Optimal control in multi-agents model

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Motivations

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Model how the individuals change their mind

Settings

- Two products:  $P \mapsto +1$; $M \mapsto -1$
- N individuals
- $x_i \in [-1, 1]$: opinion of the individual i-th, $i = 1, \ldots, N$

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- N individuals
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Evolution of the opinion for each individual x_i, $i = 1, \ldots N$

$$\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)) \quad + P_i(t)(1 - x_i(t)) - M_i(t)(1 + x_i(t))$$

interactions between individuals

external factors (e.g. advertising)

Model

Interactions between individuals:

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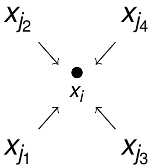

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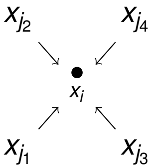

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External factors:

$$\dot{x}_i(t) = P_i(t)(1 - x_i(t)) - M_i(t)(1 + x_i(t));$$

$P_i(t), M_i(t) \in [0, 1]$:
P leads the individuals toward $+1$, M leads the individuals toward -1

Optimal control

Aim

try to understand the best strategy that a seller should have in order to maximize his sales

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- $M_i(t)$ known (strategy), $T > 0$ fixed final time
- Find $u_i(t) : [0, T] \to [0, 1]$ such that $\int_0^T \sum_{i=1}^N u_i(t)dt \leq C_1$

$$\dot{x}_i(t) = \sum_{j=1}^N a_{ij}(x_j(t) - x_i(t)) + u_i(t)(1 - x_i(t)) - M_i(t)(1 + x_i(t))$$

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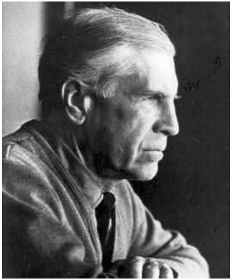
Goal

to maximize the number of individuals in 1 and minimize the cost:

$$\min \sum_{i=1}^{N} (1 - x_i(T))^2 + \int_0^T \sum_{i=1}^{N} u_i(t)^2 dt$$

Optimal control theory



L. Pontryagin

R. Bellman

- Developed in 1950s
- It is an extension of the calculus of variations
- It deals with systems that can be controlled, i.e. whose evolution can be influenced by some external agent

Definitions

$$\text{Let } \dot{x}(t) = f(x(t), u(t), t), \quad x(0) = x_0 \tag{1}$$

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  - **open-loop strategy**: $u = u(t)$
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for each initial point x_0 there are many trajectories depending on the choice of the control parameter u

Hypothesis

What we need:

- set points that can be reached (controllability)

  If controllability to find a final point x_f is granted then one can try to reach x_f minimizing some cost,
  thus defining an optimal control problem: $\min \Psi(u)$

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- final time T **fixed** or free
- set of admissible controls and set of admissible trajectories

Definitions

Given a final time $T > 0$, find a control $u : [0, T] \to [0, \infty]$ (eventually with some constraints) which minimize the pay-off functional Ψ:

$$\Psi(x, u) = \Phi(x(T)) + \int_0^T L(t, x(t), u(t))dt$$

under the constraint $\dot{x} = f(x, u, t)$.

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If $L = 0$: Mayer problem; if $L \neq 0$: otherwise Bolza problem.

Example 1: unitary mass on a 1D-line

- Point of unitary mass moving on a one dimensional line
- Control an external bounded force
- x position of the point
- u control

$$\ddot{x} = u, \quad x \in \mathbb{R}, |u| \leq C$$

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$$x_1 = x, \quad x_2 = \dot{x}_1$$
$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u$$

Goal: Drive the point to the origin with zero velocity in minimum time from the original position (x_1^0, x_2^0)

Example 2: reproductive strategies in social insects[1]

Let T be the length of the season

- $w(t)$: number of workers at time t
- $q(t)$: number of queens at time t
- $u(t)$: fraction of colony effort devoted to increasing work force
- $s(t)$: known rate at which each worker contributes to the bee economy

$$\dot{w}(t) = -\nu w(t) + bs(t)u(t)w(t), \quad w(0) = w_0$$
$$\dot{q}(t) = -\nu q(t) + c(1 - u(t))s(t)w(t), \quad q(0) = q_0$$

Goal: maximize the number of the queens: $\Psi(u(\cdot)) = q(T)$

[1] Caste and Ecology in Social Insects, by G. Oster and E. O. Wilson

Basic problem

Find u^* which minimize the pay-off, i.e.

$$\Psi(u^*(\cdot)) \leq \Psi(u(\cdot))$$

for all $u \in \mathcal{U}$.

Questions:

- does an optimal control u^* exist?
- how can we characterize an optimal control mathematically?
- how can we construct an optimal control?

Legendre Transformation

Standard problem in Calculus of Variations: find a curve x^* which minimize

$$I(x(\cdot)) = \int_0^T L(x(t), \dot{x}(t))dt, \quad x(0) = x_0, x(T) = x_T$$

where L, smooth function, is the Lagrangian.

If a C^2 minimizer $x^*(\cdot)$ exists, it satisfies the Euler Lagrange equations (EL)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}(x^*(t), \dot{x}^*(t)) = \frac{\partial L}{\partial x_i}(x^*(t), \dot{x}^*(t))$$

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Difficulty: second order ODEs
Solution: transform the (EL) into a system of ODEs (Hamiltonian equations) via the Legendre transform i.e. decouple the problem to the corresponding level sets

Hamiltonian equations

Steps:

- reduce the system (EL) into a system of $2n$ first order ODEs introducing $u := \dot{x}$
- change coordinates $(x, u) \to (x, p)$, $p_i = \frac{\partial L}{\partial u_i} =: \Phi_i(x, u)$
- define the Hamiltonian $H(x, p) := p\Phi^{-1}(x, p) - L(x, \Phi^{-1}(x, p))$

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We get (H)

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}$$

a solution for (EL) is a solution for (H) and $t \longmapsto H(x(t), p(t))$ is constant

Generalization of Classical Calculus of Variations

$$\min \int_0^T L(x(t), \dot{x}(t))dt, \quad x(0) = x_0, x(T) = x_f$$

- with non-holonomic constrains of the kind $\dot{x} = f(x, u), u \in U$
- the Lagrangian L is a function of (x, u) instead of $(x, \dot{x})$

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Tool: Pontryagin maximum principle (PMP)

- it generalizes the Euler- Lagrange equation and the Weierstrass condition of Calculus of Variation to variational problem with non-holonomic constraints
- it provides a pseudo-Hamiltonian formulation of the variational problem in the case when the standard Lagrange transformation is not well-defined

Constraints and Lagrange multipliers

If u^* is an optimal control, then there exists a function p^*, called the costate, that satisfies a certain maximization principle.

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Setup:

- ODE $\dot{x}(t) = f(x(t), u(t), t), \quad x(0) = x_0$
- Payoff functional: $\Psi(x(T, u)) = \Phi(x(T)) + \int_0^T L(x(t), u(t))dt$

The Pontryagin Maximum Principle asserts the existence of a function $p^*(t)$, which together with the optimal trajectory $x^*(t)$, satisfies an analogue of Hamilton's ODE, given by
$H(x, p, u) = f(x, u) \cdot p + L(x(t), u(t))$

Pontryagin Maximum Principle

Find the optimal solution to the problem

$$\min_{u \in \mathcal{U}} \Psi(x(T,u)) = \min \Phi(x(T)) + \int_0^T L dt$$

subject to $\dot{x} = f(t, x(t), u(t)), \quad x(0) = x_0$.

Theorem

Assume u^* is optimal and x^* is the corresponding trajectory. Then there exists a function $p^* : [0, T] \to \mathbb{R}^n$ such that

$$\dot{x}^*(t) = \frac{\partial H}{\partial p}(x^*(t), p^*(t), u*(t))$$

$$\dot{p}^*(t) = -\frac{\partial H}{\partial x}(x^*(t), p^*(t), u*(t))$$

and $H(x^*(t), p^*(t), u^*(t)) = \min_{u \in \mathcal{U}} H(x^*(t), p^*(t), u)$. In addition the mapping $t \longmapsto H(x^*(t), p^*(t), u^*(t))$ is constant. And the terminal condition is $p^*(T) = \nabla\Phi(x^*(T))$.

Example 3: control of production and consumption

$x(t)$: output produced at time $t \geq 0$ by a given factory
$u(t)$: fraction of output reinvested at time $t \geq 0$

$$\dot{x} = ku(t)x(t), \quad x(0) = x_0$$

with $k > 0$ modelling the growth rate of our reinvestment.

Payoff functional:

$$\Psi(u(\cdot)) = \int_0^T (1 - u(t))x(t)dt$$

Goal: maximize the total consumption of the output

Pontryagin maximum principle

Difficulties:

- the maximization condition not always provide a unique solution
- PMP gives two-points boundary value problem with some boundary condition given at initial time (state) and some at final time (covector)
- integrate a pseudo-Hamiltonian system
- even if one is able to find all the solutions to the PMP, it remains the problem of selecting among them the optimal trajectory

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Advantages:

- necessary optimality condition: sometimes sufficient (convex problems)
- invariant with respect to a broad class of transformations (reformulations) of the problem
- does not require prior evaluation of the pay-off functional

Open-loop strategies with L^1 constraint

Given $T > 0$, find $u : [0, T] \to [0, 1]$ such that $\int_0^T \sum_{i=1}^N u_i(t)dt \leq C_1$ which minimizes Ψ:

$$\begin{aligned}\Psi(x, u) =& \Phi(x(T)) + \varepsilon \int_0^T L(t, x(t), u(t))dt \\ =& \frac{1}{N} \sum_{i=1}^{N} (1 - x_i(T))^2 + \varepsilon \int_0^T \sum_{i=1}^{N} u_i^2 dt\end{aligned}$$

subject to

$$\dot{x}(t) = \sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)) + u_i(t)(1 - x_i(t)) - M_i(t)(1 + x_i(t))$$

$$x_i(0) = x_i^0.$$

Existence of optimal solution

Under certain hypothesis on:

- the set of admissible controls (compact)
- the function f, the cost function and the running cost (continuous)

we get the existence of optimal solution.

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we get the existence of optimal solution.

Goal: derive necessary conditions in order that a trajectory $x^*(t) = x^*(t, u^*(t))$ be optimal where u^* is a bounded admissible control

Pontryagin maximum principle

Theorem

Let f and L be continuous in all variables and continuously differentiable w.r.t. t, x. Let the bounded control $u^* : [0, T] \to U$ be optimal. Then there exists a nontrivial adjoint vector $p = (p_1, \dots, p_n)$ and constants λ_0, λ with $\lambda_0 \geq 0$ such that, for almost every $t \in [0, T]$

$$\dot{p}_i(t) = -\sum_{i=1}^{N} p_j(t) \frac{\partial f_j}{\partial x_i}(t, x^*(t), u^*(t)) - \lambda_0 \frac{\partial L}{\partial x_i}(t, x^*(t), u^*(t))$$

and

$$p(t) f(t, x^*(t), u^*) + \lambda_0 L(t, x^*(t), u^*) = \min_{\omega \text{ adm}} \{p(t) f(t, x^*(t), \omega) + \lambda_0 L(t, x^*(t), \omega)\}$$

Optimal control u^*

$$\min_{\omega \text{ adm}} \sum_{i=1}^{N} \Big[p_i(t)\omega_i(t)(1 - x_i^*(t)) + \lambda\omega_i(t) + \epsilon\lambda_0\omega_i^2(t) \Big]$$

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If $\lambda_0 = 0$

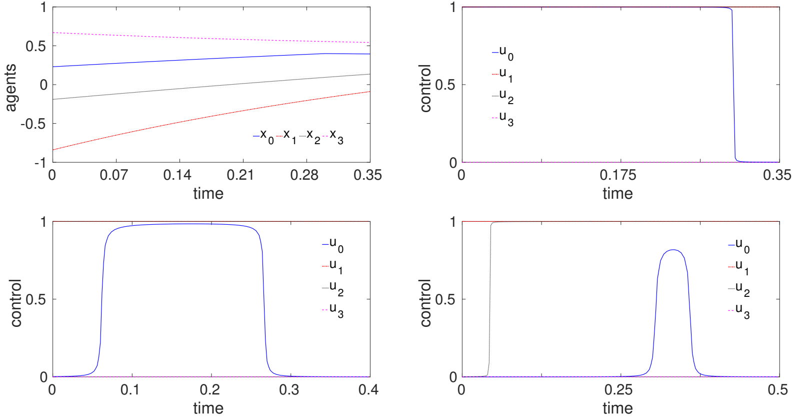
$$u_i^*(t) = \begin{cases} 0 & \text{if } \lambda \geq -p_i(t)(1 - x_i(t)) \\ -\lambda - p_i(t)(1 - x_i(t)) & \text{if } \lambda < -p_i(t)(1 - x_i(t)) \end{cases} \quad (1)$$

If $\lambda_0 > 0$

$$u_i^*(t) = \begin{cases} 0 & \text{if } \lambda \geq -p_i(t)(1 - x_i(t)) \\ \min\{C_\infty, \frac{-p_i(t)(1-x_i(t))-\lambda}{2\varepsilon\lambda_0}\} & \text{if } \lambda < -p_i(t)(1 - x_i(t)) \end{cases} \quad (2)$$

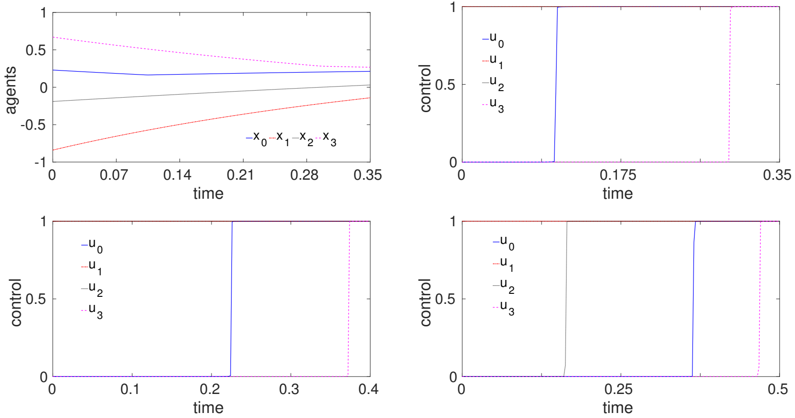
Numerical simulations

$M = 0$: only aggregation



Numerical simulations

$$M = \exp^{-t/25}$$



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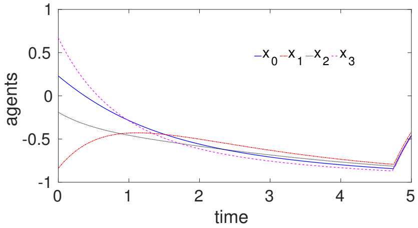

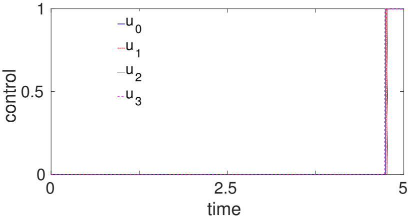

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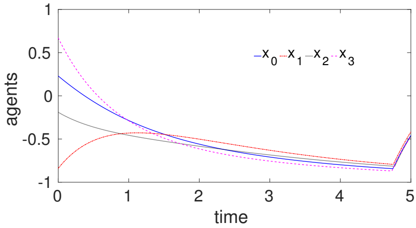

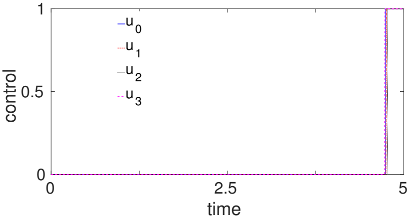

$$t^* \approx T - \frac{C_1}{N}$$

Outlook and open questions

- Uniqueness of u
- Individuals in -1, $+1$ do not change their mind
- Feedback strategies: $u_i = u_i(t, x)$
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