{c_1}\left(\|\phi\|_H + \|f\|_{L^2(V^*)}\right) \\ &\le \max(1, 2\sqrt{c_1}) \left(\|\phi\|_H + \|f\|_{L^2(V^*)}\right)^2 , \end{aligned}$$

which gives (5.9).

By (5.2a)

$$\begin{aligned} \|y_t(t)\|_{V^*} &\le \|f(t)\|_{V^*} + \nu \|y(t)\|_V + \|y(t)\|_H \|y(t)\|_V \\ &\le \left(\nu + \|y\|_{C(H)}\right) \|y(t)\|_V + \|f(t)\|_{V^*} . \end{aligned}$$

Hence, we have

$$\begin{aligned} \|y_t\|_{L^2(V^*)}^2 &\le 2\int_0^T 2\left(\nu^2 + \|y\|_{C(H)}^2\right) \|y(t)\|_V^2 + \|f(t)\|_{V^*}^2 \, dt \\ &\le (2 + 4\nu^2 c_1) \left(\|\phi\|_H + \|f\|_{L^2(V^*)}\right)^2 \\ &\quad + 4c_1 \max(1, \sqrt{c_1}) \left(\|\phi\|_H + \|f\|_{L^2(V^*)}\right)^4 . \end{aligned} \tag{5.10}$$

Finally, we conclude from (5.8) and (5.10) that

$$\begin{aligned}\|y\|^2_{W(V)} &= \|y\|^2_{L^2(V)} + \|y_t\|^2_{L^2(V^*)}\\ &\le (c_1 + 2 + 4\nu^2 c_1)\left(\|\phi\|_H + \|f\|_{L^2(V^*)}\right)^2\\ &\quad +4c_1 \max(1, \sqrt{c_1})\left(\|\phi\|_H + \|f\|_{L^2(V^*)}\right)^4,\end{aligned}$$

which gives the claim with $c_\circ^2 = \max(c_1 + 2 + 4\nu^2 c_1, 4c_1 \max(1, \sqrt{c_1}))$. □

3. The Distributed Control Problem

Let $\omega \subseteq Q$ be an open set with positive measure, let $V = H_0^1(0,1)$, $H = L^2(0,1)$, $f \in L^2(H)$, $\phi \in H$ and $B \in L(L^2(\omega), L^2(H))$ be given by

$$Bq = \begin{cases} q & \text{in } Q_\circ, \\ 0 & \text{in } Q \setminus \omega \,. \end{cases}$$

For a control $u \in L^2(\omega)$ the state $y \in W(V)$ is given by the weak solution of

$$\begin{cases} y_t - \nu y_{xx} + y y_x = f + Bu & \text{in } L^2(V^*)\,, \\ y(0) = \phi & \text{in } H\,, \end{cases} \tag{5.11}$$

which exists by Theorem 2.2.

We are also given an observation operator $C \in L(W(V), \mathcal{V})$, where $\mathcal{V}$ is a real Hilbert space. With every control we associate the cost of tracking type

$$J(y,u) = \frac{1}{2}\|Cy - z\|^2_{\mathcal{V}} + \frac{\sigma}{2}\|u\|^2_{L^2(\omega)}\,,$$

where $z \in \mathcal{V}$ is a desired state and $\sigma > 0$ is fixed.

The control problem then is

$$\min J(y,u) \quad \text{such that } (y,u) \text{ solves (5.11)}. \tag{5.12}$$

In the context of Chapter 2, Section 2, we set $\mathcal{Y} = W(V)$, $\mathcal{U} = L^2(\omega)$ and $Y = L^2(V) \times H$ and identify with $L^2(V^*) \times H$ the dual of Y. We introduce the operator

$$e = (e_1, e_2) : W(V) \times L^2(\omega) \to L^2(V^*) \times H\,,$$

whose action is defined by

$$\begin{aligned}\langle e_1(y,u), \varphi\rangle_{L^2(V^*),L^2(V)} &= \langle y_t, \varphi\rangle_{L^2(V^*),L^2(V)}\\ &\quad + \int_0^T\int_0^1 \nu y_x \varphi_x + (y y_x - f - Bu)\,\varphi\, dx dt\end{aligned}$$

for $\varphi \in L^2(V)$ and set $e_2(y,u) = y(0) - \phi$. Then we write (5.12) in the following form

$$(\mathbf{P_3}) \qquad \min J(y,u) \quad \text{subject to } e(y,u) = 0.$$

The Lagrangian associated with $(\mathbf{P_3})$ is

$$L(y,u,\lambda,\mu) = J(y,u) + \langle e_1(y,u), \lambda\rangle_{L^2(V^*),L^2(V)} + \langle e_2(y,u), \mu\rangle_H\,.$$

3.1. The Existence of an Optimal Solution.

THEOREM 3.1. *There exists a solution of* $(\mathbf{P_3})$.

PROOF. Let $(y,u) \in X$ satisfy the equation $e(y,u)=0$. We have

$$J(y,u) \geq \frac{\sigma}{2} \|u\|^2_{L^2(\omega)} .$$

From Lemma 2.7 we conclude that $\|y\|_{W(V)} \to \infty$ yields $\|u\|_{L^2(\omega)} \to \infty$. Hence,

$$J(y,u) \longrightarrow +\infty \quad \text{for} \quad \|(y,u)\|_X \to \infty \tag{5.13}$$

such that J is unbounded for unbounded sequences in X. As in the proof of Theorem 3.1 on page 79 we achieve that J is weakly lower semi-continuous. Since $J(y,u) \geq 0$ for all $(y,u) \in X$ there exists $\zeta \geq 0$ with

$$\zeta = \inf \left\{ J(y,u) \,:\, (y,u) \in X \text{ with } e(y,u) = 0 \right\} .$$

This implies the existence of a minimizing sequence $\{(y^n,u^n)\}_{n\in\mathbb{N}}$ in X such that $\zeta = \lim_{n\to\infty} J(y^n,u^n)$ and $e(y^n,u^n)=0$ for all $n \in \mathbb{N}$. Since J is radically unbounded (see (5.13)), there exists an element $(y^*,u^*) \in X$ with

$$y^n \rightharpoonup y^* \quad \text{as } n \to \infty \quad \text{in } W(V)\,, \tag{5.14a}$$
$$u^n \rightharpoonup u^* \quad \text{as } n \to \infty \quad \text{in } L^2(\omega)\,. \tag{5.14b}$$

We infer from (5.14a) that

$$\lim_{n\to\infty} \int_0^T \langle y^n_t(t) - y^*_t(t), \varphi(t) \rangle_{V^*,V} \, dt = 0 \qquad \text{for all } \varphi \in L^2(V)\,.$$

Since $V \hookrightarrow\hookrightarrow L^\infty(0,1)$ holds, we apply Theorem 3.10 on page 11 by choosing $B_0 = V$, $B_1 = L^\infty(0,1)$ and $B_2 = H$. Hence, we have $W(V) \hookrightarrow\hookrightarrow L^2(L^\infty)$ such that $y^n \to y^*$ strongly in $L^2(L^\infty)$. As the sequence $\{y^n\}_{n\in\mathbb{N}}$ converges weakly, $\|y^n\|_{W(V)}$ is bounded [**Kre78**, p. 258]. Consequently, $\|y^n\|_{C(H)}$ is also bounded by Remark 3.13 on page 12. Thus, it follows by Hölder's inequality that

$$
\begin{aligned}
\Big|\int_0^T\int_0^1 & (y^n y_x^n - y^* y_x^*)\,\varphi\,dxdt\Big| \\
&= \frac{1}{2}\Big|\int_0^T\int_0^1 (y^* y^* - y^n y^n)\,\varphi_x\,dxdt\Big| \\
&\le \frac{1}{2}\int_0^T\int_0^1 |\,(y^* - y^n)\,y^n\varphi_x|\,dxdt + \frac{1}{2}\int_0^T\int_0^1 |\,(y^* - y^n)\,y^*\varphi_x|\,dxdt \\
&\le \frac{1}{2}\int_0^T \|y^*(t) - y^n(t)\|_{L^\infty}\,\|y^n(t)\|_H\,\|\varphi(t)\|_{H_0^1}\,dt \\
&\quad + \frac{1}{2}\int_0^T \|y^*(t) - y^n(t)\|_{L^\infty}\,\|y^*(t)\|_H\,\|\varphi(t)\|_{H_0^1}\,dt \\
&\le \frac{1}{2}\,\|y^* - y^n\|_{L^2(L^\infty)}\,\|y^n\|_{C(H)}\,\|\varphi\|_{L^2(V)} \\
&\quad + \frac{1}{2}\,\|y^* - y^n\|_{L^2(L^\infty)}\,\|y^*\|_{C(H)}\,\|\varphi\|_{L^2(V)} \\
&\stackrel{n\to\infty}{\longrightarrow} 0 \qquad \text{for all } \varphi \in L^2(V)\,.
\end{aligned}
$$

As (5.14b) implies

$$
\int_0^T\int_0^1 (Bu^n - Bu^*)\,\varphi\,dxdt \stackrel{n\to\infty}{\longrightarrow} 0 \qquad \text{for all } \varphi \in L^2(V)
$$

we conclude that $e_1(y^*, u^*) = 0$ in $L^2(V^*)$. From $y^* \in W(V)$ we derive that $y^*(0) \in H$ by Lemma 3.12 on page 12. Since $y^n \stackrel{n\to\infty}{\rightharpoonup} y^*$ in $W(V)$ we infer that $y^n(0) \stackrel{n\to\infty}{\rightharpoonup} y^*(0)$ in H and thus

$$
\langle y^n(0) - y^*(0), \psi\rangle_H \stackrel{n\to\infty}{\longrightarrow} 0 \qquad \text{for all } \psi \in H\,.
$$

Thus, $e(y^*, u^*) = 0$ in $L^2(V^*) \times H$. □

3.2. The Fréchet-Derivatives of J and e. In Section 3.1 we have proved that the optimal control problem ($\mathbf{P_3}$) has a solution. The requirements of Condition 1 on page 21 are clearly met by the mapping J. Let $(y, u) \in X$ be arbitrary. Then the actions of the Fréchet-derivatives of J are given by

$$
\begin{aligned}
\langle J'(y,u), (v,q)\rangle_{X^*,X} &= \langle Cy - z, Cv\rangle_{\mathcal{Y}} + \sigma\langle u, q\rangle_{L^2(\omega)}\,, \\
\langle J''(y,u)(w,p), (v,q)\rangle_{X^*,X} &= \langle Cw, Cv\rangle_{\mathcal{Y}} + \sigma\langle p, q\rangle_{L^2(\omega)}
\end{aligned}
$$

for any directions $(v, q), (w, p) \in X$. Thus, the mapping $(y, u) \mapsto J''(y, u)$ is Lipschitz-continuous on the whole space X.

For arbitrary $(y,u) \in X$ we compute the following directional derivative of e_1:

$$\begin{aligned}\langle e_1'(y,u)(v,q),\varphi\rangle_{L^2(V^*),L^2(V)} &= \langle v_t,\varphi\rangle_{L^2(V^*),L^2(V)} \\ &\quad + \int_0^T \int_0^1 \nu v_x \varphi_x + ((yv)_x - Bq)\,\varphi\,dxdt\,.\end{aligned}$$

We draw from Remark 3.13 on page 12

$$\begin{aligned}\|e_1(y+v,u+q) - e_1(y,u) \\ -e_1'(y,u)(v,q)\|_{L^2(V^*)} &= \sup_{\|\varphi\|_V=1} \int_0^T \int_0^1 v v_x \varphi\,dxdt \\ &\leq \|v\|_{C(H)}\,\|v\|_{L^2(V)} \\ &\leq c_E\,\|v\|_{W(V)}^2\end{aligned}$$

and the directional derivative $e_1'(y,u)$ is the Fréchet-derivative of e_1 at (y,u). The action of the second Fréchet-derivative of e_1 is given by

$$\langle e_1''(y,u)(w,p)(v,q),\varphi\rangle_{L^2(V^*),L^2(V)} = \int_0^T \int_0^1 (vw)_x \varphi\,dxdt$$

for all $\varphi \in L^2(V)$ and any directions $(w,p),(v,q) \in X$.

For the operator e_2 we calculate the Fréchet-derivatives

$$e_2'(y,u)(v,q) = v(0)\,, \qquad e_2''(y,u)(w,p)(v,q) = 0$$

for any directions $(w,p),(v,q) \in X$ such that the mapping $(y,u) \mapsto e''(y,u)$ is also Lipschitz-continuous on X.

3.3. The Regular Point Condition. Now we are going to prove that the operator $e'(y,u)$ is surjective for all $(y,u) \in W(V) \times L^2(\omega)$. In particular, Condition 2 on page 22 is satisfied.

THEOREM 3.2. *The operator $e'(y,u)$ is surjective for all $(y,u) \in X$.*

PROOF. The operator $e'(y,u)$ is surjective if and only if for all $(g,h) \in L^2(V^*) \times H$ there exists $(v,q) \in W(V) \times L^2(\omega)$ such that

$$e_1'(y,u)(v,q) = g \quad \text{in } L^2(V^*)\,, \tag{5.15a}$$

$$e_2'(y,u)(v,q) = h \quad \text{in } H\,. \tag{5.15b}$$

We introduce the bilinear form $a(t;\cdot,\cdot) : V \times V \to \mathbb{R}$ for $t \in [0,T]$ a.e. by

$$a(t;\eta,\xi) = \int_0^1 \nu\eta'\xi' - y(t)\eta\xi'\,dx\,.$$

Setting $q = 0$ equations (5.15) yield

$$\frac{d}{dt}\langle v(t),\xi\rangle_H + a(t;v(t),\xi) = \langle g(t),\xi\rangle_{V^*,V} \tag{5.16a}$$

for all $\xi \in V$ and $t \in [0,T]$ a.e. and the initial condition

$$v(0) = h\,. \tag{5.16b}$$

Here $v(t)$ and $g(t)$ denote the functions $g(t,\cdot)$ and $v(t,\cdot)$ considered as a function of x only when t is fixed. For the existence of a solution v to problem (5.16) we have to prove that the bilinear form $a(t;\cdot,\cdot)$ is continuous for $t \in [0,T]$ a.e. and satisfies

$$a(t,\eta,\eta) \geq \rho \left\|\eta\right\|_V^2 - \tau \left\|\eta\right\|_H^2 \tag{5.17}$$

for all $\eta \in V$, for two constants $\rho > 0$, $\tau \geq 0$ and for $t \in [0,T]$ a.e.

As

$$|a(t;\eta,\xi)| \leq \left(\nu + \|y\|_{C(H)}\right) \|\eta\|_V \|\xi\|_V \qquad \text{for } t \in [0,T] \text{ a.e.}$$

holds, the bilinear form $a(t;\cdot,\cdot)$ is continuous for $t \in [0,T]$ a.e. If $y = 0$ holds we conclude that $a(t;\eta,\eta) = \nu \left\|\eta\right\|_V^2$, i.e. setting $\rho = \nu$ and $\tau = 0$ condition (5.17) is valid. On the other hand $y \neq 0$ leads to $\|y\|_{C(H)} > 0$. Thus, Young's and Agmon's inequality result in

$$\begin{aligned}
a(t;\eta,\eta) &= \nu \left\|\eta\right\|_V^2 - \int_0^1 y(t)\eta\eta'\,dx \\
&\geq \nu \left\|\eta\right\|_V^2 - \|y(t)\|_H \|\eta\|_{L^\infty} \|\eta\|_V \\
&\geq \nu \left\|\eta\right\|_V^2 - c_A \|y\|_{C(H)} \|\eta\|_H^{1/2} \|\eta\|_V^{3/2} \\
&\geq \nu \left\|\eta\right\|_V^2 - c_A \|y\|_{C(H)} \left(\frac{1}{4\varepsilon^4} \|\eta\|_H^2 + \frac{3\sqrt[3]{\varepsilon^4}}{4} \|\eta\|_V^2 \right)
\end{aligned}$$

for all $\eta \in V$ and $t \in [0,T]$ a.e. Choosing

$$\varepsilon = \left(\frac{2\nu}{3c_A \|y\|_{C(H)}} \right)^{\frac{3}{4}} \neq 0$$

we infer that

$$a(t;\eta,\eta) \geq \frac{\nu}{2} \|\eta\|_V^2 - \frac{c_A}{4\varepsilon^4} \|y\|_{C(H)} \|\eta\|_H^2$$

for all $\eta \in V$ and $t \in [0,T]$ a.e. Setting $\rho = \frac{\nu}{2}$ and $\tau = \frac{c_A}{4\varepsilon^4} \|y\|_{C(H)}$ condition (5.17) is satisfied. Using [**DL92**, p. 512-513] yields the existence of an element $v \in W(V)$ satisfying (5.15), which is even unique. $\square$

Remark 3.3. The proof of Theorem 3.2 shows that $e'(y,u)|_{\mathcal{Y}\times\{0\}}$ is already surjective.

3.4. The First-Order Necessary Optimality Conditions. If the point $(y^*, u^*) \in X$ is an optimal solution to $(\mathbf{P_3})$ we infer from Theorem 3.2 the existence of Lagrange multipliers $\lambda^* \in L^2(V)$ and $\mu^* \in H$ satisfying the first-order necessary optimality conditions. In particular,

$$\frac{\partial L}{\partial y}(y^*, u^*, \lambda^*, \mu^*) = 0 \quad \text{in } W(V)\,, \tag{5.18a}$$

$$\frac{\partial L}{\partial u}(y^*, u^*, \lambda^*, \mu^*) = 0 \quad \text{in } L^2(\omega)\,. \tag{5.18b}$$

We are going to show that the Lagrange multipliers are unique and specify their regularity properties.

Equation (5.18a) is equivalent to

$$\begin{aligned} 0 = \langle Cy^* - z, Cv\rangle_{\mathcal{V}} &+ \int_0^T \langle v_t(t), \lambda^*(t)\rangle_{V^*,V}\,dt \\ &+ \int_0^T\int_0^1 \nu v_x\lambda_x^* + (vy^*)_x\lambda^*\,dxdt + \int_0^1 v(0)\mu^*\,dx \end{aligned} \tag{5.19}$$

for all $v \in W(V)$, specifically for $v(t) = \chi(t)w$, where $w \in V$ and $\chi \in C_0^\infty(0,T)$. We find

$$\int_0^T \langle v_t(t), \lambda^*(t)\rangle_{V^*,V}\,dt = \left\langle \int_0^T \lambda^*(t)\dot\chi\,dt, w\right\rangle_H = -\left\langle \int_0^T \lambda_t^*(t)\chi\,dt, w\right\rangle_H,$$

where λ_t^* denotes the distributional derivatives of λ^* with respect to t. The remaining terms in (5.19) leads to

$$\begin{aligned} &\langle Cy^* - z, Cv\rangle_{\mathcal{V}} + \int_0^T\int_0^1 \nu\lambda_x^* v_x - y^* v\lambda_x^*\,dxdt \\ &= \left\langle \int_0^T \left(C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z) - \nu\lambda_{xx}^* - y^*\lambda_x^*\right)\chi\,dt, w\right\rangle_{V^*,V}. \end{aligned}$$

Inserting these expressions into (5.19) yields

$$\left\langle \int_0^T \lambda_t^*(t)\chi, w\right\rangle_H = \left\langle \int_0^T \left(C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z) - \nu\lambda_{xx}^* - y^*\lambda_x^*\right)\chi\,dt, w\right\rangle_H$$

for all $w \in H_0^1(0,1)$ and $\chi \in C_0^\infty(0,T)$. Since

$$C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z) - \nu\lambda_{xx}^* - y^*\lambda_x^* \in L^2(V^*)$$

holds and the set

$$\{v\,:\, v(t) = \chi(t)w \text{ for } \chi \in C_0^\infty(0,1) \text{ and } w \in V\}$$

is dense in $L^2(V)$, we conclude that $\lambda_t^* \in L^2(V^*)$ and consequently $\lambda^* \in W(V)$. We notice that for all $v \in W(V)$

$$\int_0^T \frac{d}{dt}\langle \lambda^*(t), v(t)\rangle_H\,dt = \langle \lambda_t^*, v\rangle_{L^2(V^*),L^2(V)} + \langle v_t, \lambda^*\rangle_{L^2(V^*),L^2(V)}\,. \tag{5.20}$$

Hence, we may apply (5.19) and (5.20) and have

$$\begin{aligned}
0 &= \left\langle -\lambda_t^* - \nu\lambda_{xx}^* - y^* v \lambda_x^* + C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z), v \right\rangle_{L^2(V^*),L^2(V)} \\
&\quad + \int_0^T \frac{d}{dt} \langle \lambda^*(t), v(t) \rangle_H \, dt + \langle v(0), \mu^* \rangle_H \\
&= \langle \lambda^*(T), v(T) \rangle_H - \langle \lambda^*(0), v(0) \rangle_H + \langle v(0), \mu^* \rangle_H \\
&= \langle \lambda^*(T), v(T) \rangle_H + \langle \mu^* - \lambda^*(0), v(0) \rangle_H .
\end{aligned}$$

Choosing v appropriately we conclude that

$$\lambda^*(T) = 0 \quad \text{and} \quad \mu^* = \lambda^*(0) . \tag{5.21}$$

Thus, λ^* is the weak solution of the backward differential equation

$$\lambda_t^* = -\nu\lambda_{xx}^* - y^*\lambda_x^* + C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z) \qquad \text{in } L^2(V^*) \tag{5.22a}$$

with the initial condition at $t = T$

$$\lambda^*(T) = 0 . \tag{5.22b}$$

Let $\tau = T - t$ and $\rho^*(\tau) = \lambda^*(t)$ for $t \in [0, T]$. Then problem (5.22) can be transformed into the forward differential equation

$$\rho_\tau^* - \nu\rho_{xx}^* - y^*\rho_x^* = -C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z) \qquad \text{in } L^2(V^*) \tag{5.23a}$$

with the initial condition at $\tau = 0$

$$\rho^*(0) = 0 . \tag{5.23b}$$

LEMMA 3.4. *The Lagrange-multipliers λ^* and μ^* associated with the optimal point (y^*, u^*) are unique. Moreover, $\lambda^* \in W(V)$ holds.*

PROOF. The existence of a unique ρ^* solving problem (5.23) is analogously shown as the proof of the existence of a (unique) v solving problem (5.15) (see the proof of Theorem 3.2). Consequently, $\lambda^*(t) = \rho^*(\tau) \in W(V)$ is the unique solution of problem (5.22), and μ^* is given by $\mu^* = \lambda^*(0)$. □

Now we are going to derive an estimate for the Lagrange-multiplier λ^*. If $y^* = 0$ holds we directly obtain

$$\|\lambda^*\|_{L^2(V)} = \|\rho^*\|_{L^2(V)} \le \frac{1}{\nu} \|C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z)\|_{L^2(V^*)} \tag{5.24}$$

from $\rho^*(\tau) = \lambda^*(t)$ and (5.23a). On the other hand, if $y^* \neq 0$ holds we set

$$c_\circ = \frac{27}{32\nu^3} c_A^4 \|y^*\|_{C(H)}^4 , \tag{5.25}$$

where the constant c_A is given by Agmon's inequality. Let w^* be given by $\rho^*(\tau) = w^*(\tau)\exp(c_\circ\tau)$. Then w^* satisfies

$$\begin{aligned}
\nu \|w^*\|_{L^2(V)}^2 &- \int_0^T \int_0^1 |y^* w_x^* w^*| \, dx d\tau \\
&+ c_\circ \|w^*\|_{L^2(H)}^2 \le \|C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z)\|_{L^2(V^*)} \|w^*\|_{L^2(V)} .
\end{aligned}$$

Analogously as in the proof of Theorem 3.2 we derive that

$$\int_0^T \int_0^1 |y^* w_x^* w^*| \, dx d\tau \leq \frac{\nu}{2} \|w^*\|^2_{L^2(V)} + c_\circ \|w^*\|^2_{L^2(H)} \tag{5.26}$$

such that

$$\|w^*\|_{L^2(V)} \leq \frac{2}{\nu} \|C^* \mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z)\|_{L^2(V^*)}$$

follows. From $\rho_x^*(\tau) = \lambda_x^*(t)$ and $\rho^*(\tau) = w^*(\tau)\exp(c_\circ \tau)$ we conclude that

$$\|\lambda^*\|_{L^2(V)} \leq \frac{2}{\nu} \exp(c_\circ \tau) \|C^* \mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z)\|_{L^2(V^*)}. \tag{5.27}$$

Hence, combining (5.24) and (5.24) we have proved the next lemma.

LEMMA 3.5. *If $C^* \mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z) \in L^2(V^*)$ then the Lagrange multiplier λ^* satisfies*

$$\|\lambda^*\|_{L^2(V)} \leq \frac{2}{\nu} \exp(c_\circ T) \|C^* \mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z)\|_{L^2(V^*)},$$

where $c_\circ$ is given by (5.25).

Note that equation (5.18b) leads to $\sigma u^* = \lambda^*$ in ω.

3.5. The Second-Order Sufficient Optimality Condition. Before we discuss the second-order sufficient optimality condition we prove an auxiliary lemma.

LEMMA 3.6. *Let $(y, u) \in X$ and $(v, q) \in \ker e'(y, u)$. Then there exists a constant $c_{\ker} > 0$ such that*

$$\|v\|^2_{W(V)} \leq c_{\ker} \|q\|^2_{L^2(\omega)}$$

holds.

PROOF. If $y = 0$ then $(v, q) \in \ker e'(y, u)$ leads to $v(0) = 0$ and

$$\langle v_t, \varphi \rangle_{L^2(V^*), L^2(V)} + \int_0^T \int_0^1 \nu v_x \varphi_x - (Bq)\varphi \, dx dt = 0$$

for all $\varphi \in L^2(V)$. In particular, the choice $\varphi = v$ implies that

$$\|v\|_{L^2(V)} \leq \frac{1}{\nu} \|q\|_{L^2(\omega)}. \tag{5.28}$$

Thus, we conclude from (5.28) that

$$\begin{aligned} \|v_t\|_{L^2(V^*)} &= \sup_{\|\varphi\|_{L^2(V)}=1} \int_0^T \int_0^1 -\nu v_x \varphi_x + (Bq)\varphi \, dx dt \\ &\leq \nu \|v\|_{L^2(V)} + \|q\|_{L^2(\omega)} \\ &\leq 2 \|q\|_{L^2(\omega)}. \end{aligned} \tag{5.29}$$

Combining (5.28) and (5.29) the claim follows with $c_{\ker} = 4 + \frac{1}{\nu^2}$.

On the other hand, if $y \neq 0$, then $(v,q) \in \ker e'(y,u)$ implies $v(0) = 0$ and

$$\langle v_t, \varphi \rangle_{L^2(V^*),L^2(V)} + \int_0^T \int_0^1 \nu v_x \varphi_x - y v \varphi_x - (Bq)\varphi \, dx dt = 0 \tag{5.30}$$

for all $\varphi \in L^2(V)$. Let $c_\circ > 0$ be given

$$c_\circ = \frac{27}{32\nu^3} \, c_A^4 \, \|y\|_{C(H)}^4 \, ,$$

where c_A is determined by Agmon's inequality. Then we derive as in the proof of Lemma 3.5 that

$$\|v\|_{L^2(V)} \leq \frac{2}{\nu} \, \exp(c_\circ T) \, \|q\|_{L^2(\omega)} \, . \tag{5.31}$$

Using (5.30) and (5.31) we derive that

$$\begin{aligned} \|v_t\|_{L^2(V^*)} &\leq \left(\nu + \|y\|_{C(H)}\right) \|v\|_{L^2(V)} + \|q\|_{L^2(\omega)} \, , \\ &\leq \left(\left(2 + \tfrac{2}{\nu} \, \|y\|_{C(H)}\right) \exp(c_\circ T) + 1\right) \|q\|_{L^2(\omega)} \, . \end{aligned} \tag{5.32}$$

Combining (5.31) and (5.32) the claim follows with

$$c_{\mathrm{ker}} = \left(\left(2 + \frac{2}{\nu} \, \|y\|_{C(H)}\right) \exp(c_\circ T) + 1\right)^2 + \frac{4}{\nu^2} \, \exp(2 c_\circ T)$$

□

Now we consider the second-order sufficient optimality condition, i.e. we give a sufficient criterion which implies Condition 3 on page 22 holds for $(\mathbf{P_3})$.

THEOREM 3.7. *Let the constants $c_\circ$ and c_E be given by (5.25) and Remark 3.13 on page 12, respectively.*

a) *If $\mathcal{V} = W(V)$, $C = I$ and*

$$\|y^* - z\|_{L^2(H)}^2 < \frac{\nu}{2 c_E} \, \exp(-c_\circ T)$$

hold, then Condition 3 on page 22 is satisfied. Moreover, $L''(y^*, u^*, \lambda^*, \mu^*)$ is coercive on the whole space X.

b) *Let $\mathcal{V} = L^2(H)$ and C be the injection of $W(V)$ into $L^2(H)$. If*

$$\|y^* - z\|_{L^2(H)}^2 < \frac{\sigma \nu}{4 c_{\mathrm{ker}} c_E} \exp(-c_\circ T)$$

is valid, then Condition 3 on page 22 holds.

PROOF. a) Using Remark 3.13 on page 12 we conclude that

$$\begin{aligned}
&\langle L''(y^*,u^*,\lambda^*,\mu^*)(v,q),(v,q)\rangle_{X^*,X}\\
&= \|v\|^2_{W(V)} + \sigma\,\|q\|^2_{L^2(\omega)} - \int_0^T\int_0^1 \lambda^*_x v^2\,dxdt\\
&\geq \|v\|^2_{W(V)} + \sigma\,\|q\|^2_{L^2(\omega)} - \int_0^T \|\lambda^*(t)\|_V\,\|v(t)\|_H\,\|v(t)\|_{L^\infty}\,dt\\
&\geq \|v\|^2_{W(V)} + \sigma\,\|q\|^2_{L^2(\omega)} - \|v\|_{C(H)}\,\|\lambda^*\|_{L^2(V)}\,\|v\|_{L^2(V)}\\
&\geq \Big(1 - c_E\,\|\lambda^*\|_{L^2(V)}\Big)\,\|v\|^2_{W(V)} + \sigma\,\|q\|^2_{L^2(\omega)}
\end{aligned}$$

for all $(v,q)\in X$. Applying Lemma 3.5 leads to

$$\|\lambda^*\|_{L^2(V)} \leq \frac{2}{\nu}\,\exp(c_\circ T)\,\|y^*-z\|_{L^2(H)}\,.$$

Setting

$$\kappa = \min\left(1 - \frac{2}{\nu}\,c_E\exp(c_\circ T)\|y^*-z\|_{L^2(H)}, \sigma\right) > 0$$

we derive that

$$\langle L''(y^*,u^*,\lambda^*,\mu^*)(v,q),(v,q)\rangle_{X^*,X} \geq \kappa\,\|(v,q)\|^2_X \text{ for all } (v,q)\in X\,.$$

b) Using Lemma 3.6 we obtain for all $(v,q)\in\ker e'(y^*,u^*)$

$$\begin{aligned}
&\langle L''(y^*,u^*,\lambda^*,\mu^*)(v,q),(v,q)\rangle_{X^*,X}\\
&\geq \|v\|^2_{L^2(H)} + \sigma\,\|q\|^2_{L^2(\omega)} - \frac{2}{\nu}\,c_E\exp(c_\circ T)\,\|y^*-z\|_{L^2(H)}\,\|v\|^2_{W(V)}\\
&\geq \frac{\sigma}{2}\,\|q\|^2_{L^2(\omega)} + \frac{\sigma}{2c_{\mathrm{ker}}}\,\|v\|^2_{W(V)} - \frac{2}{\nu}\,c_E\exp(c_\circ T)\,\|y^*-z\|_{L^2(H)}\,\|v\|^2_{W(V)}\,.
\end{aligned}$$

Setting

$$\kappa = \min\left(\frac{\sigma}{2c_{\mathrm{ker}}} - \frac{2}{\nu}\,c_E\exp(c_\circ T)\,\|y^*-z\|_{L^2(H)}, \frac{\sigma}{2}\right) > 0$$

we conclude that

$$\langle L''(y^*,u^*,\lambda^*,\mu^*)(v,q),(v,q)\rangle_{X^*,X} \geq \kappa\,\|(v,q)\|^2_X$$

for all $(v,q)\in\ker e'(y^*,u^*)$.

□

3.6. The Scheme of Algorithm 1. In this section we shall formulate Algorithm 1 for control problem ($\mathbf{P_3}$).

a) Choose $(y^0,u^0,(\lambda^0,\mu^0))\in\mathcal{Y}\times\mathcal{U}\times Y$, $c\geq 0$ and put $n=0$.

b) Calculate $e^n \in L^2(V)$ such that

$$\langle e^n(t), \varphi\rangle_V = \langle y_t^n(t), \varphi\rangle_{V^*,V} + \int_0^1 \nu\, y_x^n(t)\varphi' + (y^n(t)\, y_x^n(t) - f - Bu^n)\, \varphi\, dx \tag{5.33}$$

for all $\varphi \in V$ and $t \in [0,T]$ a.e. and define

$$\tilde{\lambda}^n = \lambda^n + c\, e^n\,, \quad \tilde{\mu}^n = \mu^n + c\,(y^n(0) - \phi)\,.$$

c) Solve for $(\bar{y}, \bar{u}, (\bar{\lambda}, \bar{\mu})) \in \mathcal{Y} \times \mathcal{U} \times Y$ the linear system

$$\begin{aligned}
&\langle L''(y^n, u^n, \tilde{\lambda}^n, \tilde{\mu}^n)(\bar{y}, \bar{u}) + e'(y^n, u^n)^*(\bar{\lambda}, \bar{\mu}), (v, q)\rangle_{X^*,X} \\
&\quad = \langle L''(y^n, u^n, \tilde{\lambda}^n, \tilde{\mu}^n)(y^n, u^n) + e'(y^n, u^n)^*(\tilde{\lambda}^n, \tilde{\mu}^n), (v, q)\rangle_{X^*,X} \\
&\qquad - \langle L'(y^n, u^n, \tilde{\lambda}^n, \tilde{\mu}^n), (v, q)\rangle_{X^*,X} \\
&\langle e'(y^n, u^n)(\bar{y}, \bar{u}), \varphi\rangle_{Y^*,Y} \\
&\quad = \langle e'(y^n, u^n)(y^n, u^n) - e(y^n, u^n), \varphi\rangle_{Y^*,Y}
\end{aligned}$$

for all $(v, q, \varphi) \in \mathcal{Y} \times \mathcal{U} \times Y$.

d) We set $(y^{n+1}, u^{n+1}, \lambda^{n+1}, \mu^{n+1}) = (\bar{y}, \bar{u}, \bar{\lambda}, \bar{\mu})$, $n = n + 1$ and go back to b).

4. The Neumann Boundary Value Problem

4.1. Existence and Uniqueness. Let $T > 0$, $Q = (0,T) \times (0,1)$, $V = H^1(0,1)$ and $H = L^2(0,1)$. We shall denote by $y(t)$ and $f(t)$ the functions $y(t,\cdot)$ respectively $f(t,\cdot)$ considered as a function of x only when t is fixed.

We are going to investigate the *unsteady Burgers equation*

$$y_t - \nu y_{xx} + y y_x = f \qquad \text{in } L^2(V^*) \tag{5.34a}$$

with initial conditions

$$y(0) = \phi \qquad \text{in } H \tag{5.34b}$$

and with inhomogeneous Neumann boundary conditions

$$y_x(t,0) = g_0\,, \quad y_x(t,1) = g_1 \quad \text{for } t \in [0,T] \text{ a.e. }, \tag{5.34c}$$

where $f \in L^2(V^*)$, $g_0, g_1 \in L^2(0,T)$ and $\phi \in H$ are given and $\nu > 0$ is the viscosity parameter.

DEFINITION 4.1. A function $y \in W(V)$ is called a weak solution of (5.34) if

$$y(0) = \phi \qquad \text{in } H \tag{5.35a}$$

and

$$\frac{d}{dt}\langle y(t),\varphi\rangle_H + \nu\langle y(t),\varphi\rangle_V - \nu\langle y(t),\varphi\rangle_H + b(y(t),y(t),\varphi) \\ = \langle f(t),\varphi\rangle_{V^*,V} + \nu g_1(t)\varphi(1) - \nu g_0(t)\varphi(0) \tag{5.35b}$$

are valid for all $\varphi \in V$ and $t \in [0,T]$ a.e.

THEOREM 4.2. *Let $g_0, g_1 \in L^2(0,T)$, $\phi \in H$ and $f \in L^2(V^*)$. Then there exists a constant $T_\circ = T_\circ(\phi, f, g_0, g_1, \nu) > 0$ such that* (5.34) *has a unique weak solution y in the interval $[0, T_\circ]$.*

PROOF. We apply the Galerkin procedure for the proof. Since V is a separable Hilbert space there exists an orthonormal basis $\{\psi_j\}_{j\in\mathbb{N}}$ of V, [**Kre78**, p. 171]. We fix $m \in \mathbb{N}$ and introduce $V_m = \operatorname{Span}\{\psi_1, \dots, \psi_m\} \subset V$. An approximate weak solution y_m of (5.34) is defined as follows:

$$y_m(t) = \sum_{i=1}^{m} g_{im}(t)\psi_i \tag{5.36a}$$

satisfies

$$\frac{d}{dt}\langle y_m(t),\psi_j\rangle_H + \nu\langle y_m(t),\psi_j\rangle_V - \nu\langle y_m(t),\psi_j\rangle_H + b(y(t),y(t),\psi_j) \\ = \langle f(t),\psi_j\rangle_{V^*,V} + \nu g_1(t)\psi_j(1) - \nu g_0(t)\psi_j(0) \tag{5.36b}$$

for $j = 1, \dots, m$ and $t \in [0,T]$ a.e. and

$$y_m(0) = \phi_m\,, \tag{5.36c}$$

where ϕ_m is the L^2-orthogonal projection in H of ϕ onto the space V_m. Hence, the estimate $\|\phi_m\|_H \le \|\phi\|_H$ holds. Equation (5.36b) forms a nonlinear differential system for the functions $g_{1m}, \dots, g_{mm}$:

$$\sum_{i=1}^{m}\langle\psi_i,\psi_j\rangle_H\, g'_{im}(t) + \nu\sum_{i=1}^{m}\langle\psi'_i,\psi'_j\rangle_H\, g_{im}(t) + \sum_{i,l=1}^{m}\int_0^1 \psi_i\psi_j\psi'_l\,dx\, g_{im}(t)g_{lm}(t) \\ = \langle f(t),\psi_j\rangle_{V^*,V} + \nu g_1(t)\psi_j(1) - \nu g_0(t)\psi_j(0)\,.$$

Inverting the nonsingular matrix with elements $\langle\psi_i,\psi_j\rangle_H$, $1 \le i,j \le m$, we can write the differential equation in the usual form:

$$g'_{im}(t) + \nu\sum_{j=1}^{m}\alpha_{ij}\, g_{jm}(t) + \sum_{j,k=1}^{m}\alpha_{ijk}g_{jm}(t)g_{km}(t) \\ = \sum_{j=1}^{m}\beta_{ij}\Big(\langle f(t),\psi_j\rangle_{V^*,V} + \nu g_1(t)\psi_j(1) - \nu g_0(t)\psi_j(0)\Big)\,, \tag{5.37a}$$

where $\alpha_{ij}, \alpha_{ijk}, \beta_{jk} \in \mathbb{R}$.

Initial condition (5.36c) is equivalent to the m scalar initial conditions

$$g_{im}(0) = \text{the } i\text{-th component of } \phi_m. \tag{5.37b}$$

The nonlinear differential system (5.37) with a Lipschitz-nonlinearity (quadratic in this case) has a maximal solution defined in some interval around $t = 0$, [**Heu91**, p. 139]. Now we have proved that $y_m(t)$ exists for some interval around $t = 0$, we would like to find the lower bound on the maximal interval of existence of positive time. From the theory of ordinary differential equations we infer that in this case the solution exists as long as it is finite. Therefore, we are going to find estimates on the size of $y_m(t)$. We multiply (5.36b) by $g_{jm}(t)$ and add these equations for $j = 1, \dots, m$. So, we derive that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|y_m(t)\|_H^2 + \nu\|y_m(t)\|_V^2 - \nu\|y_m(t)\|_H^2 + b(y_m(t), y_m(t), y_m(t)) \\ \le \|f(t)\|_{V^*}\|y_m(t)\|_V + \nu\left(|g_0(t)| + |g_1(t)|\right)\|y_m(t)\|_V.\end{aligned} \tag{5.38}$$

Using Young's inequality we find that

$$\|f(t)\|_{V^*}\|y_m(t)\|_V \le \frac{3}{2\nu}\|f(t)\|_{V^*}^2 + \frac{\nu}{6}\|y_m(t)\|_V^2$$

and

$$\nu\left(|g_0(t)| + |g_1(t)|\right)\|y_m(t)\|_V \le 3\nu\left(|g_0(t)|^2 + |g_1(t)|^2\right) + \frac{\nu}{6}\|y_m(t)\|_V^2.$$

By applying the Young and the Agmon inequality again we derive that

$$b(y_m(t), y_m(t), y_m(t)) \le \frac{729c_A}{32\nu^3}\|y_m(t)\|_H^6 + \frac{\nu}{6}\|y_m(t)\|_V^2.$$

So we infer from (5.38) that

$$\begin{aligned}&\frac{d}{dt}\|y_m(t)\|_H^2 + \nu\|y_m(t)\|_V^2 \\ &\quad\le \frac{3}{\nu}\|f(t)\|_{V^*}^2 + 6\nu\left(|g_0(t)|^2 + |g_1(t)|^2\right) \\ &\qquad + 2\nu\|y_m(t)\|_H^2 + \frac{729c_A}{16\nu^3}\|y_m(t)\|_H^6.\end{aligned} \tag{5.39}$$

We define $c_1 = \max(2\nu, \frac{729c_A}{16\nu^3}) > 0$ and $w_m(t) = 1 + \|y_m(t)\|_H^2 \ge 1$. It follows from (5.39) that

$$\begin{aligned}\frac{d}{dt}w_m(t) &= \frac{d}{dt}(1 + \|y_m(t)\|_H^2) \\ &\le c_1\|y_m(t)\|_H^2(1 + \|y_m(t)\|_H^2)^2 + \frac{3}{\nu}\|f(t)\|_{V^*}^2 \\ &\quad + 6\nu\left(|g_0(t)|^2 + |g_1(t)|^2\right) \\ &\le c_1 w_m^3(t) + \frac{3}{\nu}\|f(t)\|_{V^*}^2 + 6\nu\left(|g_0(t)|^2 + |g_1(t)|^2\right) \\ &\le \left(c_1 + \frac{3}{\nu}\|f(t)\|_{V^*}^2 + 6\nu\left(|g_0(t)|^2 + |g_1(t)|^2\right)\right) w_m^3(t).\end{aligned}$$

Consequently,

$$\int_{w_m(0)}^{w_m(t)} \frac{dz}{z^3} \leq \int_0^t c_1 + \frac{3}{\nu} \|f(s)\|_{V^*}^2 + 6\nu \left(|g_0(s)|^2 + |g_1(s)|^2 \right) ds \,,$$

and thus

$$\frac{2}{w_m^2(0)} - \frac{2}{w_m^2(t)} \leq c_1 t + \frac{3}{\nu} \|f\|_{L^2(0,t;V^*)}^2 + 6\nu \left(\|g_0\|_{L^2(0,t)}^2 + \|g_1\|_{L^2(0,t)}^2 \right) .$$

By the dominated convergence theorem [**RS80**, p. 17] there exists $T_* \in (0,T]$ such that

$$\|f\|_{L^2(0,t;V^*)} \leq \frac{\sqrt{\nu}}{\sqrt{6}(1+\|\phi\|_H^2)}$$

and

$$\|g_i\|_{L^2(0,t)} \leq \frac{1}{\sqrt{24\nu}(1+\|\phi\|_H^2)} \,, \quad i = 0, 1 \,,$$

for $t \in (0, T_*]$. From this we conclude that

$$\begin{aligned}
\frac{1}{w_m^2(t)} &\geq \frac{1}{w_m^2(0)} - \frac{c_1}{2} t - \frac{3}{2\nu} \|f\|_{L^2(0,t;V^*)}^2 - 3\nu \left(\|g_0\|_{L^2(0,t)}^2 + \|g_1\|_{L^2(0,t)}^2 \right) \\
&\geq \frac{1}{(1+\|\phi\|_H^2)^2} - \frac{c_1}{2} t - \frac{1}{2(1+\|\phi\|_H^2)^2} \\
&= \frac{1 - c_1(1+\|\phi\|_H^2)^2 t}{2(1+\|\phi\|_H^2)^2}
\end{aligned}$$

for $t \in (0, T_*]$. Finally, we derive that

$$w_m^2(t) \leq \frac{2(1+\|\phi\|_H^2)^2}{1 - c_1(1+\|\phi\|_H^2)^2 t} \qquad \text{for } t \in (0, T_*] \,.$$

Setting $T_\circ = \min(T_*, \frac{1}{2c_1(1+\|\phi\|_H^2)^2})$ we obtain that

$$\|y_m(t)\|_H^2 \leq 1 + 2\|y_m(t)\|_H^2 + \|y_m(t)\|_H^4 = w_m^2(t) \leq 4(1+\|\phi\|_H^2)^2 = c_2$$

for every $m \geq 0$ and for all $t \in [0, T_\circ]$. Thus,

$$y_m \in L^\infty([0,T_\circ]; H) \quad \text{for all } m \geq 0 \,. \tag{5.40}$$

By integrating (5.39) over the interval $[0, T_\circ]$ and using (5.40) it follows that

$$\begin{aligned}
\|y_m(T_\circ)\|_H^2 + \nu \int_0^{T_\circ} \|y_m(s)\|_V^2 \, dt \;\; &\leq \;\; \frac{3}{\nu} \|f\|_{L^2(V^*)}^2 + \|\phi\|_H^2 \\
&\quad + 6\nu \left(\|g_0\|_{L^2(0,T)}^2 + \|g_1\|_{L^2(0,T)}^2 \right) \\
&\quad + \frac{729 c_A c_2^3 T_\circ}{16\nu^3} + 2\nu c_2 T_\circ \,.
\end{aligned}$$

Hence, y_m is uniformly bounded, i.e. independent of m, in the $L^2([0,T_\circ];V)$-norm. Now we are able to derive a bound for $\|(y_m)_t\|_{L^2([0,T_\circ];V^*)}$.

$$\begin{aligned}
\|(y_m)_t\|_{L^2([0,T_\circ];V^*)} &= \int_0^{T_\circ} \Big(\sup_{\|\varphi\|_V=1} \langle (y_m)_t(t),\varphi\rangle_{V^*,V}\Big)^2\,dt \\
&= \int_0^{T_\circ} \Big(\sup_{\|\varphi\|_V=1} \langle f(t),\varphi\rangle_{V^*,V} + \nu\,(g_1(t)\varphi(1) - g_0(t)\varphi(0)) \\
&\qquad -\nu\langle y_m(t),\varphi\rangle_V + \nu\langle y_m(t),\varphi\rangle_H \\
&\qquad -b(y_m(t),y_m(t),y_m(t))\Big)^2\,dt \\
&\le \int_0^{T_\circ} \Big(\|f(t)\|_{V^*} + \nu\,(|g_0(t)| + |g_1(t)|) + \nu\,\|y_m(t)\|_V \\
&\qquad +\nu\,\|y_m(t)\|_H + \frac{1}{2}\,\|y_m(t)\|_H^2 + \frac{1}{2}\,\|y_m(t)\|_V^2\Big)^2\,dt \\
&\le 2\,\|f\|^2_{L^2(V^*)} + 8\nu\left(\|g_0\|^2_{L^2(0,T)} + \|g_1\|^2_{L^2(0,T)}\right) \\
&\qquad +8\nu\,\|y_m\|^2_{L^2([0,T_\circ];V)} + 16\nu\,\|y_m\|^2_{L^\infty([0,T_\circ];H)} \\
&\qquad +16\,\|y_m(t)\|^4_{L^\infty([0,T_\circ];H)} + 32\,\|y_m\|^4_{L^2([0,T_\circ];V)}\,.
\end{aligned}$$

Therefore, y_m is bounded in $W([0,T_\circ];V)$. It follows that there exists a subsequence, which we again denote by $\{y_m\}_{m\in\mathbb{N}}$ and an element $y \in W([0,T_\circ];V)$ such that $y_m \rightharpoonup y$ in $W([0,T_\circ];V)$ as $m \to \infty$. Analysis similar to that in the proof of Theorem 4 in [**LMT96**, p. 13] shows that

$$\begin{aligned}
\frac{d}{dt}\,\langle y(t),\psi_j\rangle_H + \nu\langle y(t),\psi_j\rangle_V - \nu\langle y(t),\psi_j\rangle_H + b(y(t),y(t),\psi_j) \\
= \langle f(t),\varphi\rangle_{V^*,V} + \nu g_1(t)\psi_j(1) - \nu g_0(t)\psi_j(0)
\end{aligned} \tag{5.41}$$

for all $\varphi = \psi_1,\psi_2,\ldots$. Thus, this equation holds for any finite linear combination of the ψ_j's, and by an continuity argument (5.41) is still true for any $\varphi \in V$. In the sequel we shall prove that the weak solution $y \in W([0,T_\circ];V)$ is unique. Let y_1 and y_2 be any two weak solutions of (5.34) in the interval $[0,T_\circ]$. Thus, we derive for $y = y_1 - y_2$ the differential equation

$$y_t - \nu\, y_{xx} + y_1 y_x - y(y_2)_x = 0$$

with homogeneous boundary conditions $y_x(t,0) = y_x(t,1) = 0$ and with homogeneous initial condition $y(0) = 0$. Then the uniqueness follows by the same method as in the proof of Theorem 4 in [**LMT96**, p. 13]. □

REMARK 4.3. The proof of Theorem 4.2 leads to the existence of two constants $c_\circ > 0$ and $c_1 > 0$ such that

$$\|y\|_{W([0,T_\circ];V)} \le c_\circ\left(\|g_0\|^2_{L^2(0,T)} + \|g_1\|^2_{L^2(0,T)}\right) + c_1 \quad \text{for all } g_0, g_1\,.$$

The constant c_1 depends on right-hand side f and the initial condition ϕ.

5. The Boundary Control Problem

As in the previous section we set $V = H^1(0,1)$ and $H = L^2(0,1)$. Let $f \in L^2(H)$ and $\phi \in V$. For controls $u_0, u_1 \in L^2(0,T)$ the state $y \in W(V)$ is given by the weak solution of

$$\left\{\begin{array}{ll} y_t - \nu y_{xx} + y y_x = f & \text{in } L^2(V^*)\,, \\ y_x(\cdot,0) = u_0,\; y_x(\cdot,1) = u_1 & \text{in } L^2(0,T)\,, \\ y(0,\cdot) = \phi & \text{in } H\,. \end{array}\right. \tag{5.42}$$

We are also given an observation operator $C \in L(W(V), \mathcal{V})$, where $\mathcal{V}$ is a real Hilbert space. With every controls u_0 and u_1 we associate the cost of tracking type

$$J(y, u_0, u_1) = \frac{1}{2}\|Cy - z\|_{\mathcal{V}}^2 + \frac{\alpha}{2}\|u_0\|_{L^2(0,T)}^2 + \frac{\beta}{2}\|u_1\|_{L^2(0,T)}^2\,,$$

where $z \in \mathcal{V}$ is a desired state and $\alpha, \beta > 0$ are fixed.

The control problem then is

$$\min J(y, u_0, u_1) \text{ such that } (y, u_0, u_1) \text{ solves (5.42).} \tag{5.43}$$

In the context of Chapter 2, Section 2, we set $\mathcal{Y} = W(V)$, $\mathcal{U} = L^2(0,T)^2$ and $Y = L^2(V) \times H$ and identify with $L^2(V^*) \times H$ the dual of Y. We introduce the operator

$$e = (e_1, e_2) : W(V) \times L^2(0,T)^2 \to L^2(V^*) \times H\,,$$

whose action is defined by

$$\begin{aligned} \langle e_1(y, u_0, u_1), \varphi\rangle_{L^2(V^*),L^2(V)} &= \langle y_t, \varphi\rangle_{L^2(V^*),L^2(V)} \\ &\quad + \int_0^T\int_0^1 \nu y_x \varphi_x + (y y_x - f)\,\varphi\,dxdt \\ &\quad + \nu\int_0^T u_0\varphi(t,0) - u_1\varphi(t,1)\,dt \end{aligned}$$

for $\varphi \in L^2(V)$, and set $e_2(y, u_0, u_1) = y(0) - \phi$. Then we write (5.12) in the following form

$$(\mathbf{P_4}) \qquad \min J(y, u_0, u_1) \quad \text{subject to } e(y, u_0, u_1) = 0.$$

The Lagrangian associated with $(\mathbf{P_4})$ is given by

$$\begin{aligned} L(y, u_0, u_1, \lambda, \mu) &= J(y, u_0, u_1) + \langle e_1(y, u_0, u_1), \lambda\rangle_{L^2(V^*),L^2(V)} \\ &\quad + \langle e_2(y, u_0, u_1), \mu\rangle_H\,. \end{aligned}$$

5.1. The Existence of an Optimal Solution.

CONDITION 15. Let there exists a local solution to $(\mathbf{P_4})$ satisfying $y^* \in L^\infty(V)$.

REMARK 5.1. Let $\mathcal{V} = W(V)$, $f = 0$, $C = I$ and z satisfies

$$\begin{cases} z_t - \nu z_{xx} + z z_x = 0 & \text{in } L^2(V^*)\,, \\ z_x(\cdot, 0) = z_x(\cdot, 1) = 0 & \text{in } L^2(0, T)\,, \\ z(0, \cdot) = \phi & \text{in } H\,. \end{cases}$$

Then $z \in W(V) \cap L^\infty(V)$ follows [**LMT96**]. Obviously, a solution to $(\mathbf{P_4})$ is given by $y^* = z$, $u_0 = u_1 = 0$.

5.2. The Fréchet-Derivatives of J and e. By Condition 15 problem $(\mathbf{P_4})$ has a solution. The requirements of Condition 1 on page 21 are clearly met by the mapping J. Let $(y, u_0, u_1) \in X$ be arbitrary. Then the action of the Fréchet-derivatives of J are given by

$$\begin{aligned} \langle J'(y, u_0, u_1), (v, q_0, q_1) \rangle_{X^*, X} &= \langle Cy - z, Cv \rangle_{\mathcal{V}} + \alpha \langle u_0, q_0 \rangle_{L^2(0,T)} \\ &\quad + \beta \langle u_1, q_1 \rangle_{L^2(0,T)} \end{aligned}$$

and

$$\begin{aligned} \langle J''(y, u_0, u_1)(w, p_0, p_1), (v, q_0, q_1) \rangle_{X^*, X} &= \langle Cw, Cv \rangle_{\mathcal{V}} + \alpha \langle p_0, q_0 \rangle_{L^2(0,T)} \\ &\quad + \beta \langle p_1, q_1 \rangle_{L^2(0,T)} \end{aligned}$$

for all directions $(v, q_0, q_1), (w, p_0, p_1) \in X$. Thus, the mapping $(y, u_0, u_1) \mapsto J''(y, u_0, u_1)$ is Lipschitz-continuous on the space X.

As we discussed in Section 3.2, the operator $e(y, u_0, u_1)$ is twice continuously Fréchet-differentiable in X. The action of the Fréchet-derivatives are given by

$$\begin{aligned} \langle e_1'(y, u_0, u_1)(v, q_0, q_1), \varphi \rangle_{L^2(V^*), L^2(V)} &= \langle v_t, \varphi \rangle_{L^2(V^*), L^2(V)} \\ &\quad + \int_0^T \int_0^1 \nu v_x \varphi_x + (yv)_x \, \varphi \, dx dt \\ &\quad + \nu \int_0^T q_0 \varphi(t, 0) - q_1 \varphi(t, 1) \, dt\,, \\ e_2'(y, u_0, u_1)(v, q_0, q_1) &= v(0) \end{aligned}$$

and

$$\begin{aligned} \langle e_1''(y, u_0, u_1)(w, p_0, p_1)(v, q_0, q_1), \varphi \rangle_{L^2(V^*), L^2(V)} &= \int_0^T \int_0^1 (vw)_x \, \varphi \, dx dt\,, \\ e_2''(y, u_0, u_1)(w, p_0, p_1)(v, q_0, q_1) &= 0 \end{aligned}$$

for all directions $(v, q_0, q_1), (w, p_0, p_1) \in X$. This implies that the mapping $(y, u_0, u_1) \mapsto e''(y, u_0, u_1)$ is also Lipschitz-continuous on X.

5.3. The Regular Point Condition.

THEOREM 5.2. *Let Condition 15 hold. Then the operator $e'(y, u_0, u_1)$ is surjective.*

PROOF. The operator $e'(y,u_0,u_1)$ is surjective if and only if for all $(g,h) \in L^2(V) \times L^2(0,1)$ there exists $(v,q_0,q_1) \in W(V) \times L^2(0,T)^2$ such that

$$e_1'(y,u)(v,q_0,q_1) = g \quad \text{in } L^2(V^*)\,, \tag{5.44a}$$

$$e_2'(y,u)(v,q_0,q_1) = h \quad \text{in } H\,. \tag{5.44b}$$

We introduce the bilinear form $a(t;\cdot,\cdot) : V \times V \to \mathbb{R}$ for $t \in [0,T]$ a.e. by

$$a(t;\eta,\xi) = \int_0^1 \nu\eta'\xi' + y(t)\eta'\xi + y'(t)\eta\xi\,dx\,.$$

Setting $q_0 = q_1 = 0$ equations (5.44) yield

$$\frac{d}{dt}\langle v(t),\xi\rangle_{L^2} + a(t;v(t),\xi) = \langle g(t),\xi\rangle_{V^*,V} \tag{5.45a}$$

for all $\xi \in V$ and $t \in [0,T]$ a.e. and the initial condition

$$v(0) = h\,. \tag{5.45b}$$

For the existence of a solution v to problem (5.45) we have to prove that the bilinear form $a(t;\cdot,\cdot)$ is continuous for $t \in [0,T]$ a.e. and satisfies

$$a(t,\eta,\eta) \geq \rho\,\|\eta\|_V^2 - \tau\,\|\eta\|_H^2 \tag{5.46}$$

for all $\eta \in V$, for two constants $\rho > 0$, $\tau \geq 0$ and for $t \in [0,T]$ a.e.

As

$$|a(t;\eta,\xi)| \leq \Big(\nu + \|y\|_{C(H)} + \|y\|_{L^\infty(V)}\Big)\,\|\eta\|_V\,\|\xi\|_V \qquad \text{for } t \in [0,T] \text{ a.e.}$$

holds, the bilinear form $a(t;\cdot,\cdot)$ is continuous for $t \in [0,T]$ a.e. If $y = 0$ holds we conclude that $a(t;\eta,\eta) = \nu\,\|\eta\|_V^2 - \nu\,\|\eta\|_H^2$, i.e. setting $\rho = \nu$ and $\tau = \nu$ condition (5.46) is valid. Otherwise, $y \neq 0$ leads to $\|y\|_{C(H)} > 0$. Thus, using Agmon's and Young's inequality we have

$$\begin{aligned}
a(t;\eta,\eta) &= \nu\,\|\eta\|_V^2 - \nu\,\|\eta\|_H^2 - \int_0^1 y(t)\eta'\eta\,dx - \int_0^1 y'(t)\eta^2\,dx \\
&\geq \nu\,\|\eta\|_V^2 - \nu\,\|\eta\|_H^2 - \|y(t)\|_H\,\|\eta\|_{L^\infty}\,\|\eta'\|_H - \|y(t)\|_V\,\|\eta\|_H\,\|\eta\|_{L^\infty} \\
&\geq \nu\,\|\eta\|_V^2 - \nu\,\|\eta\|_H^2 - c_A\,\|y\|_{C(H)}\,\|\eta\|_H^{1/2}\,\|\eta\|_V^{3/2} \\
&\quad - c_A\,\|y\|_{L^\infty(V)}\,\|\eta\|_H^{3/2}\,\|\eta\|_V^{1/2} \\
&\geq \nu\,\|\eta\|_V^2 - \frac{3\sqrt[3]{\varepsilon^4}}{4}\,c_A\,\|y\|_{C(H)}\,\|\eta\|_V^2 - \frac{\delta^4}{4}\,c_A\,\|y\|_{L^\infty(V)}\,\|\eta\|_V^2 \\
&\quad - \left(\nu + \frac{c_A}{4\varepsilon^4}\,\|y\|_{C(H)} + \frac{3}{4\sqrt[3]{\delta^4}}\,\|y\|_{L^\infty(V)}\right)\|\eta\|_H^2
\end{aligned}$$

for all $\eta \in V$ and $t \in [0,T]$ a.e. Choosing

$$\varepsilon = \left(\frac{\nu}{3c_A\,\|y\|_{C(H)}}\right)^{\frac{3}{4}} \neq 0\,, \qquad \delta = \left(\frac{\nu}{c_A\,\|y\|_{L^\infty(V)}}\right)^{\frac{1}{4}} \neq 0$$

and

$$\rho = \frac{\nu}{2}, \qquad \tau = \nu + \frac{c_A}{4\varepsilon^4}\, \|y\|_{C(H)} + \frac{3}{4\sqrt[3]{\delta^4}}\, \|y\|_{L^\infty(V)}$$

condition (5.46) is also satisfied for $y \neq 0$. Using [**DL92**, p. 512-513] yields the existence of an element $v \in W(V)$ satisfying (5.44), which is even unique. ☐

Remark 5.3. To achieve Condition 2 on page 22 for the boundary control problem ($\mathbf{P_4}$) the state $y \in W(V)$ has to be more regular. It can be proved that the weak solution of (5.34) belongs to $L^\infty([0, T_\circ]; V)$ if $f \in L^2(H)$, $\phi \in V$ and $g_0, g_1 \in H^1(0, T)$ hold.

5.4. The First-Order Necessary Optimality Conditions. If the point $(y^*, u_0^*, u_1^*) \in X$ is an optimal solution of ($\mathbf{P_4}$) satisfying $y^* \in L^\infty(V)$ we infer from Theorem 5.2 the existence of Lagrange-multipliers $\lambda^* \in L^2(V)$ and $\mu^* \in H$ such that $(y^*, u_0^*, u_1^*, \lambda^*, \mu^*)$ solves the first-order necessary optimality condition (2.2). In particular, we have

$$\frac{\partial L}{\partial y}(y^*, u_0^*, u_1^*, \lambda^*, \mu^*) = 0 \quad \text{in } W(V)\,, \tag{5.47a}$$

$$\frac{\partial L}{\partial u_0}(y^*, u_0^*, u_1^*, \lambda^*, \mu^*) = 0 \quad \text{in } L^2(0, T)\,, \tag{5.47b}$$

$$\frac{\partial L}{\partial u_1}(y^*, u_0^*, u_1^*, \lambda^*, \mu^*) = 0 \quad \text{in } L^2(0, T)\,. \tag{5.47c}$$

We shall prove that the Lagrange-multiplier λ^* is unique and has improved regularity properties. Using (5.47a) we derive that

$$\begin{aligned} 0 = & \int_0^T \langle v_t(t), \lambda^*(t)\rangle_{V^*,V}\, dt + \langle Cy^* - z, Cv\rangle_{\mathcal{V}} \\ & + \int_0^T \int_0^1 \nu v_x \lambda_x^* + (vy^*)_x \lambda^*\, dxdt + \int_0^1 v(0)\mu^*\, dx \end{aligned} \tag{5.48}$$

for all $v \in W(V)$, specifically, for $v(t) = \chi(t)w$, where $w \in H_0^1(0,1)$ and $\chi \in C_0^\infty(0,T)$. We find

$$\int_0^T \langle v_t(t), \lambda^*(t)\rangle_{V^*,V}\, dt = \left\langle \int_0^T \lambda^*(t)\dot{\chi}\, dt, w \right\rangle_{L^2} = -\left\langle \int_0^T \lambda_t^*(t)\chi\, dt, w \right\rangle_{L^2},$$

where λ_t^* denotes the distributional derivatives of λ^* with respect to t. The remaining terms in (5.48) are expressed by

$$\begin{aligned} & \langle Cy^* - z, Cv\rangle_{\mathcal{V}} + \int_0^T \int_0^1 \nu \lambda_x^* v_x - y^*\lambda_x^* v\, dxdt \\ & = \left\langle \int_0^T \left(C^* \mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z) - \nu\lambda_{xx}^* - y^*\lambda_x^*\right) \chi\, dt, w \right\rangle_{V^*,V} \end{aligned}$$

for all $w \in H_0^1(0,1)$ and $\chi \in C_0^\infty(0,T)$. By density of

$$\{v : v = \chi(t)w \text{ for } \chi \in C_0^\infty(0,T) \text{ and } w \in H_0^1(0,1)\}$$

in $L^2(H)$ we infer that

$$\lambda_t^* = -\nu\lambda_{xx}^* - y^*\lambda_x^* + C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z)\,, \tag{5.49}$$

and since the right-hand side is an element of $L^2(V^*)$ we conclude that $\lambda_t^* \in L^2(V^*)$ and consequently $\lambda^* \in W(V)$. We notice that for all $v \in W(V)$

$$\int_0^T \frac{d}{dt}\langle\lambda^*(t), v(t)\rangle_{L^2}\,dt = \langle\lambda_t^*, v\rangle_{L^2(V^*),L^2(V)} + \langle v_t, \lambda^*\rangle_{L^2(V^*),L^2(V)}\,. \tag{5.50}$$

Hence, we may apply (5.49) to the first term on the right-hand side in (5.48). We find for all $v \in W(V)$

$$\begin{aligned}
0 &= \left\langle -\lambda_t^* - \nu\lambda_{xx}^* - y^*v\lambda_x^* + C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z), v\right\rangle_{L^2(V^*),L^2(V)} \\
&\quad + \int_0^T (\nu\lambda_x^*(t,1) + y^*(t,1)\lambda^*(t,1))\,v(t,1)\,dt \\
&\quad - \int_0^T (\nu\lambda_x^*(t,0) + y^*(t,0)\lambda^*(t,0))\,v(t,0)\,dt \\
&\quad + \int_0^T \frac{d}{dt}\langle\lambda^*(t), v(t)\rangle_H\,dt + \langle v(0), \mu^*\rangle_H \\
&= \int_0^T (\nu\lambda_x^*(t,1) + y^*(t,1)\lambda^*(t,1))\,v(t,1)\,dt \\
&\quad - \int_0^T (\nu\lambda_x^*(t,0) + y^*(t,0)\lambda^*(t,0))\,v(t,0)\,dt \\
&\quad + \langle\lambda^*(T), v(T)\rangle_H + \langle\mu^* - \lambda^*(0), v(0)\rangle_H\,.
\end{aligned}$$

Choosing v appropriately we conclude

$$\lambda^*(T) = 0 \quad \text{and} \quad \mu^* = \lambda^*(0)$$

and

$$\nu\lambda_x^*(t,i) + y^*(t,i)\lambda^*(t,i) = 0 \qquad \text{for } i = 0, 1\,.$$

Thus, λ^* is the weak solution of the backward differential equation

$$\lambda_t^* = -\nu\lambda_{xx}^* - y^*\lambda_x^* + C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z) \qquad \text{in } L^2(V^*) \tag{5.51a}$$

with the initial condition at $t = T$

$$\lambda^*(T) = 0\,. \tag{5.51b}$$

and with the boundary conditions

$$\nu\lambda_x^*(t,i) + y^*(t,i)\lambda^*(t,i) = 0 \qquad \text{for } i = 0, 1\,. \tag{5.51c}$$

Let $\tau = T - t$, $\rho^*(\tau, x) = \lambda^*(t, x)$ and $\tilde{y}^*(\tau, x) = y^*(t, x)$ for $(t, x) \in Q$. Then problem (5.51) can be transformed into

$$
\left\{
\begin{array}{l}
\rho_\tau^* - \nu\rho_{xx}^* - y^*\rho_x^* = -C^*\mathfrak{J}_\mathcal{V}^{-1}(C\tilde{y}^* - z) \quad \text{in } L^2(V^*)\,, \\
\rho^*(0) = 0 \quad \text{in } H\,, \\
\nu\rho_x^*(\tau, i) + \tilde{y}^*(\tau, i)\rho^*(\tau, i) = 0 \quad \text{for } i = 0, 1\,.
\end{array}
\right. \tag{5.52}
$$

LEMMA 5.4. *The Lagrange-multipliers λ^* and μ^* associated with the optimal point (y^*, u^*) are unique and $\lambda^* \in W(V)$.*

PROOF. The existence of a unique $\rho^* \in W(V)$ solving problem (5.52) is analogously shown as the proof of the existence of a (unique) v solving problem (5.15) (see the proof of Theorem 5.2). Consequently, $\lambda^*(t, x) = \rho^*(\tau, x)$ is the unique solution of (5.51), From $\mu^* = \lambda^*(0)$ we derive that μ^* is also unique. ☐

Now we are going to derive an estimate for the Lagrange multiplier λ^*. If $y^* = 0$ holds (5.52) yields

$$
\left\{
\begin{array}{l}
\rho_\tau^* - \nu\rho_{xx}^* = -C^*\mathfrak{J}_\mathcal{V}^{-1}(C\tilde{y}^* - z) \quad \text{in } L^2(V^*)\,, \\
\rho^*(0) = 0 \quad \text{in } H\,, \\
\nu\rho_x^*(\tau, i) = 0 \quad \text{for } i = 0, 1\,.
\end{array}
\right. \tag{5.53}
$$

Setting $\rho^* = w^* \exp(\nu\tau)$ we obtain from the differential equation for ρ^* the equation

$$
w_\tau^* - \nu w_{xx}^* + \nu w^* = -\exp(-\nu\tau)C^*\mathfrak{J}_\mathcal{V}^{-1}(C\tilde{y}^* - z) \quad \text{in } L^2(V^*)\,,
$$

which implies

$$
\|w^*\|_{L^2(V)} \le \frac{1}{\nu}\,\|C^*\mathfrak{J}_\mathcal{V}^{-1}(C\tilde{y}^* - z)\|_{L^2(V^*)} = \frac{1}{\nu}\,\|C^*\mathfrak{J}_\mathcal{V}^{-1}(Cy^* - z)\|_{L^2(V^*)}
$$

Consequently,

$$
\|\lambda^*\|_{L^2(V)} = \|\rho^*\|_{L^2(V)} \le \frac{1}{\nu}\,\exp(\nu T)\,\|C^*\mathfrak{J}_\mathcal{V}^{-1}(Cy^* - z)\|_{L^2(V^*)}\,. \tag{5.54}
$$

On the other hand, if $y^* \neq 0$ is valid we set

$$
c_\circ = \nu + \frac{27c_A^4}{32\nu^3}\,\|y^*\|_{C(H)}^4 + \frac{3}{4\sqrt[3]{\nu}}\,\|y^*\|_{L^\infty(V)}^{4/3}\,, \tag{5.55}
$$

where the constant c_A is given by Agmon's inequality. Let $\rho^* = w^* \exp(c_\circ\tau)$. Then we conclude from (5.52) that

$$
\left\{
\begin{array}{l}
w_\tau^* - \nu w_{xx}^* - \tilde{y}^* w_x^* + c_\circ w^* = -\exp(-c_\circ T)C^*\mathfrak{J}_\mathcal{V}^{-1}(C\tilde{y}^* - z)\,, \\
w^*(0) = 0\,, \\
\nu w_x^*(\tau, i) + \tilde{y}^*(\tau, i)w^*(\tau, i) = 0 \ \text{ for } i = 0, 1\,.
\end{array}
\right. \tag{5.56}
$$

We multiply the differential equation by w^* and integrate over the interval $(0,1)$. It follows that

$$\begin{aligned}\frac{d}{dt}\|w(\tau)\|_H^2 + \nu\,\|w(\tau)\|_V^2 - \nu\,\|w(\tau)\|_H^2 + \int_0^1 (\tilde{y}^* w)_x w\,dx \\ = \exp(-c_\circ T) C^* \mathfrak{J}_{\mathcal{V}}^{-1}(C\tilde{y}^* - z)\,.\end{aligned}$$

Now analysis similar to the proof of Theorem 5.2 yields the estimate

$$\|w^*\|_{L^2(V)} \le \frac{2}{\nu}\,\|C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z)\|_{L^2(V^*)}\,.$$

Hence,

$$\|\lambda^*\|_{L^2(V)} = \|\rho^*\|_{L^2(V)} \le \frac{2}{\nu}\,\exp(c_\circ T)\,\|C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z)\|_{L^2(V^*)}\,. \tag{5.57}$$

Combining (5.54) and (5.57) we have proved the following result.

LEMMA 5.5. *The Lagrange multiplier λ^* satisfies*

$$\|\lambda^*\|_{L^2(V)} \le \frac{2}{\nu}\,\exp(c_\circ T)\,\|C^*\mathfrak{J}_{\mathcal{V}}^{-1}(Cy^* - z)\|_{L^2(V^*)}\,,$$

where $c_\circ$ is given by (5.55).

Finally, we consider (5.47b) and (5.47c). We derive that the equations $\alpha u_0^*(t) = -\nu\lambda^*(t,0)$ and $\beta u_1^*(t) = \nu\lambda^*(t,1)$ hold.

5.5. The Second-Order Sufficient Optimality Condition. Before we discuss the second-order sufficient optimality condition we prove an auxiliary lemma.

LEMMA 5.6. *Let $(y,u_0,u_1) \in X$ satisfying $y \in L^\infty(V)$ and let $(v,q_0,q_1) \in \ker e'(y,u_0,u_1)$. Then there exists a constant $c_{\mathrm{ker}} > 0$ such that*

$$\|v\|_{W(V)}^2 \le c_{\mathrm{ker}}\left(\|q_0\|_{L^2(0,T)}^2 + \|q_1\|_{L^2(0,T)}^2\right)\,,$$

PROOF. If $y = 0$ holds $(v,q_0,q_1) \in \ker e'(y,u_0,u_1)$ leads to $v(0) = 0$ and

$$\begin{aligned}\langle v_t,\varphi\rangle_{L^2(V^*),L^2(V)} + \int_0^T\int_0^1 \nu v_x\varphi_x\,dxdt \\ +\nu\int_0^T q_0(t)\varphi(t,0) - q_1(t)\varphi(t,1)\,dt = 0\end{aligned} \tag{5.58}$$

for all $\varphi \in L^2(V)$. Let $v = w\exp(\nu t)$. Then we infer from (5.58) that

$$\begin{aligned}\langle w_t,\varphi\rangle_{L^2(V^*),L^2(V)} + \int_0^T\int_0^1 \nu w_x\varphi_x + \nu w\varphi\,dx \\ +\nu\int_0^T \exp(-\nu t)\left(q_0(t)\varphi(t,0) - q_1(t)\varphi(t,1)\right)dt = 0\,.\end{aligned}$$

Thus, $\varphi = w$ and $w(0) = 0$ lead to

$$\begin{aligned}
\|w\|^2_{L^2(V)} &\leq \int_0^T |q_0(t)\varphi(t,0) - q_1(t)\varphi(t,1)|\,dt \\
&\leq \int_0^T |q_0(t)|\,\|w(t)\|_{L^\infty} + |q_1(t)|\,\|w(t)\|_{L^\infty}\,dt \\
&\leq \Big(\|q_0\|_{L^2(0,T)} + \|q_1\|_{L^2(0,T)}\Big)\,\|w\|_{L^2(V)}\,.
\end{aligned}$$

From $\|v\|_{L^2(V)} \leq \exp(\nu T)\|w\|_{L^2(V)}$ we infer that

$$\|v\|_{L^2(V)} \leq \exp(\nu T)\,\Big(\|q_0\|_{L^2(0,T)} + \|q_1\|_{L^2(0,T)}\Big)\,. \tag{5.59}$$

We conclude from (5.58) and (5.59) that

$$\begin{aligned}
\|v_t\|_{L^2(V^*)} &= \sup_{\|\varphi\|_V=1} \int_0^T \langle v_t, \varphi\rangle_{V^*,V}\,dx \\
&= \sup_{\|\varphi\|_V=1} \int_0^T \int_0^1 -\nu v_x\varphi_x\,dx - \nu q_0(t)\varphi(t,0) + q_1(t)\varphi(t,1)\,dt \\
&\leq \nu\,\Big(\|v\|_{L^2(V)} + \|q_0\|_{L^2(0,T)} + \|q_1\|_{L^2(0,T)}\Big) \\
&\leq \nu\,(1+\exp(\nu T))\,\Big(\|q_0\|_{L^2(0,T)} + \|q_1\|_{L^2(0,T)}\Big)
\end{aligned}$$

such that the claim follows with $c_{\text{ker}} > 2\exp(2\nu T) + 2\nu^2(1+\exp(\nu T))^2$.

On the other hand, let $y \neq 0$. So, $(v, q_0, q_1) \in \ker e'(y, u_0, u_1)$ leads to $v(0) = 0$ and

$$\begin{aligned}
&\langle v_t, \varphi\rangle_{L^2(V^*),L^2(V)} + \int_0^T \int_0^1 \nu v_x\varphi_x + (yv)_x\varphi\,dx \\
&\qquad + \nu \int_0^T q_0(t)\varphi(t,0) - q_1(t)\varphi(t,1)\,dt = 0
\end{aligned} \tag{5.60}$$

for all $\varphi \in L^2(V)$. Let $c_\circ$ be given by

$$c_\circ = \nu + \|y\|_{L^\infty(V)} + \frac{27c_A^4}{32\nu^3}\,\|y\|^4_{C(H)}\,,$$

where c_A is determined by the Agmon inequality. Then we derive as in the proof of Theorem 5.2

$$\|v\|_{L^2(V)} \leq 2\exp(c_\circ T)\,\Big(\|q_0\|_{L^2(0,T)} + \|q_1\|_{L^2(0,T)}\Big)\,. \tag{5.61}$$

Applying (5.60) and (5.61) leads to

$$\begin{aligned}
\|v_t\|_{L^2(V^*)} &= \sup_{\|\varphi\|_V=1} \int_0^T \langle v_t(t), \varphi\rangle_{V^*,V}\, dx \\
&= \sup_{\|\varphi\|_V=1} \int_0^T \int_0^1 -\nu v_x \varphi'\, dx - \nu q_0(t)\varphi(t,0) + q_1(t)\varphi(t,1)\, dt \\
&\le \Big(\nu + \|y\|_{L^\infty(L^\infty)} + \|y\|_{C(H)}\Big) \|v\|_{L^2(V)} \\
&\quad +\nu \Big(\|q_0\|_{L^2(0,T)} + \|q_1\|_{L^2(0,T)}\Big) \\
&\le \Big(\nu + 2\exp(c_\circ T)(\nu + \|y\|_{L^\infty(L^\infty)} + \|y\|_{C(H)})\Big) \\
&\quad \cdot \Big(\|q_0\|_{L^2(0,T)} + \|q_1\|_{L^2(0,T)}\Big)\,,
\end{aligned}$$

such that the claim follows with

$$c_{\ker} = 4\exp(2c_\circ T) + 2\Big(\nu + 2\exp(c_\circ T)(\nu + \|y\|_{L^\infty(L^\infty)} + \|y\|_{C(H)})\Big)^2\,.$$

□

Now we consider the second-order sufficient optimality condition.

THEOREM 5.7. *Let $y^* \in L^\infty(V)$ and the constant $c_\circ > 0$ be given by (5.55).*

a) *If $\mathcal{V} = W(V)$, $C = I$ and*

$$\|y^* - z\|_{L^2(H)} < \frac{\nu}{4c_E}\exp(-c_\circ T)\,,$$

where c_E is determined by Remark 3.13 on page 12. Then Condition 3 on page 22 is satisfied. Moreover, $L''(y^*, u_0^*, u_1^*, \lambda^*, \mu^*)$ is coercive on the whole space X.

b) *Let $\mathcal{V} = L^2(H)$ and C be the injection of $W(V)$ into $L^2(L^2)$. If the residue $\|y^* - z\|_{L^2(H)}$ is small enough, Condition 3 on page 22 holds.*

PROOF. a) Using Remark 3.13 on page 12 we conclude

$$\begin{aligned}
&\langle L''(y^*, u_0^*, u_1^*, \lambda^*, \mu^*)(v, q_0, q_1), (v, q_0, q_1)\rangle_{X^*,X} \\
&= \|v\|^2_{W(V)} + \alpha\|q_0\|^2_{L^2(0,T)} + \beta\|q_1\|^2_{L^2(0,T)} + 2\int_0^T\int_0^1 \lambda^* v v_x\, dxdt \\
&\ge \|v\|^2_{W(V)} + \alpha\|q_0\|^2_{L^2(0,T)} + \beta\|q_1\|^2_{L^2(0,T)} \\
&\quad -\int_0^T \|\lambda^*(t)\|_{L^\infty}\|v(t)\|_H\|v(t)\|_V\, dt \\
&\ge \|v\|^2_{W(V)} + \alpha\|q_0\|^2_{L^2(0,T)} + \beta\|q_1\|^2_{L^2(0,T)} - 2\|v\|_{C(L^2)}\|\lambda^*\|_{L^2(V)}\|v\|_{L^2(V)} \\
&\ge \Big(1 - 2c_E\|\lambda^*\|_{L^2(V)}\Big)\|v\|^2_{W(V)} + \alpha\|q_0\|^2_{L^2(0,T)} + \beta\|q_1\|^2_{L^2(0,T)}
\end{aligned}$$

for all $(v, q_0, q_1) \in X$. Applying Lemma 5.5 leads to

$$\|\lambda^*\|_{L^2(V)} \leq \frac{2}{\nu} \exp(c_\circ T) \, \|y^* - z\|_{L^2(H)} \, .$$

Setting

$$\kappa = \min \left(1 - \frac{4}{\nu} \, c_E \exp(c_\circ T) \|y^* - z\|_{L^2(H)}, \alpha, \beta \right) > 0$$

we derive:

$$\langle L''(y^*, u_0^*, u_1^*, \lambda^*, \mu^*)(v, q_0, q_1), (v, q_0, q_1) \rangle_{X^*,X} \geq \kappa \, \|(v, q_0, q_1)\|_X^2$$

for all $(v, q_0, q_1) \in X$.

b) Using Lemma 3.6 we obtain for all $(v, q_0, q_1) \in \ker e'(y^*, u_0^*, u_1^*)$

$$\begin{aligned}
&\langle L''(y^*, u_0^*, u_1^*, \lambda^*, \mu^*)(v, q_0, q_1), (v, q_0, q_1) \rangle_{X^*,X} \\
&\quad \geq \|v\|_{L^2(H)}^2 + \alpha \, \|q_0\|_{L^2(0,T)}^2 + \beta \, \|q_1\|_{L^2(0,T)}^2 \\
&\qquad - \frac{4}{\nu} \, c_E \exp(c_\circ T) \, \|y^* - z\|_{L^2(H)} \, \|v\|_{W(V)}^2 \\
&\quad \geq \frac{\min(\alpha, \beta)}{2} \left(\|q_0\|_{L^2(0,T)}^2 + \|q_1\|_{L^2(0,T)}^2 \right) + \frac{\min(\alpha, \beta)}{2c_{\text{ker}}} \, \|v\|_{W(V)}^2 \\
&\qquad - \frac{4}{\nu} \, c_E \exp(c_\circ T) \, \|y^* - z\|_{L^2(H)} \, \|v\|_{W(V)}^2 \, .
\end{aligned}$$

Let

$$\|y^* - z\|_{L^2(H)} < \frac{\nu \min(\alpha, \beta)}{8 c_{\text{ker}} c_E} \exp(-c_\circ T) \, .$$

Setting

$$\kappa = \min \left(\frac{\min(\alpha, \beta)}{2c_{\text{ker}}} - \frac{4}{\nu} \, c_E \exp(c_\circ T) \, \|y^* - z\|_{L^2(H)}, \frac{\alpha}{2}, \frac{\beta}{2} \right) > 0$$

we derive:

$$\langle L''(y^*, u_0^*, u_1^*, \lambda^*, \mu^*)(v, q_0, q_1), (v, q_0, q_1) \rangle_{X^*,X} \geq \kappa \, \|(v, q_0, q_1)\|_X^2$$

for all $(v, q_0, q_1) \in \ker e'(y^*, u_0^*, u_1^*)$.

□

5.6. The Scheme of Algorithm 1. In this section we shall formulate Algorithm 1 for control problem ($\mathbf{P_4}$).

a) Choose $(y^0, (u_0^0, u_1^0), (\lambda^0, \mu^0)) \in \mathcal{Y} \times \mathcal{U} \times Y$, $c \geq 0$ and put $n = 0$.

b) Calculate $e^n \in L^2(V)$ such that

$$\begin{aligned}
\langle e^n(t), \varphi \rangle_V \;&=\; \langle y_t^n(t), \varphi \rangle_{V^*,V} + \int_0^1 \nu \, y_x^n(t) \varphi' + (y^n(t) \, y_x^n(t) - f) \, \varphi \, dx \\
&\quad + \nu \, (u_0^n(t) \varphi(0) - u_1^n(t) \varphi(1))
\end{aligned}$$

for all $\varphi \in V$ and $t \in [0,T]$ a.e. and define

$$\tilde{\lambda}^n = \lambda^n + c\, e^n\,, \quad \tilde{\mu}^n = \mu^n + c\,(y^n(0) - \phi)\,.$$

c) Solve for $(\bar{y}, (\bar{u}_0, \bar{u}_1), (\bar{\lambda}, \bar{\mu})) \in \mathcal{Y} \times \mathcal{U} \times Y$ the linear system

$$\begin{aligned}
&\langle L''(y^n, u_0^n, u_1^n, \tilde{\lambda}^n, \tilde{\mu}^n)(\bar{y}, \bar{u}_0, \bar{u}_1) + e'(y^n, u_0^n, u_1^n)^*(\bar{\lambda}, \bar{\mu}), (v, q)\rangle_{X^*,X} \\
&\quad = \langle L''(y^n, u_0^n, u_1^n, \tilde{\lambda}^n, \tilde{\mu}^n)(y^n, u_0^n, u_1^n) + e'(y^n, u_0^n, u_1^n)^*(\tilde{\lambda}^n, \tilde{\mu}^n), (v, q)\rangle_{X^*,X} \\
&\qquad - \langle L'(y^n, u_0^n, u_1^n, \tilde{\lambda}^n, \tilde{\mu}^n), (v, q)\rangle_{X^*,X}
\end{aligned}$$

$$\begin{aligned}
&\langle e'(y^n, u_0^n, u_1^n)(\bar{y}, \bar{u}_0, \bar{u}_1), \varphi\rangle_{Y^*,Y} \\
&\quad = \langle e'(y^n, u_0^n, u_1^n)(y^n, u_0^n, u_1^n) - e(y^n, u_0^n, u_1^n), \varphi\rangle_{Y^*,Y}
\end{aligned}$$

for all $(v, (q_0, q_1), \varphi) \in \mathcal{Y} \times \mathcal{U} \times Y$.

d) We set $(y^{n+1}, u_0^{n+1}, u_1^{n+1}, \lambda^{n+1}, \mu^{n+1}) = (\bar{y}, \bar{u}_0, \bar{u}_1, \bar{\lambda}, \bar{\mu})$, $n = n+1$ and go back to b).

6. The Numerical Treatment of the Control Problems

6.1. The Discretization. Applying the *implicit Euler method* the parabolic boundary value problems are approximated by a sequence of elliptic boundary value problems. Then each elliptic boundary value problem can be solved by piecewise linear approximations, see Chapter 4, Section 6.

To explain the procedure we discuss the approximation of the unsteady Burgers equation with homogeneous Dirichlet boundary conditions, see problem (5.2). Let $l \in \mathbb{N}$ be given. We introduce a discretization of the time interval $[0,T]$ by putting $k = \frac{T}{l}$ and $t_j = jk$ for $j = 0, \ldots, l$. A weak solution $y \in W(H_0^1)$ to (5.2) satisfies (5.3). By y^j, $j = 0, \ldots, l$, we denote the approximations of $y(t_j, \cdot)$. Then the implicit Euler method leads to $y^0 = \phi$ and

$$\left\langle \frac{y^j - y^{j-1}}{k}, \varphi \right\rangle_{H^{-1},H_0^1} + \nu\langle y^j, \varphi\rangle_{H_0^1} + b(y^j, y^j, \varphi) = \langle f(t_j), \varphi\rangle_{H^{-1},H_0^1}$$

for all $\varphi \in H_0^1(0,1)$ and for $1 \le j \le l$.

We use the same spatial discretization as in Example 3.1 on page 27. We defined the piecewise linear functions $\varphi_0, \ldots, \varphi_m$. Then we compute approximations $y_h^j \in \operatorname{Span}\{\varphi_1, \ldots, \varphi_{m-1}\} \subset H_0^1(0,1)$ of y^j for $j = 1, \ldots, l$, where y_h^j satisfies

$$\left\langle \frac{y_h^j - y_h^{j-1}}{k}, \varphi_i \right\rangle_{H^{-1},H_0^1} + \nu\langle y_h^j, \varphi_i\rangle_{H_0^1} + b(y_h^j, y_h^j, \varphi_i) = \langle f(t_j), \varphi_i\rangle_{H^{-1},H_0^1}$$

for $i = 1, \ldots, m-1$.

6.2. Preconditioned Conjugate Gradient Method. To solve the saddle-point system in Algorithm 2-c) on page 38 we apply a *preconditioned conjugate gradient method* [**BS90**, p. 302]. Let us suppose that the saddle-point problem leads to the equation $\mathbf{M}\cdot\mathbf{x} = -\mathbf{r}$, where $\mathbf{M}$ denotes the matrix representation of the operator M_h and the vector $\mathbf{x}$ contains the discretization of the state and the control variable and of the Lagrange multipliers. The matrix $\mathbf{B}$ denotes the preconditioner.

a) Choose a start up value $\mathbf{x}_0$, calculate $\mathbf{r}_0 = -\mathbf{r} - \mathbf{M}\cdot\mathbf{x}_0$, $\mathbf{q}_1 = \mathbf{B}^{-1}\cdot\mathbf{r}_0$ and set $\mathbf{p}_1 = \mathbf{q}_1$.

b) For $i = 1$ to $i_{\max}$ do

$$\begin{aligned} \alpha_i &= \frac{\mathbf{r}_{i-1}^T\mathbf{q}_i}{\mathbf{p}_i^T\mathbf{M}\cdot\mathbf{p}_i}\,; \\ \mathbf{x}_i &= \mathbf{x}_{i-1} + \alpha_i\mathbf{p}_i\,; \end{aligned}$$

if **accurate** then **exit**;

$$\begin{aligned} \mathbf{r}_i &= \mathbf{r}_{i-1} - \alpha_i\mathbf{M}\cdot\mathbf{p}_i\,; \\ \mathbf{q}_{i+1} &= \mathbf{B}^{-1}\cdot\mathbf{r}_i\,; \\ \beta_{i+1} &= \frac{\mathbf{r}_i^T\mathbf{q}_{i+1}}{\mathbf{r}_{i-1}^T\mathbf{q}_i}\,; \\ \mathbf{p}_{i+1} &= \mathbf{q}_{i+1} + \beta_{i+1}\mathbf{p}_i\,; \end{aligned}$$

end for.

6.3. Numerical Examples. We carried out numerous tests for control problems $(\mathbf{P_3})$ and $(\mathbf{P_4})$ and focused our interest on the asymptotic mesh-independence.

The programs were written in MATLAB, and realized on an IBM RS/6000 590-computer.

In all numerical examples we took the preconditioner

$$\mathbf{B}_p = \begin{pmatrix} 0 & 0 & \mathbf{P}^* \\ 0 & \frac{1}{\sigma}\mathbf{I} & 0 \\ \mathbf{P} & 0 & 0 \end{pmatrix}. \tag{5.62}$$

Here, the matrix $\mathbf{P}$ is the discretization of the heat operator $y_t - \nu y_{xx}$ with homogeneous Dirichlet boundary conditions at $x = 0$ and $x = 1$, and the matrix $\mathbf{P}^*$ is the discretization of the adjoint heat operator $-y_t - \nu y_{xx}$ with homogeneous Neumann boundary conditions at $x = 0$ and $x = 1$.

EXAMPLE 6.1 $(\mathbf{P_3})$. Let $T = 1$, $\omega = Q$, $\sigma = 0.1$, $\nu = 0.01$, $\mathcal{V} = L^2(H)$ and let C be the injection of $W(V)$ into $L^2(H)$. Further, we put

$$z(t,x) = \left(\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}\right)\left(\left(t - \frac{1}{2}\right)^2 - \frac{1}{4}\right)$$

and $f = z_t - \nu z_{xx} + z z_x$. The distributed control problem is given by

$$(5.63) \qquad \begin{cases} \text{minimize} & \dfrac{1}{2}\|y - z\|^2_{L^2(H)} + \dfrac{\sigma}{2}\|u\|^2_{L^2(Q)} \\ \text{subject to} & \begin{cases} y_t - \nu y_{xx} + y y_x = f + u\,, & 0 < t, x < 1\,, \\ y(t,0) = y(t,1) = 0\,, & 0 < t < 1\,, \\ y(0,x) = 0\,, & 0 < x < 1\,. \end{cases} \end{cases}$$

We see that $(y^*, u^*) = (z, 0)$ is the optimal solution of (5.63). For the augmented Lagrangian-SQP method we choose $c = 50$. If we choose $h = k = \frac{1}{30}$ and $\mathbf{B} = \mathbf{I}$ (no preconditioning), then the CPU-time of the program is nine-times as long as in the case of $\mathbf{B} = \mathbf{B}_p$. The start-up value y^0_{hk} of Algorithm 2 is shown in Figure 1. Further we choose $u^0_h = 1$ and $\lambda^0_h = (x - \frac{1}{2})^2$. In

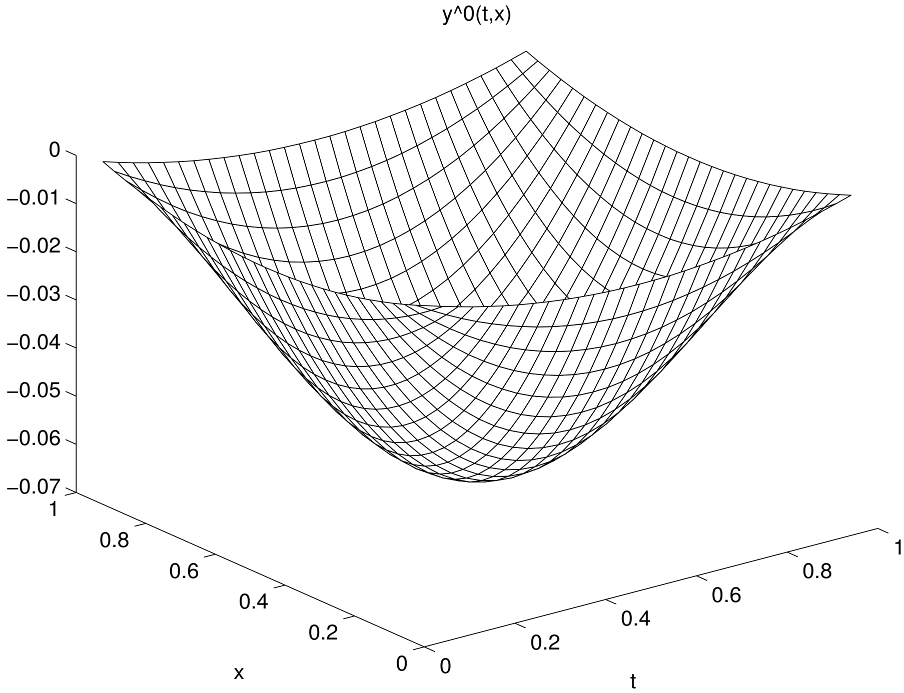

FIGURE 1. Start-up value y^0_{hk} for $h, k = \frac{1}{30}$

Figure 2 and Figure 3 we give the graph of y^*_{hk} and u^*_{hk}, respectively. Note that the iteration y^n_{hk} has to "pass through 0.

In Table 1 we specify the smallest iteration number $n = n_{hk}(\varepsilon)$ such that

$$\|(y^n_{hk}, u^n_{hk}, \lambda^n_{hk}, \mu^n_{hk}) - (y^*_{hk}, u^*_{hk}, \lambda^*_{hk}, \mu^*_{hk})\|_{Z_{hk}} < \varepsilon\,.$$

We proved in Theorem 2.1 on page 61 that a strong mesh-independence principle is valid for $n_{hk}(\varepsilon)$ and the corresponding iteration count $n(\varepsilon)$ of the infinite dimensional method. But the used approximation scheme does not generally satisfy the assumptions of Theorem 2.1 on page 61. However, Table

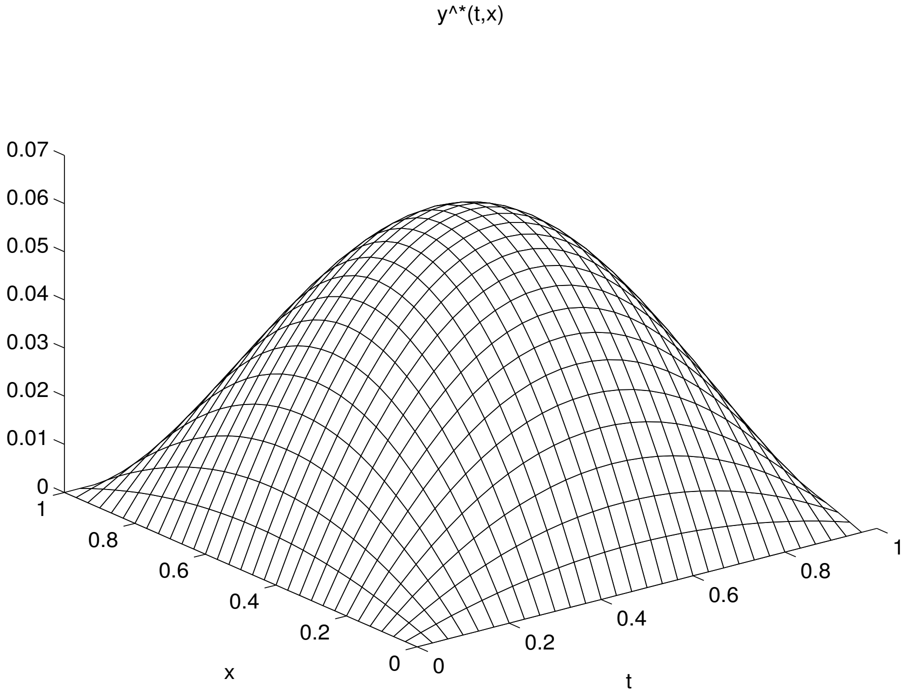

FIGURE 2. Solution y_{hk}^* for $h, k = \frac{1}{30}$

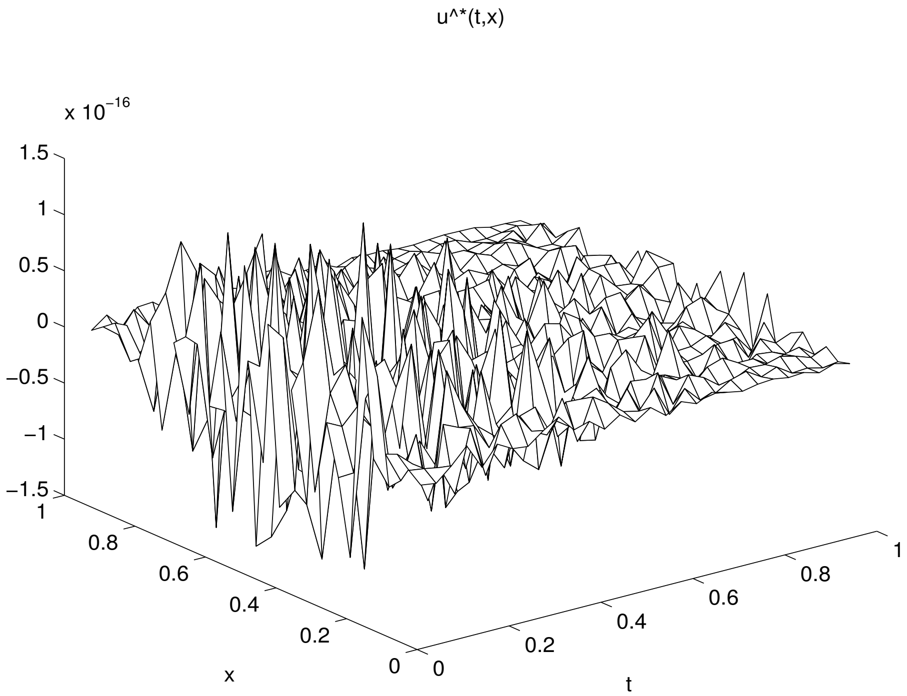

FIGURE 3. Solution u_{hk}^* for $h, k = \frac{1}{30}$

1 illustrates that the asymptotic mesh-independence can be observed numerically for $h, k \leq \frac{1}{30}$.

l	30	40	60	80	100	160	200	250
m	30	40	60	80	100	160	200	250
$\varepsilon = 10^{0}$	3	3	3	3	3	4	4	4
$\varepsilon = 10^{-2}$	5	5	6	6	6	6	5	5
$\varepsilon = 10^{-4}$	6	6	7	7	7	7	6	6
$\varepsilon = 10^{-6}$	6	7	7	7	7	7	7	6
$\varepsilon = 10^{-8}$	7	7	8	8	8	7	7	7
$\varepsilon = 10^{-10}$	7	7	8	8	8	8	7	7
$\varepsilon = 10^{-12}$	7	8	8	8	8	8	8	7
$\varepsilon = 10^{-14}$	8	8	8	9	8	8	8	8
$\varepsilon = 10^{-16}$	8	8	9	9	9	8	8	8

TABLE 1.

EXAMPLE 6.2 ($\mathbf{P_3}$). We choose again $T = 1$ and $\omega = Q$. Further, let $\sigma = 1$, $\mathcal{V} = L^2(H)$, $z = 0$ and let C be the injection of $W(V)$ into $L^2(H)$. We consider the distributed control problem

$$\left\{\begin{array}{ll} \text{minimize} & \dfrac{1}{2}\,\|y - z\|^2_{L^2(H)} + \dfrac{\sigma}{2}\,\|u\|^2_{L^2(Q)} \\ \text{subject to} & \left\{\begin{array}{ll} y_t - \nu y_{xx} + y y_x = u\,, & 0 < t, x < 1\,, \\ y(t,0) = y(t,1) = 0\,, & 0 < t < 1\,, \\ y(0,x) = \sin(4\pi x)\,, & 0 < x < 1\,. \end{array}\right. \end{array}\right. \tag{5.64}$$

For the viscosity parameter we choose $\nu = 0.1$, $\nu = 0.01$ and $\nu = 0.005$. To solve (5.64) we shall again apply Algorithm 2. The start-up values are given by $y^0_{hk} = 0$, $u^0_{hk} = 0$ and $\lambda^0_{hk} = 0$. We present the graphs of y^*_{hk} and u^*_{hk} in Figures 4-9. If we choose $\nu = 0.05$, $h = \frac{1}{50}$ and $\mathbf{B} = \mathbf{I}$ (no preconditioning), then the CPU-time of the program is thirty-five-times as long as in the case of $\mathbf{B} = \mathbf{B}_p$.

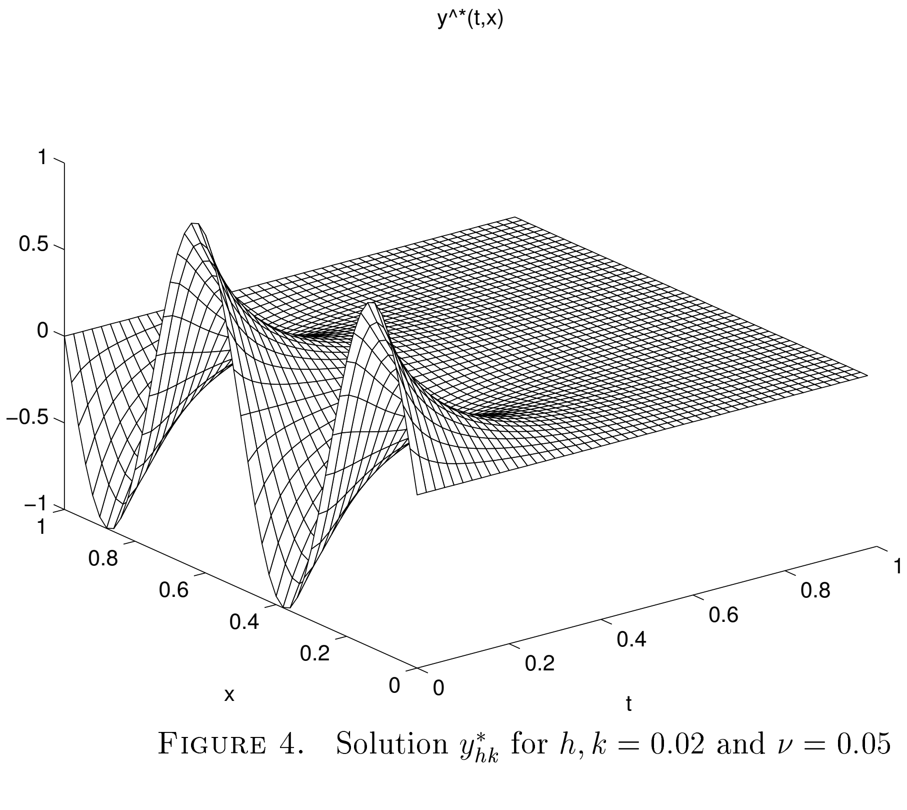

FIGURE 4. Solution y^*_{hk} for $h, k = 0.02$ and $\nu = 0.05$

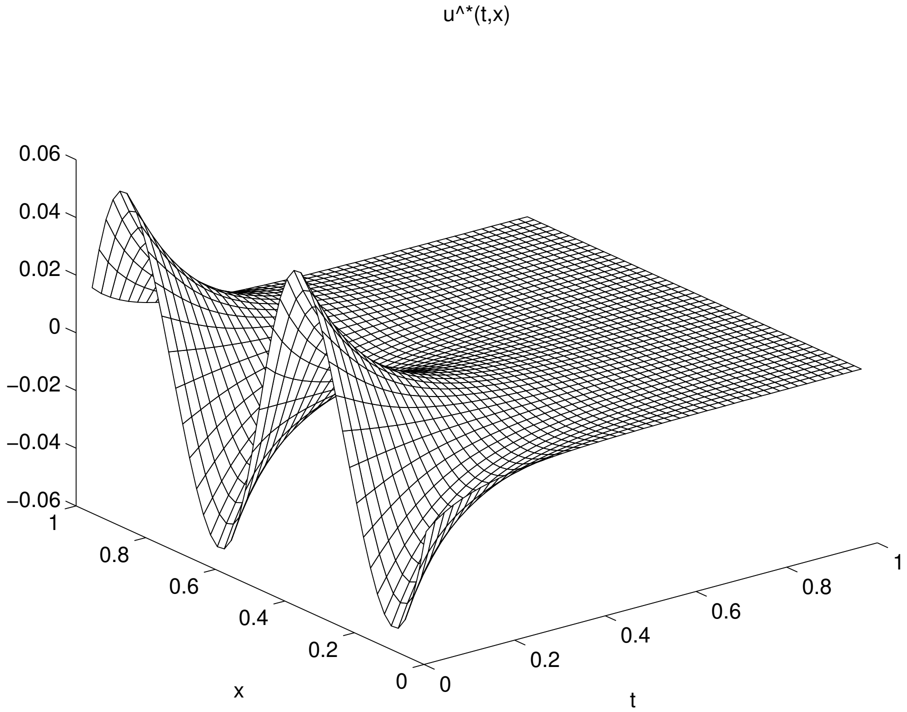

FIGURE 5. Solution u^*_{hk} for $h, k = 0.02$ and $\nu = 0.05$

Comparing Figures 4, 6 and 8 we see that the difference $y^*_{hk} - z_{hk}$ becomes larger if we decrease the viscosity parameter ν. The values of the cost functional are given in Table 2.

ν	0.05	0.01	0.005
$J_{hk}(y^*_{hk}, u^*_{hk})$	0.0149	0.0468	0.0686

TABLE 2. Values of the cost functional for $h, k = 0.01$ and for different ν

The larger the viscosity parameter is the larger the influence of the control on the state.

For the three values of ν we observe that the asymptotic mesh-independence principle hold. Again we specify the smallest iteration number $n = n_{hk}(\varepsilon)$ such that

$$\|F_{hk}(y^n_{hk}, u^n_{hk}, \tilde{\lambda}^n_{hk}, \tilde{\mu}^n)\|_{Z^*_{hk}} < \varepsilon$$

holds. We recall that F_{hk} is defined in (2.29). Let us point out that mesh-independence also holds numerically, if we fix m or l and increase l or m, respectively.

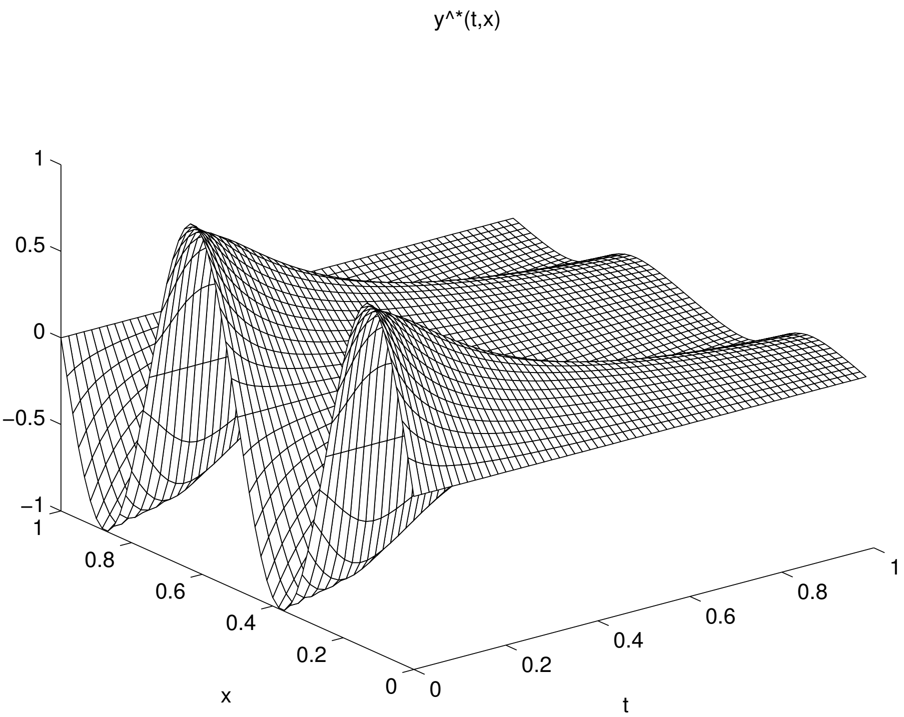

FIGURE 6. Solution y^*_{hk} for $h, k = \frac{1}{60}$ and $\nu = 0.01$

l	50	75	100	125	150	175	200	225	250	275	300
m	50	75	100	125	150	175	200	225	250	275	300
$\varepsilon = 10^{-1}$	1	1	1	1	1	1	1	1	1	1	1
$\varepsilon = 10^{-2}$	2	2	2	2	2	2	2	2	2	2	2
$\varepsilon = 10^{-3}$	3	3	3	3	3	3	2	2	2	2	2
$\varepsilon = 10^{-4}$	3	3	3	3	3	3	3	3	3	3	3
$\varepsilon = 10^{-5}$	4	4	3	4	4	3	3	3	3	3	3
$\varepsilon = 10^{-6}$	5	5	5	5	5	5	5	5	5	5	5
$\varepsilon = 10^{-7}$	5	5	5	5	5	5	5	5	5	5	5
$\varepsilon = 10^{-8}$	5	5	5	5	5	5	5	5	5	5	5
$\varepsilon = 10^{-9}$	5	5	5	5	5	5	5	5	5	5	5
$\varepsilon = 10^{-10}$	6	6	6	6	6	6	5	6	5	5	5
$\varepsilon = 10^{-11}$	6	7	6	6	6	7	6	6	6	6	6
$\varepsilon = 10^{-12}$	7	7	6	6	6	7	7	7	7	6	6

TABLE 3. $\nu = 0.05$

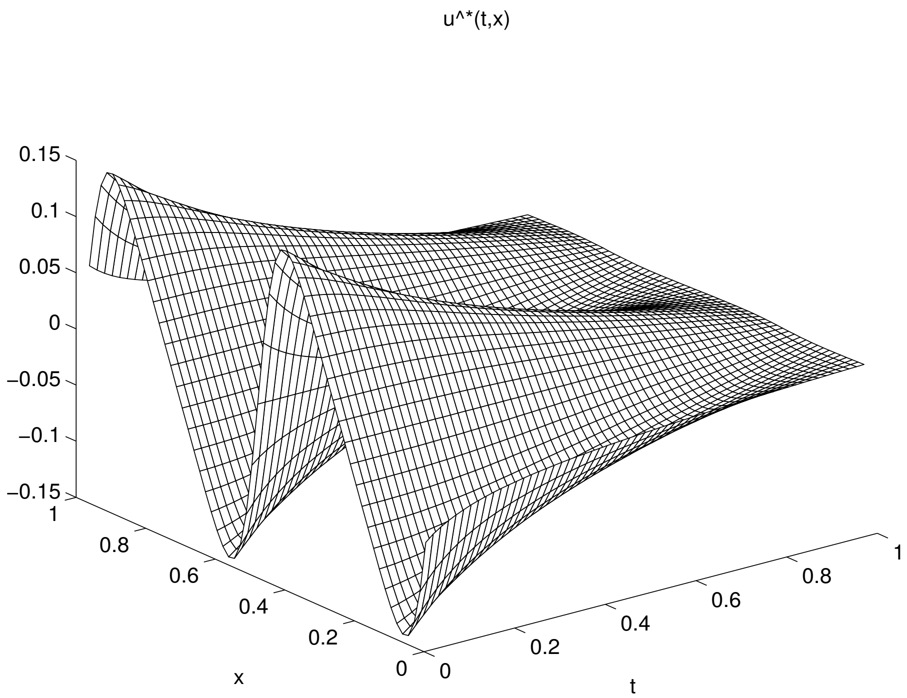

FIGURE 7. Solution u^*_{hk} for $h, k = \frac{1}{60}$ and $\nu = 0.01$

l	60	75	100	125	150	175	200	225	250	275	300
m	60	75	100	125	150	175	200	225	250	275	300
$\varepsilon = 10^{-1}$	1	1	1	1	1	1	1	1	1	1	1
$\varepsilon = 10^{-2}$	2	2	2	2	2	2	2	2	2	2	2
$\varepsilon = 10^{-3}$	3	3	3	3	3	3	2	2	2	2	2
$\varepsilon = 10^{-4}$	4	4	4	4	3	3	3	3	3	3	3
$\varepsilon = 10^{-5}$	5	5	4	4	4	4	4	4	4	4	4
$\varepsilon = 10^{-6}$	6	6	5	5	5	5	5	5	5	5	5
$\varepsilon = 10^{-7}$	6	6	6	6	6	6	6	6	6	6	5
$\varepsilon = 10^{-8}$	6	7	6	6	6	6	6	6	6	6	6
$\varepsilon = 10^{-9}$	7	7	7	6	7	7	7	7	6	6	6
$\varepsilon = 10^{-10}$	7	7	7	7	7	7	7	7	7	7	7
$\varepsilon = 10^{-11}$	7	7	7	7	7	7	7	7	7	7	7
$\varepsilon = 10^{-12}$	7	7	7	7	7	7	7	7	7	7	7

TABLE 4. $\nu = 0.01$

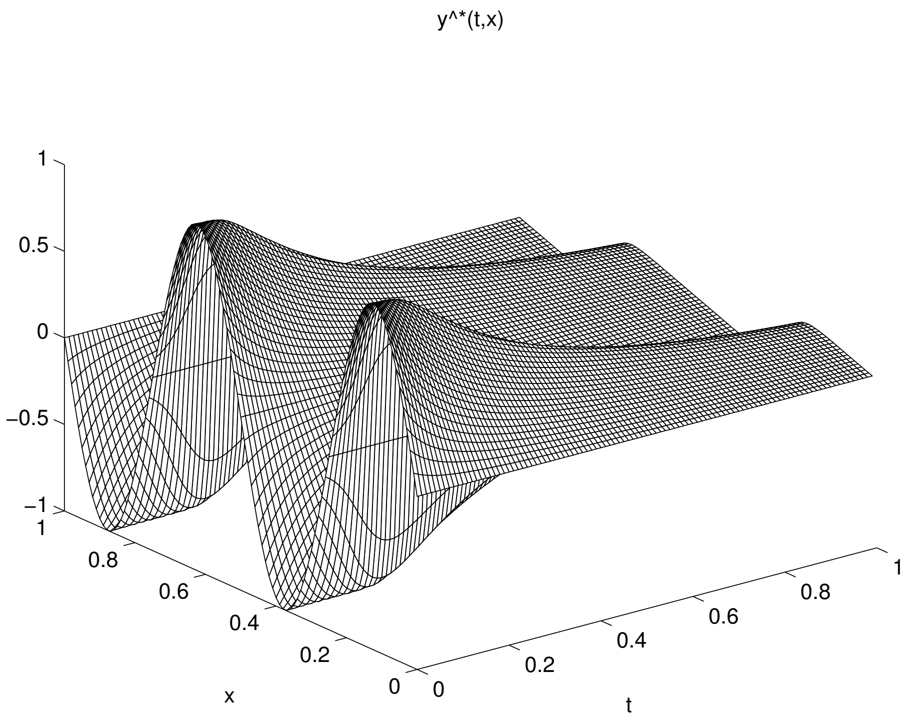

FIGURE 8. Solution y^*_{hk} for $h, k = 0.01$ and $\nu = 0.005$

l	100	125	150	175	200	225	250	275	300
m	100	125	150	175	200	225	250	275	300
$\varepsilon = 10^{-1}$	1	1	1	1	1	1	1	1	1
$\varepsilon = 10^{-2}$	2	2	2	2	2	2	2	2	2
$\varepsilon = 10^{-3}$	3	3	3	3	2	2	2	2	2
$\varepsilon = 10^{-4}$	5	5	5	4	4	4	4	4	4
$\varepsilon = 10^{-5}$	6	6	6	6	5	5	5	5	5
$\varepsilon = 10^{-6}$	6	6	6	6	6	6	6	6	6
$\varepsilon = 10^{-7}$	8	8	7	8	7	7	8	8	7
$\varepsilon = 10^{-8}$	8	8	8	8	8	8	8	8	8
$\varepsilon = 10^{-9}$	8	8	8	8	9	8	8	8	8
$\varepsilon = 10^{-10}$	8	8	8	8	9	9	9	9	9
$\varepsilon = 10^{-11}$	9	9	9	9	9	9	9	9	9
$\varepsilon = 10^{-12}$	9	9	9	9	9	9	9	9	9

TABLE 5. $\nu = 0.005$

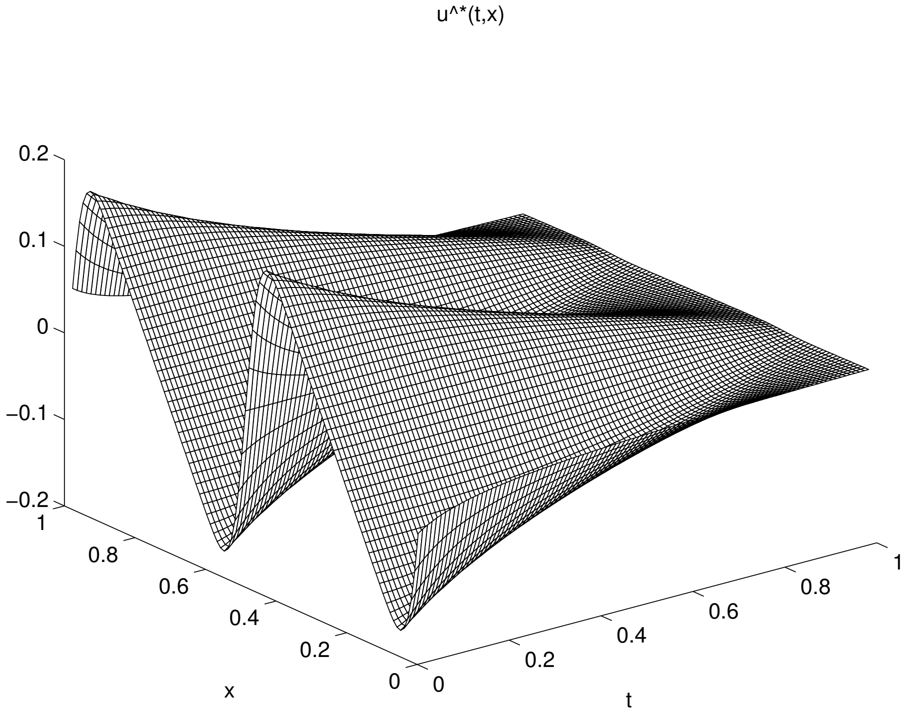

FIGURE 9. Solution u_{hk}^{*} for $h, k = 0.01$ and $\nu = 0.005$

EXAMPLE 6.3 ($\mathbf{P_4}$). Let $T = 1$, $\sigma = 0.01$, $\mathcal{V} = L^2(H)$, $z = 0$ and let C be the injection of $W(V)$ into $L^2(H)$. Further, let $\nu = 0.1$ and $z = 0.35$. The boundary control problem is given by

$$(5.65)\quad \begin{cases} \text{minimize} & \dfrac{1}{2}\,\|y - z\|^2_{L^2(H)} + \dfrac{\sigma}{2}\,\left(\|u_0\|^2_{L^2(0,T)} + \|u_1\|^2_{L^2(0,T)}\right) \\ \text{subject to} & \begin{cases} y_t - \nu y_{xx} + y y_x = 0\,, & 0 < t, x < 1\,, \\ y_x(t,0) = u_0(t),\; y_x(t,1) = u_1(t)\,, & 0 < t < 1\,, \\ y(0,x) = x^2(1-x)^2\,, & 0 < x < 1\,. \end{cases} \end{cases}$$

The viscosity parameter is given by $\nu = 0.1$. Although we do not know if (5.65) has a local solution we calculate a numerical solution to the problem. We set $c = 20$ in Algorithm 2. The start-up values are $y^0_{hk} = z_{hk}$, $u^0_{0,k} = u^0_{1,k} = 0$ and $\lambda^0_{hk} = 0$, where z_{hk} denotes a discretization of the desired state z. Note that in the infinite dimensional case the choice $y^0 = z$ guarantees that the second-order sufficient optimality condition is satisfied by Theorem 5.7-b).

The graphs of $u^*_{0,k}$, $u^*_{1,k}$, y^*_{hk} and λ^*_{hk} are presented in Figures 10-13.

If we choose $h = 0.01$ and $\mathbf{B} = \mathbf{I}$ (no preconditioning), then the program does not converge.

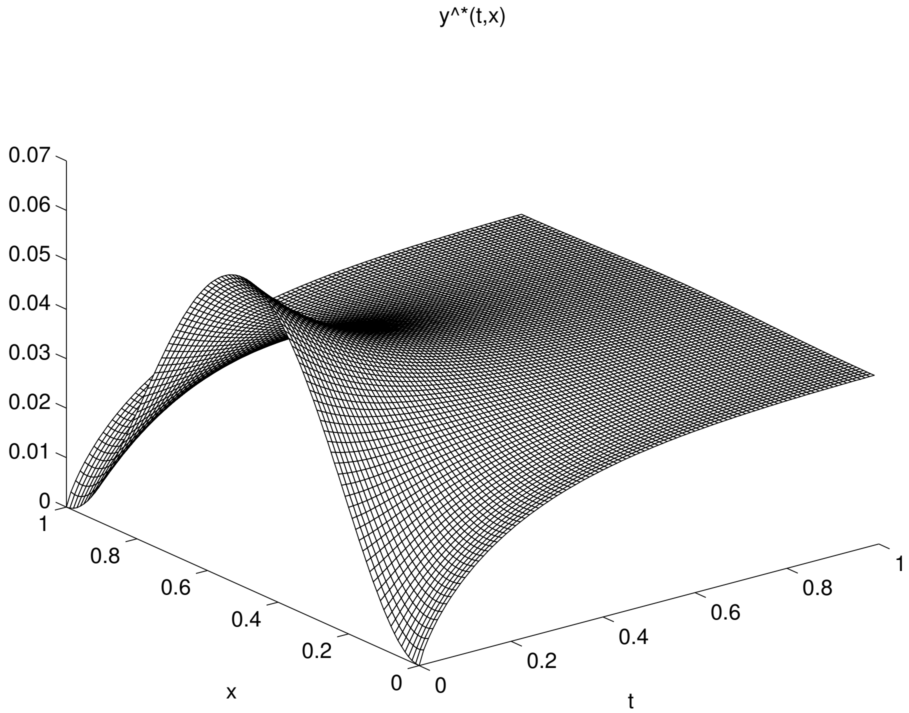

FIGURE 10. Solution y^*_{hk} for $h, k = 0.01$

Table 6 illustrates that the asymptotic mesh-independence can again be observed numerically for $h \le 0.01$, where we specify the smallest iteration

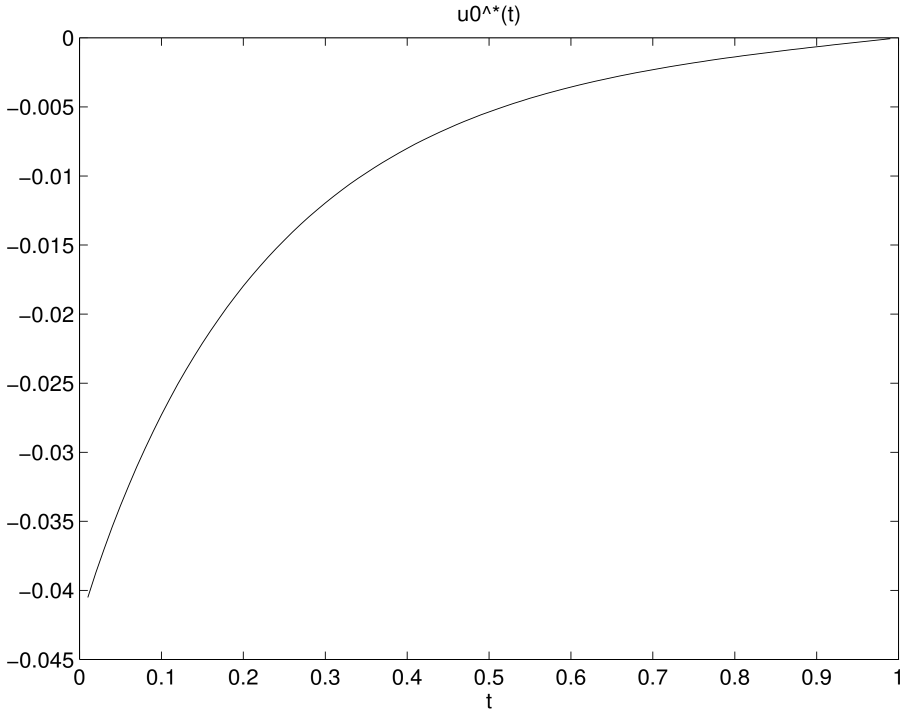

FIGURE 11. Solution $u_{0,k}^*$ for $k = 0.01$

number $n = n_{hk}(\varepsilon)$ with

$$\|F_{hk}(y_{hk}^n, u_{0,k}^n, u_{1,k}^n, \tilde{\lambda}_{hk}^n, \tilde{\mu}^n)\|_{Z_{hk}^*} < \varepsilon\,.$$

l	100	125	150	175	200	225	250	275	300	350	400
m	100	125	150	175	200	225	250	275	300	350	400
$\varepsilon = 10^{-1}$	1	1	1	1	1	1	1	1	1	1	1
$\varepsilon = 10^{-2}$	1	1	1	1	1	1	1	1	1	1	1
$\varepsilon = 10^{-3}$	2	2	2	2	2	2	2	2	2	2	2
$\varepsilon = 10^{-4}$	2	2	2	2	2	2	2	2	2	2	2
$\varepsilon = 10^{-5}$	3	3	3	3	3	3	2	2	2	2	2
$\varepsilon = 10^{-6}$	4	4	4	4	3	3	3	3	3	3	3
$\varepsilon = 10^{-7}$	5	5	5	5	5	5	5	5	5	4	4
$\varepsilon = 10^{-8}$	5	5	5	5	5	5	5	5	5	5	5
$\varepsilon = 10^{-9}$	6	6	6	5	5	5	5	5	5	5	5
$\varepsilon = 10^{-10}$	7	7	7	6	6	6	6	6	6	6	6

TABLE 6.

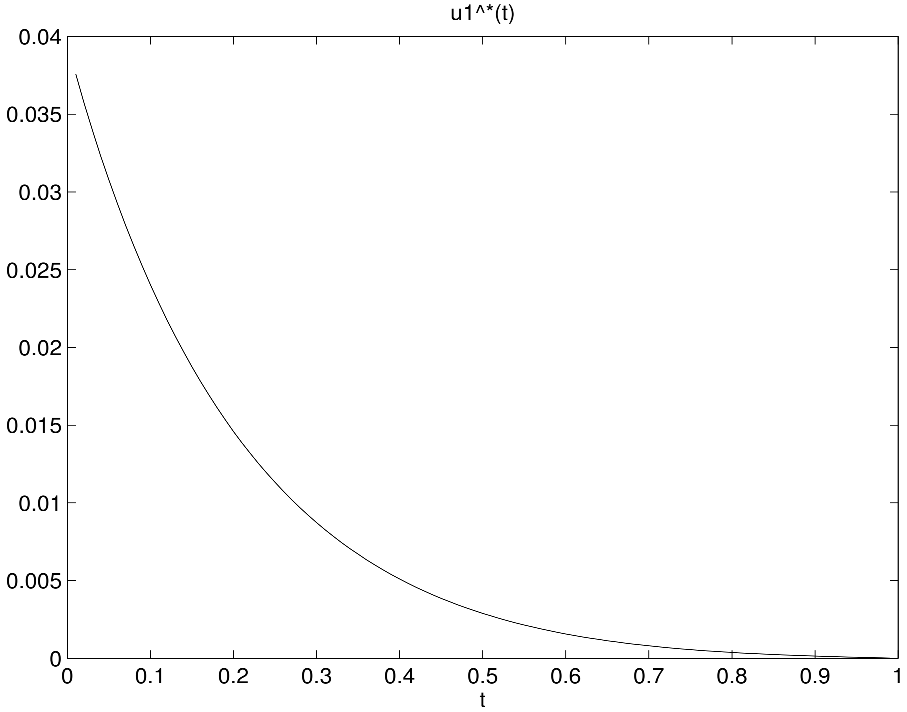

FIGURE 12. Solution $u_{1,k}^*$ for $k = 0.01$

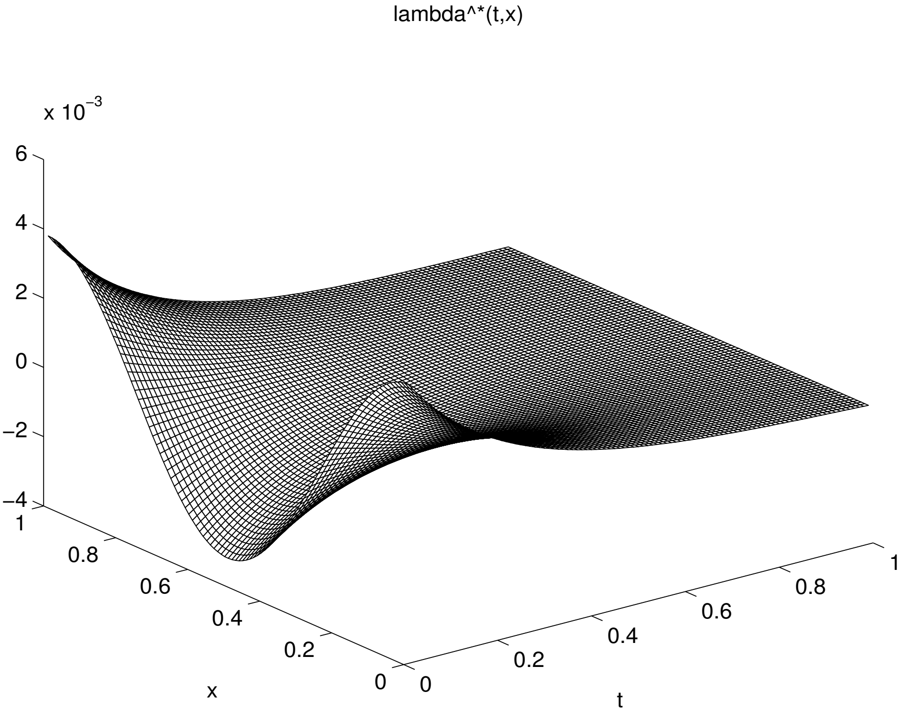

FIGURE 13. Solution λ_{hk}^* for $h, k = 0.01$

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