STRUCTURAL PROPERTIES OF SUBADDITIVE SEQUENCES WITH APPLICATIONS TO FACTORIZATION THEORY AND ADDITIVE COMBINATORICS

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Abstract. Let $\mathfrak{X} = (X_k)_{k \ge 1}$ be a sequence of eventually non-empty subsets of $\mathbf{Z}$ such that $X_h + X_k \subseteq X_{h+k}$ for all $h, k \in \mathbf{N}^+$ (i.e., $\mathfrak{X}$ is subadditive). We say that $\mathfrak{X}$ is a periodic AAPS if there exist $d, \mu \in \mathbf{N}^+$, $M \in \mathbf{N}$, $X_0', X_0'', \ldots, X_{\mu-1}', X_{\mu-1}'' \subseteq [\![0, M]\!]$, and $x_1, x_2, \ldots \in \mathbf{Z}$ such that, for all large k,

$$X_k = \left(\inf X_k + X'_{k \bmod \mu}\right) \uplus \mathscr{P}_k \uplus \left(\sup X_k - X''_{k \bmod \mu}\right) \subseteq x_k + d \cdot \mathbf{Z},$$

where $\mathscr{P}_k := (x_k + d \cdot \mathbf{Z}) \cap [\![\inf X_k + M, \sup X_k - M]\!]$ is an arithmetic progression (with difference d).

Among other things, we obtain sufficient and necessary conditions for $\mathfrak{X}$ to be a periodic AAPS, and from this we derive an all-inclusive proof of three fundamental results in additive combinatorics and factorization theory that at first have little to nothing in common with each other: a 1972 theorem of Nathanson on the (asymptotic) structure of the n-fold sumset of a finite set of integers; a weaker form of a theorem of Geroldinger and Halter-Koch on the structure of the sets of lengths of the powers of a fixed element in a cancellative commutative monoid that is finitely generated up to associates; and a recent theorem of the author on the structure of unions of sets of lengths in monoids (and rings) with accepted elasticity.

1. Introduction

Current developments in factorization theory (viz., the study of phenomena of non-uniqueness related to factorizations in monoid-like and ring-like structures) have revealed new connections of the field with arithmetic combinatorics, see [3, 18] and [1, 4]. In the present paper, we further contribute to this line of research, by inquiring into the properties of subadditive sequences (of sets), an "additive model" we use to gain a better understanding of fundamental results encountered in the two aforementioned areas.

More specifically, let H be a monoid with identity 1_H, and $H^\times$ the group of units of H (basic notations and terminology will be explained in § 1.1). We define $\mathsf{L}_H(1_H) := \{0\} \subseteq \mathbf{N}$, and given $x \in H \smallsetminus \{1_H\}$, we denote by $\mathsf{L}_H(x)$ the set of all $k \in \mathbf{N}^+$ such that $x = a_1 \cdots a_k$ for some atoms of H, where $a \in H$ is an *atom* (or *irreducible element*) if $a \notin H^\times$ and there do not exist $x, y \in H \smallsetminus H^\times$ with $a = xy$. Accordingly, we refer to the elements of the family

$$\mathscr{L}(H) := \{\mathsf{L}_H(x) : x \in H\} \smallsetminus \{\varnothing\} \subseteq \mathcal{P}(\mathbf{N})$$

as the *sets of lengths* of H, and to $\mathscr{L}(H)$ as the *system of sets of lengths* of H: These are by far the most studied of all invariants still used, after decades since their introduction in [9], in the investigation of

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various phenomena arising, say, from the non-uniqueness of essentially different factorizations in a wide assortment of (commutative and non-commutative) rings where every non-zero non-unit can be expressed as a (finite) product of irreducibles, see [10] for a survey.

In particular, it is a central problem in factorization theory to understand whether sets of lengths exhibit any non-trivial structure in specific classes of monoids (and unital rings). The next result, for instance, has been a landmark achievement in the study of the arithmetic of Krull monoids (and domains), see [11, § 4.9 and Theorem 4.9.6] for a proof of a slightly weaker statement.

Theorem 1.1. *Let H be a cancellative, commutative monoid that is finitely generated up to associates. Then there exist $M, k_0 \in \mathbf{N}$ and $\mu \in \mathbf{N}^+$ with the property that, for each $a \in H$, the following hold: There are $x, d \in \mathbf{N}^+$ and $L_0', L_0'', \ldots, L_{\mu-1}', L_{\mu-1}'' \subseteq [\![0, M-1]\!]$ (all depending on a) such that, for every $k \ge k_0$,*

$$\mathsf{L}_H(a^k) = \big(\inf \mathsf{L}_H(a^k) + L'_{k \bmod \mu}\big) \uplus \mathscr{P}_k \uplus \big(\sup \mathsf{L}_H(a^k) - L''_{k \bmod \mu}\big) \subseteq kx + d \cdot \mathbf{Z},$$

where $\mathscr{P}_k := (kx + d \cdot \mathbf{Z}) \cap [\![\inf \mathsf{L}_H(a^k) + M, \sup \mathsf{L}_H(a^k) - M]\!]$.

In particular, Theorem 1.1 implies the following weaker result, cf. [11, Corollary 4.9.7.1]:

Corollary 1.2. *Let H be a cancellative, commutative monoid that is finitely generated up to associates, and let $a \in H$. Then there can be found $M \in \mathbf{N}$, $\mu, x, d \in \mathbf{N}^+$, and $L_0', L_0'', \ldots, L_{\mu-1}', L_{\mu-1}'' \subseteq [\![0, M-1]\!]$ (all depending on a) such that, for every large k,*

$$\mathsf{L}_H(a^k) = \big(\inf \mathsf{L}_H(a^k) + L'_{k \bmod \mu}\big) \uplus \mathscr{P}_k \uplus \big(\sup \mathsf{L}_H(a^k) - L''_{k \bmod \mu}\big) \subseteq kx + d \cdot \mathbf{Z},$$

where $\mathscr{P}_k := (kx + d \cdot \mathbf{Z}) \cap [\![\inf \mathsf{L}_H(a^k) + M, \sup \mathsf{L}_H(a^k) - M]\!]$.

Conclusions along the lines of Theorem 1.1 have also been obtained by A. Foroutan and W. Hassler in the (alternative) setting of C-monoids, see [6, Theorem 3.1].

On the other hand, it is practically difficult to obtain a comprehensive description of $\mathscr{L}(H)$ even in cases where H is a "nice monoid". E.g., a long-standing question, known as the *Characterization Problem*, asks to show that, if H (respectively, H') is the multiplicative monoid of the ring of integers of a number field with (ideal) class group G (respectively, G') and $|G|$ is sufficiently large, then $\mathscr{L}(H) = \mathscr{L}(H')$ only if G and G' are isomorphic; see [10, § 6] for terminology and supplementary information.

Therefore, further invariants, derived from but smoother than $\mathscr{L}(H)$, have been also intensively studied. Among all of them, a prominent role is still played by unions of sets of lengths, first considered by S.T. Chapman and W.W. Smith in [2].

In detail, we denote by $\mathscr{U}_k$, for a fixed $k \in \mathbf{N}^+$, the set of all $\ell \in \mathbf{N}^+$ such that $a_1 \cdots a_k = b_1 \cdots b_\ell$ for some atoms $a_1, \ldots, a_k, b_1, \ldots, b_\ell \in H$. The sets $\mathscr{U}_k$ are given the name of *unions of sets of lengths*, since it is clear that $\mathscr{U}_k = \bigcup \{L \in \mathscr{L}(H) : k \in L\}$ for all $k \in \mathbf{N}$: Understanding their "structure" is a central problem in the theory of non-unique factorization.

Most notably, it is known that in a wealth of cases, see, e.g., [8, Theorem 4.2], [10, Theorem 13], and [3, Theorem 3.5], H satisfies what is called the *Structure Theorem for Unions*: There exist $d \in \mathbf{N}^+$ and $M \in \mathbf{N}$ such that, for all but finitely many k,

$$(k + d \cdot \mathbf{Z}) \cap [\![\inf \mathscr{U}_k(H) + M, \sup \mathscr{U}_k(H) - M]\!] \subseteq \mathscr{U}_k(H) \subseteq k + d \cdot \mathbf{Z}.$$

The Structure Theorem has been the state of the art in the topic for several years, modulo the fact that, for important but rather special families of monoids, the sets $\mathscr{U}_k$ are arithmetic progressions, if not even intervals as in the case of the ring of integers of a number field or, more in general, of a commutative Krull monoid each of whose ideal classes contains a prime, see [13, Theorem 1] and [7, Theorem 4.1].

But late work [18] has revealed that, in many interesting situations, H satisfies the following condition, sharper than the Structure Theorem and, hence, referred to as the *Strong Structure Theorem for Unions*: There exist $d, \mu \in \mathbf{N}^+$, $M \in \mathbf{N}$, and $\mathscr{U}_0', \mathscr{U}_0'', \ldots, \mathscr{U}_{\mu-1}', \mathscr{U}_{\mu-1}'' \subseteq [\![0, M-1]\!]$ such that, for all large k,

$$\mathscr{U}_k = \left(\inf \mathscr{U}_k + \mathscr{U}_{k \bmod \mu}'\right) \uplus \mathscr{P}_k \uplus \left(\sup \mathscr{U}_k - \mathscr{U}_{k \bmod \mu}''\right) \subseteq k + d \cdot \mathbf{Z},$$

where $\mathscr{P}_k := (k + d \cdot \mathbf{Z}) \cap [\![\inf \mathscr{U}_k(H) + M, \sup \mathscr{U}_k(H) - M]\!]$. Indeed, [18, Theorem 1.2] yields that:

Theorem 1.3. *Every elastic monoid satisfies the Strong Structure Theorem for Unions.*

Here, we say that a monoid H is *elastic* (or has accepted elasticity) if the supremum of the set

$$\{m/n : a_1 \cdots a_m = b_1 \cdots b_n \text{ for some atoms } a_1, \ldots, a_m, b_1, \ldots, b_n \in H\} \subseteq \mathbf{Q}^+$$

is attained or zero: This is true, e.g., for all commutative Krull monoids (and domains) with finite class group; for some maximal orders in central simple algebras over global fields; for a wide class of weakly Krull commutative domains, such as orders in number fields with finite elasticity; and for all numerical monoids (see [18, Example 3.4] for references and further examples).

Roughly speaking, Theorems 1.1 and 1.3 imply a regularity in the (asymptotic) structure of certain sequences $(X_k)_{k \ge 0}$ of subsets of $\mathbf{N}$, showing that the "boundaries" of the sets X_k repeat, up to a shift, with a certain periodicity, while the remaining parts are arithmetic progressions with the same difference. In this sense, the two results not only share a high degree of similarity to each other, but are also reminiscent of the following 1972 theorem of M.B. Nathanson, see [14] and [15, Theorem 1.1].

Theorem 1.4. *Let $X \subseteq \mathbf{N}$ be a finite set with $0 \in X$ and $\gcd(X) = 1$. Then there exist $X', X'' \subseteq \mathbf{N}$, $a, b \in \mathbf{N}$, and $k_0 \in \mathbf{N}^+$ such that $kX = X' \uplus [\![a, b + (k - k_0) \max X]\!] \uplus (b_k + X'')$ for $k \ge k_0$.*

Actually, one of the main contributions of this paper is a structural result for sequences of subsets of $\mathbf{Z}$ that brings Corollary 1.2 and Theorems 1.3 and 1.4 under the umbrella of a unifying framework, shedding new light on the "combinatorial nature" of the former two and paving the way, we hope, to further developments.

1.1. **Generalities.** We use $\mathbf{N}$ for the non-negative integers, $\mathbf{Z}$ for the integers, $\mathbf{Q}$ for the rationals $\mathbf{R}$ for the reals, and $\mathbf{R}^*$ for $\mathbf{R} \cup \{\pm\infty\}$ (that is, the extended real numbers). Unless otherwise stated, we reserve the letters d, k, m, and n (with or without subscripts) for positive integers, and the letters h, i, j, and κ for non-negative integers.

We let a *monoid* be a pair $(H, \otimes)$ consisting of a set H (called the *carrier* of the monoid) and an associative (binary) operation $\otimes : H \times H \to H$ for which there exists a (provably unique) element $e \in H$ (called the *identity* of the monoid) such that $e \otimes x = x \otimes e = x$ for all $x \in H$. We assume that monoid homomorphisms preserve the identity and, if there is no danger of confusion, systematically identify a monoid with its carrier.

If $(H, \otimes)$ is a monoid and $X, Y \subseteq H$, we set $X \otimes Y := \{x \otimes y : (x, y) \in X \times Y\}$, and we denote by $H^\times$ the *group of units* (or *invertible elements*) of H; accordingly, we write $x \simeq_H y$, for $x, y \in H$, if there exist $u, v \in H^\times$ such that $x = u \otimes y \otimes v$.

If $a, b \in \mathbf{R} \cup \{\pm\infty\}$ and $d \in \mathbf{N}^+$, we let $[\![a, b]\!] := \{x \in \mathbf{Z} : a \le x \le b\}$ stand for the (discrete) interval between a and b, and we take an *arithmetic progression* (shortly, AP) *with difference* d to be a set of the form $x + d \cdot [\![y, z]\!]$ with $x \in \mathbf{Z}$ and $y, z \in \mathbf{Z} \cup \{\pm\infty\}$ (note that an AP need not be finite or non-empty); in particular, we use $\mathbf{P}_d(x)$ for the bi-infinite progression $x + d \cdot \mathbf{Z}$.

If $\lambda \in \mathbf{R}$ and $X, Y \subseteq \mathbf{R}$, we denote by X^+ and X^-, respectively, the positive and the negative parts of X; and we define the sumset of X and Y by $X + Y := \{x + y : (x, y) \in X \times Y\}$, the n-fold sumset of X by $nX := \{x_1 + \cdots + x_n : x_1, \ldots, x_n \in X\}$, and the λ-dilation of X by $\lambda \cdot X := \{\lambda x : x \in X\}$. So, for instance, $\mathbf{N}^+$ is the set of all positive integers (in particular, $0 \notin \mathbf{N}^+$) and $\mathbf{Z}^- = -\mathbf{N}^+$.

Given sets X and $X_1, \ldots, X_n$, we write $\mathcal{P}(X)$ for the power set of X, and $X = X_1 \uplus \cdots \uplus X_n$ to mean that $X = X_1 \cup \cdots \cup X_n$ and $X_i \cap X_j = \varnothing$ for all distinct $i, j \in [\![1, n]\!]$. Lastly, we let $\mathfrak{S}_n$ be the group of permutations of $[\![1, n]\!]$, and we adopt the convention that $\sup \varnothing := -\infty$, $\gcd(\varnothing) = \infty - \infty = -\infty + \infty = 0 \cdot \infty = \infty \cdot 0 := 0$, and $\gcd(0) = \inf \varnothing := \infty$.

Further notations and terminology, if not explained, are standard, should be clear from the context, or are borrowed from [18].

2. SEQUENCES OF SUBSETS OF Z AS A MODEL: BASIC DEFINITIONS AND PROPERTIES

Let $\mathfrak{X} = (X_k)_{k \ge 1}$ be a sequence of subsets of $\mathbf{Z}$; note that, unless a statement to the contrary is made, the sets X_k need not be finite or non-empty. We put

$$-\mathfrak{X} := (-X_k)_{k \ge 1}, \quad \mathfrak{X}_{\ge 0} := (X_k \cap \mathbf{N})_{k \ge 1}, \quad \text{and} \quad \mathfrak{X}_{\le 0} = (X_k \cap \mathbf{Z}_{\le 0})_{k \ge 1}.$$

Moreover, for all $i, k \in \mathbf{N}^+$, we take

$$\rho_{k,i+1}(\mathfrak{X}) := \sup\bigl(X_k \setminus \{\rho_{k,1}(\mathfrak{X}), \ldots, \rho_{k,i}(\mathfrak{X})\}\bigr) \quad \text{and} \quad \lambda_{k,i+1}(\mathfrak{X}) := -\rho_{k,i+1}(-\mathfrak{X}), \tag{1}$$

where $\rho_{k,1}(\mathfrak{X}) := \sup X_k$; in particular, we set

$$\rho_k(\mathfrak{X}) := \rho_{k,1}(\mathfrak{X}) \quad \text{and} \quad \lambda_k(\mathfrak{X}) := \lambda_{k,1}(\mathfrak{X}),$$

and it is easy to verify that

$$\lambda_{k,i+1} = \inf\bigl(X_k \setminus \{\lambda_{k,1}(\mathfrak{X}), \ldots, \lambda_{k,i}(\mathfrak{X})\}\bigr) \quad \text{and} \quad \lambda_k(\mathfrak{X}) = -\rho_k(-\mathfrak{X}) = \inf X_k.$$

Then, we define the *upper* and the *lower elasticity* of $\mathfrak{X}$, respectively, by

$$\rho(\mathfrak{X}) := \sup_{k \ge 1} \frac{\rho_k(\mathfrak{X})}{k} \quad \text{and} \quad \lambda(\mathfrak{X}) := -\rho(-\mathfrak{X}) = \inf_{k \ge 1} \frac{\lambda_k(\mathfrak{X})}{k}. \tag{2}$$

We write $\wp(\mathfrak{X})$ for the greatest common divisor of the set $\{k \in \mathbf{N}^+ : X_k \ne \varnothing\}$. Lastly, we let

$$\Delta(\mathfrak{X}) := \bigcup_{k \ge 1} \Delta(X_k), \quad \delta(\mathfrak{X}) := \begin{cases} 1 & \text{if } \Delta(\mathfrak{X}) = \varnothing \\ \inf \Delta(\mathfrak{X}) & \text{otherwise} \end{cases}, \quad \text{and} \quad \mathfrak{q}(\mathfrak{X}) := \limsup_{k \to \infty} \sup \Delta(X_k),$$

where for $X \subseteq \mathbf{Z}$ we denote by $\Delta(X)$ the set of all $d \in \mathbf{N}^+$ such that $L \cap [\![x, x + d]\!] = \{x, x + d\}$ for some $x \in \mathbf{Z}$. We call $\Delta(\mathfrak{X})$ the *set of distances* of $\mathfrak{X}$; note that $\Delta(\mathfrak{X}) \subseteq \mathbf{N}^+$ and $\delta(\mathfrak{X}) \in \mathbf{N}^+ \cup \{\infty\}$.

We will usually omit the dependence of most of the above quantities on $\mathfrak{X}$ when $\mathfrak{X}$ is implied from the context, so as to write ρ for $\rho(\mathfrak{X})$, $\wp$ for $\wp(\mathfrak{X})$, δ for $\delta(\mathfrak{X})$, etc.

Remark 2.1. In contrast to [18], here we deal with subsets of the *integers* (rather than of the *non-negative* integers). For one thing, this has the advantage that definitions and results concerning a sequence $\mathfrak{X}$ of subsets of $\mathbf{Z}$ can be often "dualized" (and their length roughly cut by a half), by just switching from $\mathfrak{X}$ to $-\mathfrak{X}$: Later on, we will see many examples of this principle at work.

By and large, our primary objective in the present paper is to show that, under mild assumptions on $\mathfrak{X}$, the sets X_k become "highly structured" as k gets large. To this end, we make the following:

Definition 2.2. Let $\mathfrak{X} = (X_k)_{k\ge1}$ be a sequence of subsets of $\mathbf{Z}$. We say that $\mathfrak{X}$ is

- *primitive* if $X_k \ne \varnothing$ for every large k;
- *subadditive* if $X_h + X_k \subseteq X_{h+k}$ for all $h, k \in \mathbf{N}^+$;
- *upper elastic* if $\rho_n(\mathfrak{X}) = n\rho(\mathfrak{X})$ for some $n \in \mathbf{N}^+$;
- *lower elastic* if $-\mathfrak{X}$ is upper elastic;
- *elastic* if it is both lower and upper elastic.

It turns out that subadditive sequences of subsets of $\mathbf{Z}$ are an effective model for the study of a broad range of phenomena, and here come some examples the reader may want to keep in mind.

Example 2.3. Let $\mathscr{L} \subseteq \mathcal{P}(\mathbf{Z})$ be a *subadditive family*, meaning that, for all $X, Y \in \mathscr{L}$, there is $Z \in \mathscr{L}$ with $X + Y \subseteq Z$; a more restrictive definition (with $\mathbf{N}$ replaced by $\mathbf{Z}$) was first considered in [18, § 2], where numerous examples are also provided.

Given $k \in \mathbf{N}^+$, we denote by $\mathscr{U}_k(\mathscr{L})$ the union of all $X \in \mathscr{L}$ such that $k \in L$. Then it is not difficult to show (we omit details) that $(\mathscr{U}_k(\mathscr{L}))_{k\ge1}$ is a subadditive sequence; cf. [18, Lemma 2.7(iv)] for the case where every set in $\mathscr{L}$ is actually a subset of $\mathbf{N}$.

Example 2.4. Let H be a multiplicatively written monoid with identity 1_H; A a subset of H such that $1_H \notin \langle A\rangle_H$, where $\langle A\rangle_H$ is the subsemigroup of H generated by A; and η a function $A \to \mathbf{Z}$, which, roughly speaking, assigns an integral "weight" to each element of A, cf. [12].

Define $\mathsf{L}_H(x;\eta) := \{\eta(a_1) + \cdots + \eta(a_n) : x = a_1 \cdots a_n \text{ for some } a_1, \ldots, a_n \in A\}$ for $x \in H \smallsetminus \{1_H\} \subseteq \mathbf{Z}$, and $\mathsf{L}_H(1_H;\eta) := \{0\} \subseteq \mathbf{N}$. Then, similarly to [18, Example 2.1] (where η is non-negative), we see that

$$\mathscr{L}(H;\eta) := \{\mathsf{L}_H(x;\eta) : x \in H\} \smallsetminus \{\varnothing\} \subseteq \mathcal{P}(\mathbf{Z})$$

is a subadditive family. So, by Example 2.3, $(\mathscr{U}_k(\mathscr{L}(H;\eta)))_{k\ge1}$ is a subadditive sequence.

On the other hand, if we fix $x \in H \smallsetminus H^\times$ and, for every $k \in \mathbf{N}^+$, define $X_k := \mathsf{L}_H(x^k;\eta)$, it is easy to check that the sequence $(X_k)_{k\ge1}$ is also subadditive: Indeed, let $h, k \in \mathbf{N}^+$. If $X_h = \varnothing$ or $X_k = \varnothing$, then $X_h + X_k = \varnothing \subseteq X_{h+k}$ and we are done. Otherwise, pick $m \in X_h$ and $n \in X_k$. Then $x^h = a_1 \cdots a_m$ and $x^k = b_1 \cdots b_n$ for some $a_1, \ldots, a_m, b_1, \ldots, b_n \in A$, with the result that $m + n \in X_{h+k}$, since it is clear that $x^{h+k} \ne 1_H$ (otherwise x would be a unit).

In particular, the above conclusions apply to (ordinary) sets of lengths, which correspond to the case when $A = \mathscr{A}(H)$ and $\eta(a) = 1$ for all $a \in A$, cf. [18, Example 2.2].

Example 2.5. Let X be a (finite or infinite) subset of $\mathbf{Z}$ and, for every $k \in \mathbf{N}^+$, take $X_k := kX$. The sequence $\mathfrak{X} = (X_k)_{k\ge1}$ is clearly subadditive; and it is as well elastic, because $\rho_k = k\rho = k \sup X$ and $\lambda_k = k\lambda = k \inf X$ for all $k \in \mathbf{N}^+$.

Example 2.6. Pick $n \in \mathbf{N}^+$ and, for each $k \in \mathbf{N}^+$, let $X_k := [\![1, nk]\!]$. It is easy to check that $(X_k)_{k \ge 1}$ is an upper elastic, subadditive, primitive sequence of subsets of $\mathbf{N}$ (with $\rho = \rho_1 = n$); but it is not lower elastic, since $\lambda_1 = \lambda_2 = \cdots = 1$, and hence $\lambda = 0 \ne k\lambda_k$ for all $k \in \mathbf{N}^+$.

Structural properties of subadditive sequences of subsets of $\mathbf{Z}$ turn out to be tightly related to basic properties of subadditive sequences of extended real numbers, which leads us to:

Definition 2.7. An $\mathbf{R}^*$-valued sequence $(x_k)_{k \ge 1}$ is *subadditive* if $x_h + x_k \le x_{h+k}$ for all $h, k \in \mathbf{N}^+$.

In particular, one of the most fundamental properties of subadditive $\mathbf{R}^*$-valued sequences is exhibited by an old lemma, commonly attributed to M. Fekete [5], that has found a great deal of applications in number theory, ergodic theory, statistics, etc. (see [17, § 1.10] for a historical overview and references). Here we give a slightly stronger formulation of Fekete's result.

Lemma 2.8 (Fekete's lemma revisited)**.** *Let $(x_n)_{k \ge 1}$ be a sequence with values in $\mathbf{R}^*$ such that*

(A) *$x_{nk} \ge n x_k$ for every $n, k \in \mathbf{N}^+$;*
(B) *$x_k \ne -\infty$ and $x_h + x_k \le x_{h+k}$ for all sufficiently large $h, k \in \mathbf{N}^+$.*

Then

$$\lim_{k \to \infty} \frac{x_k}{k} = \sup_{k \ge \kappa} \frac{x_k}{k}, \quad \textit{for every } \kappa \in \mathbf{N}^+.$$

Proof. It is straightforward from (A) that $x_k/k \le x_{nk}/(nk)$ for all $n, k \in \mathbf{N}^+$, whence

$$s := \sup_{k \ge k_0} \frac{x_k}{k} = \sup_{k \ge \kappa} \frac{x_k}{k}, \quad \text{for every } \kappa \in \mathbf{N}^+.$$

Consequently, it will be enough to prove that $x_k/k \to s$ as $k \to \infty$. On the other hand, we have from (B) that there exists $k_0 \in \mathbf{N}^+$ such that

$$x_k \in \mathbf{R} \cup \{\infty\} \quad \text{and} \quad x_h + x_k \le x_{h+k}, \quad \text{for all } h, k \ge k_0. \tag{3}$$

Thus it is clear that $s \ne -\infty$, and we distinguish two cases (depending on whether $s \in \mathbf{R}$ or $s = \infty$).

CASE 1: $s \in \mathbf{R}$. Let $\varepsilon \in \mathbf{R}^+$. By definition of the supremum, $(s - \varepsilon)k_\varepsilon \le x_{k_\varepsilon}$ for some $k_\varepsilon \ge k_0$. Then, fix an index $k \ge 3k_\varepsilon$. We infer from the division algorithm that there are $q \in \mathbf{N}^+$ and $r \in [\![k_\varepsilon, 2k_\varepsilon - 1]\!]$ such that $k = qk_\varepsilon + r$. Therefore, we conclude that

$$x_k = x_{qk_\varepsilon + r} \overset{(3)}{\ge} x_{qk_\varepsilon} + x_r \overset{(A)}{\ge} q x_{k_\varepsilon} + M \ge (s - \varepsilon) q k_\varepsilon + M,$$

where $M := \min_{0 \le i < k_\varepsilon} x_{i + k_\varepsilon} \in \mathbf{R}$. It follows that

$$s \ge \frac{x_k}{k} \ge \frac{(s-\varepsilon)qk_\varepsilon}{k} + \frac{M}{k} = s - \varepsilon + \frac{M - (s - \varepsilon)r}{k} \ge s - \varepsilon + \frac{M - 2k_\varepsilon \cdot |s - \varepsilon|}{k}.$$

So, first passing to the limit as $k \to \infty$, and then to the limit as $\varepsilon \to 0^+$, we see that $x_k/k \to s$ as $k \to \infty$.

CASE 2: $s = \infty$. Let $\tau \in \mathbf{R}^+$. Similarly to the previous case, $\tau k_\tau \le x_{k_\tau}$ for some $k_\tau \ge k_0$; also, for every $k \ge 3k_\tau$ there exist $q \in \mathbf{N}^+$ and $r \in [\![k_\tau, 2k_\tau - 1]\!]$ such that $k = qk_\tau + r$, with the result that

$$\frac{x_k}{k} \ge \frac{x_{qk_\tau} + x_r}{k} \ge \frac{q x_{k_\tau} + M}{k} \ge \frac{Nqk_\tau + M}{k} \ge N + \frac{M - 2Nk_\tau}{k},$$

where $M := \min_{0 \le i < k_\tau} x_{i + k_\tau} \in \mathbf{R}$. So, first taking the limit as $k \to \infty$, and then the limit as $\tau \to \infty$, we find again that $x_k/k \to s$ as $k \to \infty$. ∎

We will formalize the link between subadditive $\mathbf{R}^*$-valued and subadditive $\mathcal{P}(\mathbf{Z})$-valued sequences in a series of lemmas and propositions, some of which provide a "sequential counterpart" of results first proved in [18, § 2] for subadditive families of subsets of $\mathbf{N}$.

We begin with a correlative of [18, Proposition 2.5 and Corollary 2.6].

Proposition 2.9. *Let $\mathfrak{X} = (X_k)_{k\ge 1}$ be a subadditive sequence of subsets of $\mathbf{Z}$ such that $\Delta(\mathfrak{X}) \ne \varnothing$, and let Δ' be a non-empty subset of $\Delta(\mathfrak{X})$ with $\gcd \Delta' \le \delta$. Then $\gcd \Delta' = \delta$. In particular, $\delta = \gcd \Delta(\mathfrak{X})$.*

Proof. Define $\delta' := \gcd \Delta'$. Since $\varnothing \ne \Delta' \subseteq \Delta(\mathfrak{X}) \subseteq \mathbf{N}^+$, it follows from [16, Theorem 1.4] that there are $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\} \subseteq \mathbf{Z}$, $d_1, \dots, d_n \in \Delta'$, and $m_1, \dots, m_n \in \mathbf{N}^+$ for which $\delta' = \sum_{i=1}^n \varepsilon_i m_i d_i$; in addition, for each $i \in [\![1, n]\!]$ we can find $k_i \in \mathbf{N}^+$ such that $d_i \in \Delta(X_{k_i})$; namely, $\{x_i, x_i + \varepsilon_i d_i\} \subseteq X_{k_i}$ for some $x_i \in X_{k_i}$. So, using that $\mathfrak{X}$ is a subadditive sequence (of subsets of $\mathbf{Z}$), we see that

$$\sum_{i=1}^n \{m_i x_i, m_i(x_i + \varepsilon_i d_i)\} \subseteq \sum_{i=1}^n m_i X_{k_i} \subseteq X_m, \tag{4}$$

where $m := \sum_{i=1}^n m_i k_i$. Set $\ell := \sum_{i=1}^n m_i x_i$. Then $\ell + \delta' = \sum_{i=1}^n m_i(x_i + \varepsilon_i d_i)$, and we obtain from (4) that ℓ and $\ell + \delta'$ are both in X_m. Thus $\delta \le \inf \Delta(X_m) \le \delta' = \gcd \Delta'$, which is enough to conclude that $\gcd \Delta' = \delta$, because $\gcd \Delta' \le \delta$ (by hypothesis). The rest is now clear, since $\gcd \Delta(\mathfrak{X}) \le \delta$. ■

Corollary 2.10. *Let $\mathfrak{X} = (X_k)_{k\ge 1}$ be a subadditive sequence of subsets of $\mathbf{Z}$ and suppose that $\Delta(\mathfrak{X}) \ne \varnothing$. The following hold:*

- (i) *If $x, y \in X_k$ for some $k \in \mathbf{N}^+$, then $\delta \mid y - x$.*
- (ii) *For every $q \in \mathbf{N}$, there exist $x \in \mathbf{Z}$ and $k \in \mathbf{N}^+$ such that $x + \delta \cdot [\![0, q]\!] \subseteq X_k$.*

Proof. (i) Let $k \in \mathbf{N}^+$ such that $X_k \ne \varnothing$ and pick $x, y \in X_k$. If $x = y$, the claim is obvious. Otherwise, there are $x_1, \dots, x_n \in \mathbf{Z}$ such that $x = x_1 < \cdots < x_n = y$ and $X_k \cap [\![x, y]\!] = \{x_1, \dots, x_n\}$, where without loss of generality we suppose $x < y$. It follows $x_{i+1} - x_i \in \Delta(X_k)$ for each $i \in [\![1, n-1]\!]$ (note that $n \ge 2$), which implies by Proposition 2.9 that $\delta \mid x_{i+1} - x_i$. So $\delta \mid y - x$, since $y - x = \sum_{i=1}^{n-1}(x_{i+1} - x_i)$.

(ii) Pick $q \in \mathbf{N}$. Since $\delta \in \Delta(\mathfrak{X}) \ne \varnothing$, there exist $x_0 \in \mathbf{Z}$ and $k_0 \in \mathbf{N}^+$ such that $\{x_0, x_0 + \delta\} \subseteq X_{k_0}$. Thus, since $\mathfrak{X}$ is a subadditive sequence (of subsets of $\mathbf{Z}$), we obtain

$$(q+1)x_0 + \delta \cdot [\![0, q+1]\!] = (q+1)\{x_0, x_0 + \delta\} \subseteq (q+1)X_{k_0} \subseteq X_{(q+1)k_0}.$$

To wit, $x + \delta \cdot [\![0, q]\!] \subseteq X_{(q+1)k_0}$, where $x := (q+1)x_0 + \delta \in \mathbf{Z}$. ■

We continue with some analogues of [18, Lemmas 2.7, 2.8, 2.10, and 2.18].

Lemma 2.11. *Let $\mathfrak{X} = (X_k)_{k\ge 1}$ be a subadditive sequence of subsets of $\mathbf{Z}$, and define, for every $i, k \in \mathbf{N}^+$, $X_{k,1} := X_k$ and $X_{k,i+1} := X_{k,i} \setminus \{\rho_{k,i}\}$. Then the following hold:*

- (i) *$\Delta(\mathfrak{X}) = \varnothing$ if and only if $|X_k| \le 1$ for all $k \in \mathbf{N}^+$.*
- (ii) *$\rho_k = \infty$, for some $k \in \mathbf{N}^+$, if and only if $\rho_{k,i} = \infty$ for all $i \in \mathbf{N}^+$.*
- (iii) *$X_{h,i} + X_{k,j} \subseteq X_{h+k,i+j-1}$ for all $i, j, h, k \in \mathbf{N}^+$; in particular, $\rho_{h,i} + \rho_{k,j} \le \rho_{h+k,i+j-1}$ provided that $X_{h,i}$ and $X_{k,j}$ are non-empty.*

Proof. (i) $\Delta(\mathfrak{X}) \ne \varnothing$ if and only if there exists $k \in \mathbf{N}^+$ such that $\Delta(X_k) \ne \varnothing$; namely, if and only if $\{x, x + d\} \subseteq X_k$ for some $x \in \mathbf{Z}$ and $d \in \mathbf{N}^+$; i.e., if and only if $|X_k| \ge 2$.

(ii) The "if" part is obvious, so let $k \in \mathbf{N}^+$ such that $\rho_k = \infty$. Then X_k is an infinite subset of $\mathbf{Z}$, and hence so are $X_{k,1}, X_{k,2}, \ldots$, because $X_{k,i+1} = X_k$ for every $i \in \mathbf{N}^+$ (by induction on i). Therefore, it is clear that $\rho_{k,1} = \rho_{k,2} = \cdots = \infty$.

(iii) Fix $i, j, h, k \in \mathbf{N}^+$; we have first to prove that $X_{h,i} + X_{k,j} \subseteq X_{h+k,i+j-1}$. If $X_{h,i}$ or $X_{k,j}$ is empty, we are done. Otherwise, let $r \in X_{h,i}$ and $s \in X_{k,j}$; it will suffice to check that $r + s \in X_{h+k,i+j-1}$. For, we see from the definition of $X_{h,i}$ and $X_{k,j}$ that

$$r \le \underbrace{\rho_{h,i} \le \cdots \le \rho_{h,1}}_{\text{(a)}} \quad \text{and} \quad s \le \underbrace{\rho_{k,j} \le \cdots \le \rho_{k,1}}_{\text{(b)}},$$

where, as implied by (ii), the inequalities labeled by (a) (respectively, by (b)) are strict if and only if $i \ge 2$ and $\rho_h < \infty$ (respectively, $j \ge 2$ and $\rho_k < \infty$). Thus, it is straightforward that

$$r + s \le \overbrace{\rho_{h,i} + \rho_{k,j} \le \cdots \le \rho_{h,1} + \rho_{k,j}}^{\text{(A)}} \le \cdots \le \rho_{h,1} + \rho_{k,1}, \tag{5}$$

where the lower brace (B) spans $\rho_{h,1} + \rho_{k,j} \le \cdots \le \rho_{h,1} + \rho_{k,1}$:

$$\underbrace{\rho_{h,1} + \rho_{k,j} \le \cdots \le \rho_{h,1} + \rho_{k,1}}_{\text{(B)}}$$

with the inequalities labeled by (A) (respectively, by (B)) being strict if and only if $i \ge 2$ (respectively, $j \ge 2$) and $\rho_h, \rho_k < \infty$. In addition, $\rho_{h,1}, \ldots, \rho_{h,i} \in X_h$ if and only if $\rho_h < \infty$, and similarly $\rho_{k,1}, \ldots, \rho_{k,j} \in X_k$ if and only if $\rho_k < \infty$. So, putting it all together and using that $X_h + X_k \subseteq X_{h+k}$ (by hypothesis), we conclude from (5) and (ii) that

$$r + s \in X_{h+k} \quad \text{and} \quad \left|X_{h+k} \cap [\![r + s, \infty]\!]\right| \ge i + j - 1.$$

It follows that $r + s \in X_{h+k,i+j-1}$ (as wished). It remains to demonstrate that $\rho_{h,i} + \rho_{k,j} \le \rho_{h+k,i+j-1}$, but this is now trivial, since $X_{h,i} + X_{k,j} \subseteq X_{h+k,i+j-1}$ and $X_{h,i}, X_{k,j} \ne \varnothing$ imply that

$$\rho_{h,i} + \rho_{k,j} = \sup X_{h,i} + \sup X_{k,j} = \sup(X_{h,i} + X_{k,j}) \le \sup X_{h+k,i+j-1} = \rho_{h+k,i+j-1}.$$

(Reecall that $\sup \varnothing := -\infty$, and note that $\sup X = -\infty$, for some $X \subseteq \mathbf{Z}$, if and only if X is empty.) ■

Lemma 2.12. *Let $\mathfrak{X} = (X_k)_{k\ge1}$ be a subadditive sequence of subsets of $\mathbf{Z}$. The following hold:*

(i) *If $\rho = -\infty$ then $X_k = \varnothing$ (and hence $\rho_k = -\infty$) for all $k \in \mathbf{N}^+$.*

(ii) *$k\rho \ge \rho_k$ and $\rho_{nk} \ge n\rho_k$ for every $n, k \in \mathbf{N}^+$.*

(iii) *If $\rho_n = n\rho$ for some $n \in \mathbf{N}^+$, then $\rho_{nk} = k\rho_n = nk\rho$ for all $k \in \mathbf{N}^+$.*

(iv) *$X_k = \varnothing$ for every $k \notin \wp \cdot \mathbf{N}^+$.*

(v) *If $\wp \ne 0$ then there exists $k_0 \in \mathbf{N}$ such that $X_{\wp k} \ne \varnothing$ for all $k \ge k_0$.*

(vi) *If $\rho_n = \infty$ for some $n \in \mathbf{N}^+$, then $\rho_{\wp k} = \infty$ for all but finitely many k.*

(vii) *Let $\Delta(\mathfrak{X}) \ne \varnothing$ and fix $i \in \mathbf{N}^+$. Then $\lambda_{\wp k,1} \le \cdots \le \lambda_{\wp k,i} < \rho_{\wp k,i} \le \cdots \le \rho_{\wp k,1}$ for all large k.*

(viii) *If $\Delta(\mathfrak{X}) \ne \varnothing$, then $\Delta(X_k) \ne \varnothing$ for all large k and $\mathfrak{q} \in \mathbf{N}^+ \cup \{\infty\}$.*

Proof. (i) is trivial by our definitions, and (ii) is an immediate consequence of (2) and Lemma 2.11(iii).

(iii) Assume that $\rho_n = n\rho$ for some $n \in \mathbf{N}^+$ and let $k \in \mathbf{N}^+$. If $\rho = -\infty$, then we get from (i) that $nk\rho = \rho_{nk} = -\infty$, and we are done. Otherwise, it follows from (ii) that $\rho_n/n \le \rho_{nk}/(nk) \le \rho = \rho_n/n$, and hence $\rho_{nk} = nk\rho$.

(iv) If $\wp = 0$, then $X_k = \varnothing$ for every $k \in \mathbf{N}^+$, and we are done. Otherwise, it is clear that $X_k \ne \varnothing$ for some $k \in \mathbf{N}^+$ only if $\wp \mid k$; or equivalently, that $X_k = \varnothing$ for every $k \notin \wp \cdot \mathbf{N}^+$.

(v) By [16, Theorem 1.4], there exist $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\} \subseteq \mathbf{Z}$, $m_1, \ldots, m_n \in \mathbf{N}^+$, and $k_1, \ldots, k_n \in K$ such that $\wp = \varepsilon_1 m_1 k_1 + \cdots + \varepsilon_n m_n k_n$, where $K := \{k \in \mathbf{N}^+ : X_k \neq \varnothing\}$ is non-empty because $\wp \neq 0$. Set

$$\ell := 1 + \frac{2}{\wp}(m_1 k_1 + \cdots + m_n k_n).$$

Then $\ell \in \mathbf{N}^+$, since $\wp \mid k_i$ for every $i \in [\![1, n]\!]$; moreover, we have

$$\wp\ell = a_1 k_1 + \cdots + a_n k_n \quad \text{and} \quad \wp(\ell + 1) = b_1 k_1 + \cdots + b_n k_n, \tag{6}$$

where, for $i \in [\![1, n]\!]$, $a_i := (2 + \varepsilon_i) m_i \in \mathbf{N}^+$ and $b_i := (2 + 2\varepsilon_i) m_i \in \mathbf{N}$. Let $k \geq (\ell - 1)\ell + 1$.

By [16, Theorem 1.7], there are $x, y \in \mathbf{N}$ with $x + y \geq 1$ such that $\ell x + (\ell + 1) y = k$. So, we derive from (6) that $\wp k = \alpha_1 k_1 + \cdots + \alpha_n k_n \geq 1$, where $\alpha_i := a_i x + b_i y \in \mathbf{N}$ for each $i \in [\![1, n]\!]$ and at least one of $\alpha_1, \ldots, \alpha_n$ is a *positive* integer. Using that $\mathfrak{X}$ is a subadditive sequence (of subsets of $\mathbf{Z}$), we can thus conclude that $\varnothing \neq \alpha_1 X_{k_1} + \cdots + \alpha_n X_{k_n} \subseteq X_{\wp k}$. This proves the claim with $k_0 = (\ell - 1)\ell + 1$.

(vi) Assume that $\rho_n = \infty$ for some $n \in \mathbf{N}^+$. Then $X_n \neq \varnothing$ and $0 \neq \wp \mid n$, and it follows from (v) that there exists $k_0 \in \mathbf{N}^+$ such that $X_{\wp k} \neq \varnothing$, and hence $\rho_{\wp k} \neq -\infty$, for all $k \geq k_0$. So, by Lemma 2.11(iii), $\rho_{\wp k} \geq \rho_{\wp k - n} + \rho_n \geq \infty$ for every $k \geq k_0 + n$ (note that $\wp k - n$ is a multiple of $\wp$).

(vii) Since $\mathfrak{X}$ is a subadditive sequence and $\Delta(\mathfrak{X})$ is non-empty, we get from Corollary 2.10(ii) that $x + \delta \cdot [\![0, 2i - 1]\!] \subseteq X_\ell$ for some $x \in \mathbf{Z}$ and $\ell \in \mathbf{N}^+$, and from (v) that there exists $k_0 \in \mathbf{N}^+$ such that $X_{\wp k} \neq \varnothing$ for $k \geq k_0$. In particular, we have that $0 \neq \wp \mid \ell$.

Accordingly, let $k \geq k_0 + \ell/\wp$. Then $k - \ell/\wp$ is an integer $\geq k_0$, and we infer from the above that $X_{\wp k - \ell} \neq \varnothing$. So, picking $y \in X_{\wp k - \ell}$ and once again appealing to the subadditivity of $\mathfrak{X}$, we find that

$$X_{\wp k} \supseteq X_{\wp k - \ell} + X_\ell \supseteq (x + y) + \delta \cdot [\![0, 2i - 1]\!].$$

This proves $|X_{\wp k}| \geq 2i$, whence we obtain that $\lambda_{\wp k, 1} \leq \cdots \leq \lambda_{\wp k, i} < \rho_{\wp k, i} \leq \cdots \leq \rho_{\wp k, 1}$.

(viii) By (vii), there exists $k_0 \in \mathbf{N}^+$ such that $|X_k| \geq 2$ for every $k \geq k_0$. It follows that $\Delta(X_k) \neq \varnothing$, and hence $\sup \Delta(X_k) \geq 1$, for all large k, which in turn shows that $\mathfrak{q} \in \mathbf{N}^+ \cup \{\infty\}$. ■

We conclude this section with a remark that will greatly simplify our work in the sequel.

Remark 2.13. Let $\mathfrak{X} = (X_k)_{k \geq 1}$ be a subadditive $\mathcal{P}(\mathbf{Z})$-valued sequence. If $\wp = 0$, then our definitions give that $X_k = \varnothing$ for all $k \in \mathbf{N}^+$; otherwise, $\wp$ is a positive integer and, by Lemma 2.12(v), the sequence $(X_{\wp k})_{k \geq 1}$ is primitive, in addition to being clearly subadditive. Moreover, we have, by Lemma 2.12(iv), that $X_k = \varnothing$ for all $k \notin \wp \cdot \mathbf{N}^+$. So we see that, when it comes to determine whether $\mathfrak{X}$ is an AAPS, we can assume without loss of generality that the sequence is not just subadditive, but also primitive.

3. STRUCTURE OF SUBADDITIVE SEQUENCES

Through this section, we investigate the structure of subadditive sequences of subsets of $\mathbf{Z}$ and carry out some critical steps towards the proof of our main theorem in § **??**. We start with the following:

Definition 3.1. Given $d \in \mathbf{N}^+$ and $M \in \mathbf{N}$, we take a set $X \subseteq \mathbf{Z}$ to be an *almost arithmetic progression* (in short, an AAP) *with difference* d *and bound* M if there exists $x \in \mathbf{Z}$ such that

$$\mathfrak{P}_d(x) \cap [\![\inf X + M, \sup X - M]\!] \subseteq X \subseteq \mathfrak{P}_d(x).$$

Accordingly, we say that a sequence $\mathfrak{X} = (X_k)_{k \geq 1}$ of subsets of $\mathbf{Z}$ is *asymptotically a sequence of* AAPs *with the same difference and bound* (in short, an AAPS) if there can be found $d \in \mathbf{N}^+$ and $M \in \mathbf{N}$ such that X_k is an AAP with difference d and bound M for all large k (in which case we may also refer to $\mathfrak{X}$ as an AAPS with difference d, or with difference d and bound M).

In addition, we call $\mathfrak{X}$ an *upper* AAPS if there are $a_1, a_2, \ldots \in \mathbf{N}$ with the property that $\big(X_k \cap [\![\rho_k - a_k, \rho_k]\!]\big)_{k \ge 1}$ is an AAPS and $a_k \to \infty$ as $k \to \infty$; and a *lower* AAPS if $-\mathfrak{X}$ is an upper AAPS.

We will see that AAPSs capture the essence of various results in the theory of non-unique factorization (with regard to the structure of sets of lengths and their unions), as well as in additive combinatorics (in relation to the structure of k-fold sumsets in the integers): Most notably, this will be the case with Corollary 1.2 and Theorems 1.3 and 1.4.

Proposition 3.2. *Let $\mathfrak{X} = (X_k)_{k \ge 1}$ be a subadditive, primitive sequence of subsets of $\mathbf{Z}$ with $\Delta(\mathfrak{X}) \ne \varnothing$, and suppose that there exist $M \in \mathbf{N}$, $d \in \mathbf{N}^+$, and $x_1, x_2, \ldots \in \mathbf{Z}$ such that $\mathfrak{P}_d(x_k) \cap [\![\lambda_k + M, \rho_k - M]\!] \subseteq X_k \subseteq \mathfrak{P}_d(x_k)$ for infinitely many k. Then $d = \delta$.*

Proof. Because $\Delta(\mathfrak{X})$ is non-empty and $\mathfrak{X}$ is a subadditive, primitive sequence, we obtain from Corollary 2.10(ii) that there are $x \in \mathbf{Z}$ and $\ell \in \mathbf{N}^+$ for which

$$x + \delta \cdot [\![0, 2M+1]\!] \subseteq X_\ell; \tag{7}$$

and from Lemma 2.12(vii) that there exists $\kappa_0 \in \mathbf{N}^+$ such that

$$X_k \ne \varnothing \quad \text{and} \quad \lambda_k + (d+2M)\delta < \rho_k - (d+2M)\delta, \quad \text{for } k \ge \kappa_0.$$

In particular, we infer from the last equation that $\lambda_k + M + 2d \le \rho_k - M$. So using that, by hypothesis, $\mathfrak{P}_d(x_k) \cap [\![\lambda_k + M, \rho_k - M]\!] \subseteq X_k \subseteq \mathfrak{P}_d(x_k)$ for infinitely many k, we find that, for some $k_0 \ge \kappa_0 + \ell$,

$$\mathcal{J}_{k_0} := X_{k_0} \cap [\![\lambda_{k_0} + M, \rho_{k_0} - M]\!] = \mathfrak{P}_d(x_{k_0}) \cap [\![\lambda_{k_0} + M, \rho_{k_0} - M]\!] \quad \text{and} \quad |\mathcal{J}_{k_0}| \ge 2. \tag{8}$$

Then $d \in \Delta(X_{k_0}) \subseteq \Delta(\mathfrak{X})$, and hence $\delta \le d$. It remains to show that $d \le \delta$.

For, note that $X_{k_0 - \ell}$ is non-empty, because $k_0 - \ell \ge \kappa_0$ (and $X_k \ne \varnothing$ for $k \ge \kappa_0$). Accordingly, pick $y \in X_{k_0 + \ell}$. Then we have from Lemma 2.11(iii) that

$$x + y + \delta \cdot [\![0, 2M+1]\!] \overset{(7)}{\subseteq} X_\ell + X_{k_0 - \ell} \subseteq X_{k_0},$$

and therefore $\lambda_{k_0} + M \le x + y + M\delta < x + y + (M+1)\delta \le \rho_{k_0} - M$. It follows that

$$x + y + M\delta + \{0, \delta\} \subseteq X_{k_0} \cap [\![\lambda_{k_0} + M, \rho_{k_0} - M]\!] \overset{(8)}{=} \mathfrak{P}_d(x_{k_0}) \cap [\![\lambda_{k_0} + M, \rho_{k_0} - M]\!],$$

which is only possible if $d \mid \delta$. So $d \le \delta$ and we are done. ∎

Corollary 3.3. *Let $\mathfrak{X} = (X_k)_{k \ge 1}$ is a subadditive, primitive sequence of subsets of $\mathbf{Z}$ with $\Delta(\mathfrak{X}) \ne \varnothing$. If $\mathfrak{X}$ is an* AAPS *with difference d for some $d \in \mathbf{N}^+$, then $d = \delta$.*

Proof. By hypothesis, there exist $d \in \mathbf{N}^+$, $M \in \mathbf{N}$, and $x_1, x_2, \ldots \in \mathbf{Z}$ such that $\mathfrak{P}_d(x_k) \cap [\![\lambda_k + M, \rho_k - M]\!] \subseteq X_k \subseteq \mathfrak{P}_d(x_k)$ for all large k. On the other hand, we infer from Lemma 2.12(vii) that $\rho_k - \lambda_k \to \infty$ as $k \to \infty$ (here we use that $\Delta(\mathfrak{X})$ is non-empty). So applying Proposition 3.2 with $a_k := \lambda_k + M$ and $b_k := \rho_k - M$ yields $d = \delta$. ∎

Now we look for sufficient and necessary conditions for a (subadditive) sequence of subsets of $\mathbf{Z}$ to be an AAPS or a periodic AAPS. We begin with a variant of [18, Lemma 2.23].

Lemma 3.4. *Given* $X, Y \subseteq \mathbf{Z}$, *the following hold:*

(i) *If* $\inf X = \inf Y$, $\sup X = \sup Y$, *and* $X \subseteq Y$, *then* $\sup \Delta(Y) \le \sup \Delta(Y)$.

(ii) $\sup \Delta(X+Y) \le \max(\sup \Delta(X), \sup \Delta(Y))$.

Proof. (i) If $\Delta(Y) = \varnothing$, then $\sup \Delta(Y) = 0$ and there is nothing left to prove. Otherwise, let $d \in \Delta(Y)$: It suffices to prove $d \le \sup \Delta(X)$. For, pick $z \in \mathbf{Z}$ such that $Y \cap \llbracket z, z+d \rrbracket = \{z, z+d\}$. Accordingly, let $x := \sup\bigl(X \cap \llbracket -\infty, z \rrbracket\bigr)$ and $y := \inf\bigl(X \cap \llbracket z+1, \infty \rrbracket\bigr)$. Clearly $x, y \in X$, since our assumptions imply that $\inf X = \inf Y \le z < z+d \le \sup Y = \sup X$. It follows $d \le y - x \in \Delta(X)$, because $X \subseteq Y$ and there exists no element in Y that is strictly in between z and $z+d$. Thus, we obtain $d \le \sup \Delta(X)$.

(ii) If $\Delta(X+Y) = \varnothing$, the conclusion is trivial. Otherwise, pick $d \in \Delta(X+Y)$, and let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $(X+Y) \cap \llbracket x_1+y_1, x_2+y_2 \rrbracket = \{x_1+y_1, x_2+y_2\}$ and $d = (x_2+y_2) - (x_1+y_1) \ge 1$.

Now, using that $\Delta(Z+k) = \Delta(Z)$ for all $Z \subseteq \mathbf{Z}$ and $k \in \mathbf{Z}$, we can assume without loss of generality that $x_1 = y_1 = 0$. It follows (up to symmetry) that $x_2 \ge 1$. Accordingly, set $x := \inf X^+$.

We derive from the above that $x \in \Delta(X) \cap X^+ \cap (X+Y)$, and since $(X+Y) \cap \llbracket 0, x_2+y_2 \rrbracket = \{0, d\}$, we conclude that $d \le x \le \sup \Delta(X)$. This finishes the proof, because $d \in \Delta(X+Y)$ was arbitrary. ■

Lemma 3.5. *Let* $\mathfrak{X} = (X_k)_{k \ge 1}$ *be a sequence of subsets of* $\mathbf{Z}$. *The following hold:*

(i) $\mathfrak{X}$ *is subadditive* (*respectively, primitive*) *if and only if so is* $-\mathfrak{X}$.

(ii) $d \in \Delta(\mathfrak{X})$ *if and only if* $d \in \Delta(-\mathfrak{X})$, *that is,* $\Delta(\mathfrak{X}) = \Delta(-\mathfrak{X})$.

(iii) $\mathfrak{X}$ *is an* AAPS *with difference* d, *for some* $d \in \mathbf{N}^+$, *if and only if so is* $-\mathfrak{X}$.

(iv) $\mathfrak{X}$ *is a* μ*-periodic* AAPS, *for some* $\mu \in \mathbf{N}^+$, *if and only if so is* $-\mathfrak{X}$.

Proof. Since $-(-\mathfrak{X}) = \mathfrak{X}$, it will be sufficient to show the "only if" part of each statement.

(i) This is trivial, because if $X + Y \subseteq Z$ for some $X, Y, Z \subseteq \mathbf{Z}$ (respectively, $X \ne \varnothing$ for some $X \subseteq \mathbf{Z}$) then $(-X) + (-Y) \subseteq -Z$ (respectively, $-X \ne \varnothing$).

(ii) Let $d \in \Delta(\mathfrak{X})$. Then there exist $k \in \mathbf{N}^+$ and $x \in \mathbf{Z}$ such that $X_k \cap \llbracket x, x+d \rrbracket = \{x, x+d\}$, which is equivalent to $(-X_k) \cap \llbracket y, y+d \rrbracket = \{y, y+d\}$, where $y := -(x+d)$. Hence $d \in \Delta(-X_k) \subseteq \Delta(-\mathfrak{X})$.

(iii) Assume that $\mathfrak{X}$ is an AAPS with difference d for some $d \in \mathbf{N}^+$, namely, there exist $M \in \mathbf{M}$ and $x_1, x_2, \ldots \in \mathbf{Z}$ such that, for all large k,

$$\mathfrak{P}_d(x_k) \cap \llbracket \lambda_k(\mathfrak{X}) + M, \rho_k(\mathfrak{X}) - M \rrbracket \subseteq X_k \subseteq \mathfrak{P}_d(x_k). \tag{9}$$

But $\mathfrak{P}_d(-x) = -\mathfrak{P}_d(x)$ for every $x \in \mathbf{Z}$, and it is easy that $\lambda_k(-\mathfrak{X}) = -\rho_k(\mathfrak{X})$ and $\rho_k(-\mathfrak{X}) = -\lambda_k(\mathfrak{X})$. So $-\mathfrak{X}$ is also an AAPS with difference d, as we get from (9) that, for all but finitely many k,

$$\begin{aligned} \mathfrak{P}_d(-x_k) \cap \llbracket \lambda_k(-\mathfrak{X}) + M, \rho_k(-\mathfrak{X}) - M \rrbracket &= (-\mathfrak{P}_d(x_k)) \cap \llbracket -\rho_k(\mathfrak{X}) + M, -\lambda_k(\mathfrak{X}) - M \rrbracket \\ &\subseteq -X_k \subseteq -\mathfrak{P}_d(x_k) = \mathfrak{P}_d(-x_k). \end{aligned} \tag{10}$$

(iv) Suppose that $\mathfrak{X}$ is a μ-periodic AAPS for some $\mu \in \mathbf{N}^+$, namely, there exist $d \in \mathbf{N}^+$, $M \in \mathbf{N}$, $x_1, x_2, \ldots \in \mathbf{Z}$, and $X_0', X_0'', \ldots, X_{\mu-1}', X_{\mu-1}'' \subseteq \llbracket 0, M-1 \rrbracket$ such that, from some k on,

$$X_k = \bigl(\lambda_k(\mathfrak{X}) + X'_{k \bmod \mu}\bigr) \uplus \mathscr{P}_k \uplus \bigl(\rho_k(\mathfrak{X}) - X''_{k \bmod \mu}\bigr) \subseteq \mathfrak{P}_d(x_k),$$

where $\mathscr{P}_k := \mathfrak{P}_d(x_k) \cap \llbracket \lambda_k(\mathfrak{X}) + K, \rho_k(\mathfrak{X}) - K \rrbracket$. So by the same considerations as in the proof of (10), we find that, for all sufficiently large k,

$$\begin{aligned} -X_k &= \bigl(-\rho_k(\mathfrak{X}) + X''_{k \bmod \mu}\bigr) \uplus (-\mathscr{P}_k) \uplus \bigl(-\lambda_k(\mathfrak{X}) - X'_{k \bmod \mu}\bigr) \\ &= \bigl(\lambda_k(-\mathfrak{X}) + X''_{k \bmod \mu}\bigr) \uplus (-\mathscr{P}_k) \uplus \bigl(\rho_k(-\mathfrak{X}) - X'_{k \bmod \mu}\bigr) \subseteq \mathfrak{P}_d(-x_k). \end{aligned}$$

Moreover, we have that, for all k,

$$-\mathscr{P}_k = (-\mathfrak{P}_d(x_k)) \cap [\![-\rho_k(\mathfrak{X}) + K, -\lambda_k(\mathfrak{X}) - K]\!] = \mathfrak{P}_d(-x_k) \cap [\![\lambda_k(-\mathfrak{X}) + K, \rho_k(-\mathfrak{X}) - K]\!].$$

This is enough to conclude that $-\mathfrak{X}$ is also a μ-periodic AAPS. ■

Lemma 3.6. *Let $\mathfrak{X} = (X_k)_{k\ge1}$ be a sequence of subsets of $\mathbf{Z}$, and let $a_1, a_2, \ldots, b_1, b_2, \ldots \in \mathbf{Z}$ such that $\lambda_k \le a_k$ and $b_k \le \rho_k$ for all sufficiently large k. Then $\mathfrak{X}$ is an* AAPS *with difference d and bound M, for some $d \in \mathbf{N}^+$ and $M \in \mathbf{N}$, only if so is the sequence $\big(X_k \cap [\![a_k, b_k]\!]\big)_{k\ge1}$.*

Proof. Assume that $\mathfrak{X}$ is an AAPS with difference d and bound M for some $d \in \mathbf{N}^+$ and $M \in \mathbf{N}$, i.e., there exist $k_0 \in \mathbf{N}^+$ and $x_1, x_2, \ldots \in \mathbf{Z}$ such that, for every $k \ge k_0$,

$$\mathfrak{P}_d(x_k) \cap [\![\lambda_k(\mathfrak{X}) + M, \rho_k(\mathfrak{X}) - M]\!] \subseteq X_k \subseteq \mathfrak{P}_d(x_k). \tag{11}$$

Accordingly, fix $k \ge k_0$ and set $Y_k := X_k \cap [\![a_k, b_k]\!]$; without loss of generality, we will suppose $\lambda_k \le a_k$ and $b_k \le \rho_k$ (by hypothesis, these inequalities hold for all large k). If $b_k - M < a_k + M$, then

$$\mathfrak{P}_d(x_k) \cap [\![\inf Y_k + M, \sup Y_k - M]\!] = \varnothing \subseteq Y_k \subseteq X_k \subseteq \mathfrak{P}_d(x_k).$$

Otherwise, we have that $\lambda_k + M \le a_k + M \le b_k - M \le \rho_k - M$, which, together with (11), implies

$$\begin{aligned} \mathfrak{P}_d(x_k) \cap [\![\inf Y_k + M, \sup Y_k - M]\!] &\subseteq \mathfrak{P}_d(x_k) \cap [\![a_k + M, b_k - M]\!] \\ &= \mathfrak{P}_d(x_k) \cap [\![\lambda_k(\mathfrak{X}) + M, \rho_k(\mathfrak{X}) - M]\!] \cap [\![a_k + M, b_k - M]\!] \\ &\subseteq X_k \cap [\![a_k + M, b_k - M]\!] \subseteq Y_k \subseteq X_k \subseteq \mathfrak{P}_d(x_k). \end{aligned}$$

We thus conclude that $(Y_k)_{k\ge1}$ is also an AAPS with difference d and bound M. ■

Theorem 3.7. *Let $\mathfrak{X}$ a subadditive, primitive sequence of subsets of $\mathbf{Z}$ such that $\Delta(\mathfrak{X}) \ne \varnothing$. Then are equivalent:*

(a) *$\mathfrak{X}$ is an* AAPS.

(b) *There are $M \in \mathbf{N}$ and $x_1, x_2, \ldots \in \mathbf{Z}$ such that, for all but finitely many k,*

$$\mathfrak{P}_\delta(x_k) \cap [\![\lambda_k + M, \rho_k - M]\!] \subseteq X_k \quad \text{and} \quad x_k \in X_k.$$

(c) *$\mathfrak{q} < \infty$ and $\mathfrak{X}$ is both an upper and a lower* AAPS *with difference δ.*

Proof. By Lemma 2.12(v), there exists $k_0 \in \mathbf{N}^+$ such that X_k is non-empty for $k \ge k_0$; accordingly, we take $(\bar{x}_k)_{k\ge1}$ to be a $\mathbf{Z}$-valued sequence such that $\bar{x}_{k_0} \in X_{k_0}, \bar{x}_{k_0+1} \in X_{k_0+1}, \ldots$ Based on these premises, we proceed to prove that (a) ⇒ (b) ⇒ (c) ⇒ (a).

(a) ⇒ (b): Since $\Delta(\mathfrak{X}) \ne \varnothing$ and $\mathfrak{X}$ is an AAPS, we get from Corollary 3.3 that there are $M \in \mathbf{N}$ and $z_1, z_2, \ldots \in \mathbf{Z}$ such that, for all large $k \ge k_0$, $\mathfrak{P}_\delta(z_k) \cap [\![\lambda_k + M, \rho_k - M]\!] \subseteq X_k \subseteq \mathfrak{P}_\delta(z_k)$; in particular, $\bar{x}_k \in \mathfrak{P}_\delta(z_k)$, which yields $\mathfrak{P}_\delta(z_k) = \mathfrak{P}_\delta(\bar{x}_k)$ and finishes the proof.

(b) ⇒ (c): By our assumptions and Lemma 2.12(vii) (applied with $i = 2M + \delta$), there is an integer $\kappa_0 \ge k_0$ such that $\mathscr{J}_k := \mathfrak{P}_\delta(x_k) \cap [\![\lambda_k + M, \rho_k - M]\!]$ is a *non-empty* subset of X_k for $k \ge \kappa_0$.

Accordingly, fix $k \ge \kappa_0$ and set $y_k := \inf \mathscr{J}_k$ and $z_k := \sup \mathscr{J}_k$ (note that $\lambda_k \le y_k \ne \infty$ and $-\infty \ne z_k \le \rho_k$, though we may have $y_k = -\infty$ or $z_k = \infty$).

It is trivial that $y_k \le \lambda_k + M + \delta$ and $\rho_k - (M + \delta) \le z_k$; moreover, we get from Corollary 2.10(i) that $X_k \subseteq \mathfrak{P}_\delta(x_k)$, with the result that $X_k \cap [\![y_k, z_k]\!] = \mathfrak{P}_\delta(x_k) \cap [\![y_k, z_k]\!]$. Thus, we find that

$$X_k = \big(X_k \cap [\![\lambda_k, y_k]\!]\big) \cup (X_k \cap [\![y_k, z_k]\!]) \cup \big(X_k \cap [\![z_k, \rho_k]\!]\big) \supseteq \{\lambda_k, y_k\} \cup \big(\mathfrak{P}_\delta(x_k) \cap [\![y_k, z_k]\!]\big) \cup \{z_k, \rho_k\}.$$

Also, we see by construction that $y_k = -\infty$ if and only if $\lambda_k = -\infty$, in which case

$$\{\lambda_k, y_k\} \cup \big(\mathfrak{P}_\delta(x_k) \cap \llbracket y_k, z_k \rrbracket\big) = \mathfrak{P}_\delta(x_k) \cap \llbracket -\infty, z_k \rrbracket;$$

and similarly, $z_k = \infty$ if and only if $\rho_k = \infty$, in which case

$$\big(\mathfrak{P}_\delta(x_k) \cap \llbracket y_k, z_k \rrbracket\big) \cup \{z_k, \rho_k\} = \mathfrak{P}_\delta(x_k) \cap \llbracket y_k, \infty \rrbracket.$$

Consequently, we derive from Lemma 3.4(i) that $\sup \Delta(X_k) \le M + \delta$, and hence $\mathfrak{q} < \infty$.

For the rest, it follows from the above that $\mathfrak{P}_\delta(x_k) \cap \llbracket \lambda_k + M, \rho_k - M \rrbracket \subseteq X_k \subseteq \mathfrak{P}_\delta(x_k)$ for $k \ge \kappa_0$, so $\mathfrak{X}$ is an AAPS with difference δ. On the other hand, we infer from Lemma 2.12(vii) that $\rho - \lambda > 0$; from Lemmas 3.5(i) and 2.11(iii) (with the latter applied first to $\mathfrak{X}$, and then to $-\mathfrak{X}$) that $\lambda_{nk} \le n\lambda_k$ and $n\rho_k \le \rho_{nk}$ for every $n, k \in \mathbf{N}^+$; and from Lemmas 3.5(i) and 2.12(ii) (again, with the latter applied first to $\mathfrak{X}$, and then to $-\mathfrak{X}$) that $\lambda_{h+k} \le \lambda_h + \lambda_k \le \rho_h + \rho_k \le \rho_{n+k}$ for all $h, k \ge k_0$. So putting it all together, we conclude from Lemma 2.8 that there are $\varepsilon \in \mathbf{R}^+$ and $k_\varepsilon \ge \kappa_0$ such that

$$\lambda_k \le (\lambda + \varepsilon)k \le (\rho - \varepsilon)k \le \rho_k, \quad \text{for } k \ge k_\varepsilon.$$

Accordingly, it follows from Lemma 3.6 (applied first with $a_k = \lambda_k$ and $b_k = (\lambda + \varepsilon)k$, and then with $a_k = (\rho - \varepsilon)k$ and $b_k = \rho_k$) that $\mathfrak{X}$ is both an upper and a lower AAPS with difference δ.

(c) $\Rightarrow$ (a): By hypothesis, there exist $\varepsilon \in \mathbf{R}^+$ such that each of the sequences $(\mathscr{J}_k)_{k \ge 1}$ and $(\mathscr{S}_k)_{k \ge 1}$ is an AAPS with difference δ, where we set, for every $k \in \mathbf{N}^+$,

$$\mathscr{I}_k := X_k \cap \llbracket \lambda_k, (\lambda + \varepsilon)k \rrbracket \quad \text{and} \quad \mathscr{S}_k := X_k \cap \llbracket (\rho - \varepsilon)k, \rho_k \rrbracket.$$

It follows that there are $\kappa_0 \in \mathbf{N}^+$, $N \in \mathbf{N}$, and $y_1, y_2, \ldots, z_1, z_2, \ldots \in \mathbf{Z}$ such that, for every $k \ge \kappa_0$,

$$\mathfrak{P}_\delta(y_k) \cap \llbracket \lambda_k + N, \ell_k - N \rrbracket \subseteq \mathscr{J}_k \subseteq \mathfrak{P}_\delta(y_k) \quad \text{and} \quad \mathfrak{P}_\delta(z_k) \cap \llbracket s_k + N, \rho_k - N \rrbracket \subseteq \mathscr{S}_k \subseteq \mathfrak{P}_\delta(z_k).$$

where, for ease of notation, we have defined $\ell_k := \sup \mathscr{I}_k$ and $s_k := \inf \mathscr{S}_k$.

On the other hand, we have by ... that $\mathfrak{q} \in \mathbf{N}^+$. So we derive from Corollary 2.10(ii) that there exist $x \in \mathbf{Z}$ and $\ell \in \mathbf{N}^+$ such that $X^* := x + \delta \cdot \llbracket 0, \mathfrak{q} \rrbracket \subseteq X_\ell$, which, together with Lemma 2.11(iii), implies

$$X_{k-\ell} + \big(x + \delta \cdot \llbracket 0, \mathfrak{q} \rrbracket\big) \subseteq X_{k-\ell} + X_\ell \subseteq X_k, \quad \text{for } k \ge k_1 + \ell. \tag{12}$$

On the other hand, it follows from our assumptions that there exist $k_0 \in \mathbf{N}^+$ and $N \in \mathbf{N}$ for which

$$1 \le \sup \Delta(X_k) \le \mathfrak{q} \quad \text{and} \quad \mathscr{P}_k := (x_k + \delta \cdot \mathbf{Z}) \cap \llbracket \rho_{k-\ell} + \ell, \rho_k - N \rrbracket \subseteq \mathscr{U}_k, \text{ for } k \ge k_0. \tag{13}$$

Fix $k \ge \ell + \max(k_0, k_1, k_{N+1})$ and set $X_k^* := X^* + X_{k-\ell}$. Then $\sup \Delta(X_{k-\ell}) \le \mathfrak{q}$, and because X^* is an AP with difference δ and $|X^*| = \mathfrak{q} + 1$, it is clear that X_k^* is also an AP with difference δ, i.e.,

$$X_k^* = \mathfrak{P}_\delta(\bar{x}_k) \cap \llbracket \inf X_k^*, \sup X_k^* \rrbracket \overset{(12)}{\subseteq} X_k. \tag{14}$$

Moreover, we have that

$$\{k, \ell + \rho_{k-\ell}\} \subseteq X_k^*, \quad \inf X_k^* = \ell + \lambda_{k-\ell}, \quad \text{and} \quad \sup X_k^* = (\ell + \delta D) + \rho_{k-\ell} \ge \ell + \rho_{k-\ell}. \tag{15}$$

But $\ell + \lambda_{k-\ell} \le k = \ell + (k - \ell) \le \ell + \rho_{k-\ell}$, and it is obvious that

$$\mathfrak{P}_\delta(x_k) \cap \llbracket k, \rho_k - N \rrbracket \subseteq \big(\mathfrak{P}_\delta(x_k) \cap \llbracket k, \ell + \rho_{k-\ell} \rrbracket\big) \cup \big(\mathfrak{P}_\delta(x_k) \cap \llbracket \ell + \rho_{k-\ell}, \rho_k - N \rrbracket\big).$$

So, we infer from (13)-(15) that $\mathfrak{P}_\delta(\bar{x}_k) \cap \llbracket k, \rho_k - N \rrbracket \subseteq X_k^* \cup \mathscr{P}_k \subseteq X_k$, which implies that $\mathfrak{X}$ satisfies the Structure Theorem for Unions.

is an AAP with difference $d_{\rm low}$ and bound N, and $X_k \cap [\![(\rho-\varepsilon)k, \rho_k]\!]$ is an AAPS (with difference δ).

Let $k \ge k_0$, and set $\mathscr{J}_k := \mathfrak{P}_\delta(x_k) \cap [\![\lambda_k + M, \rho_k - M]\!]$. It follows from our assumptions and Lemma 2.12(vii) (applied with $i = 2M + \delta$) that, for all large k, $\mathscr{J}_k$ is a *non-empty* subset of X_k, so we can pick $y_k \in \mathscr{J}_k \subseteq \mathfrak{P}_\delta(x_k)$ and conclude by Corollary 2.10(i) that $X_k \subseteq \mathfrak{P}_\delta(y_k) = \mathfrak{P}_\delta(x_k)$ (recall that, by hypothesis, $x_k \in X_k$ from some k on). At the end of the day, this means that $\mathfrak{P}_\delta(x_k) \cap [\![\lambda_k+M, \rho_k-M]\!] \subseteq X_k \subseteq \mathfrak{P}_\delta(x_k)$ for all but finitely many k, which again shows that $\mathfrak{X}$ is an AAPS. ■

Corollary 3.8. *Let $\mathfrak{X} = (X_k)_{k\ge1}$ be a sequence of subsets of $\mathbf{Z}$ with $\Delta(\mathfrak{X}) \ne \varnothing$ and $\mathfrak{q} < \infty$. Then are equivalent:*

(i) *$\mathfrak{X}$ is an upper* AAPS *with difference δ.*

(ii) *There exist $x \in \mathbf{Z}$, $\ell \in \mathbf{N}^+$, and $N \in \mathbf{N}$ such that $x + \delta \cdot [\![0, \mathfrak{q}]\!] \subseteq X_\ell$ and $X_k \cap [\![\rho_{k-\ell} + x, \rho_k - N]\!]$ is an* AAP *with difference δ for all large k.*

4. COMPLEMENTARY STRUCTURAL RESULTS

We have already observed in Remark 2.1 that dealing with sequences of subsets of $\mathbf{Z}$ results in a sort of "duality" that can be conveniently used to make proofs and definitions shorter, but would not be available if we restrict attention to $\mathbf{N}$ since the beginning. On the other hand, also working with subsets of $\mathbf{N}$ has its own advantages, were it only for the fact that every non-empty set of non-negative integers has a minimum (with respect to the usual ordering of $\mathbf{Z}$).

These naive considerations lead us to Theorem 4.3 and Corollary 4.4 below, which, together with Lemmas 4.1 and 4.2, help to clarify the bigger picture, by putting in relation various properties of a subadditive sequence $\mathfrak{X}$ of subsets of $\mathbf{Z}$ with corresponding properties of $\mathfrak{X}_{\ge0}$ and $\mathfrak{X}_{\le0}$.

Lemma 4.1. *Let $\mathfrak{X} = (X_k)_{k\ge1}$ be a subadditive sequence of subsets of $\mathbf{Z}$. The following hold:*

(i) *$(-\mathfrak{X})_{\le0} = -(\mathfrak{X}_{\ge0})$ and $(-\mathfrak{X})_{\ge0} = -(\mathfrak{X}_{\le0})$.*

(ii) *$\mathfrak{X}_{\ge0}$ and $\mathfrak{X}_{\le0}$ are subadditive.*

(iii) *If $\mathfrak{X}$ is primitive and $X_n \cap \mathbf{N}^+ \ne \varnothing$ for some $n \in \mathbf{N}^+$, then $\mathfrak{X}_{\ge0}$ is also primitive.*

(iv) *$\mathfrak{X}$ is primitive if and only if so is at least one of $\mathfrak{X}_{\ge0}$ and $\mathfrak{X}_{\le0}$.*

(v) *Suppose that $X_n \cap \mathbf{N}^+ \ne \varnothing$ for some $n \in \mathbf{N}^+$. Then $\delta(\mathfrak{X}) = \delta(\mathfrak{X}_\ge 0)$.*

(vi) *If $\mathfrak{X}_{\ge0} \ne \mathfrak{X} \ne \mathfrak{X}_{\le0}$, then $\Delta(\mathfrak{X}_{\ge0})$ and $\Delta(\mathfrak{X}_{\le0})$ are non-empty, $\delta(\mathfrak{X}) = \delta(\mathfrak{X}_{\ge0}) = \delta(\mathfrak{X}_{\ge0})$, and there exists $q \in \mathbf{N}^+$ such that $X_{qk} \cap \mathbf{N}^+ \ne \varnothing$ and $X_{qk} \cap \mathbf{Z}^- \ne \varnothing$ for all k.*

Proof. (i) It is a straightforward consequence of the definitions (we omit details).

(ii) By (i) and Lemma 3.5(i), it suffices to show that $\mathfrak{X}_{\ge0}$ is subadditive. But this is trivial, because the subadditivity of $\mathfrak{X}$ implies that, for all $h, k \in \mathbf{N}^+$,

$$(X_h \cap \mathbf{N}) + (X_k \cap \mathbf{N}) \subseteq (X_h + X_k) \cap \mathbf{N} \subseteq X_{h+k} \cap \mathbf{N}.$$

(iii) Assume that $\mathfrak{X}$ is primitive and $X_n \cap \mathbf{N}^+ \ne \varnothing$ for some n. Accordingly, pick $a \in X_n \cap \mathbf{N}^+$ and let $k_0 \in \mathbf{N}^+$ such that X_k is non-empty for $k \ge k_0$. Then, for each $r \in [\![0, n-1]\!]$ fix $x_r \in X_{nk_0+r}$. Lastly, fix $k \ge n(\bar{x} + k_0)$, where $\bar{x} := \max_{0\le i<n} |x_i|$.

By the division algorithm, we have that $k = n(q + k_0) + r$ for some $q \in \mathbf{N}$ and $r \in [\![0, n-1]\!]$. So we conclude from the subadditivity of $\mathfrak{X}$ that

$$bq + x_r \in qX_n + X_{nk_0+r} \subseteq X_{nq} + X_{nk_0+r} \subseteq X_{n(q+k_0)+r} = X_k;$$

This shows $X_k^+ \neq \varnothing$, because it is evident that $bq + x_r \ge bq - |x_r| \ge |x| - \bar{x} \ge 0$. Consequently, $\mathfrak{X}_{\ge 0}$ is primitive, for k was an arbitrary integer $\ge n(\bar{x} + k_0)$.

(iv) The "if" clause is trivial, so suppose that $\mathfrak{X}$ is primitive. If $X_k = \{0\}$ for all k, then $\mathfrak{X} = \mathfrak{X}_{\ge 0}$ and we are done. Otherwise, we note that $\mathfrak{X}$ is primitive if and only if so is $-\mathfrak{X}$, and hence we infer from (i) that, without loss of generality, $X_n \cap \mathbf{N}$ is non-empty for some $n \in \mathbf{N}^+$; by point (iii), this implies that $\mathfrak{X}_{\ge 0}$ is primitive, and again we are done.

(v) Set $\delta := \delta(\mathfrak{X})$ and $\delta^+ := \delta(\mathfrak{X}_{\ge 0})$. It is clear from our definitions that $\Delta(\mathfrak{X}_{\ge 0})$ is a subset of $\Delta(\mathfrak{X})$. Thus, if $\Delta(\mathfrak{X})$ is empty, then $\delta = \delta^+ = 0$ and we are done. Accordingly, we assume from now on that $\Delta(\mathfrak{X}) \neq \varnothing$, and hence $\{x, x+\delta\} \subseteq X_{k_0}$ for some $x \in \mathbf{Z}$ and $k_0 \in \mathbf{N}^+$. Then the subadditivity of $\mathfrak{X}$, along with the assumption that $X_n \cap \mathbf{N}^+ \neq \varnothing$, implies that there exists $y \in \mathbf{N}^+$ such that

$$ky \in X_{nk} \cap \mathbf{N}^+ \quad \text{and} \quad ky + \{x, x+\delta\} \subseteq X_{nk} + X_{k_0} \subseteq X_{nk+k_0}, \quad \text{for every } k \in \mathbf{N}^+.$$

So we see that $|x| + \{x, x+\delta\} \subseteq X_{n|x|+k_0} \cap \mathbf{N}$, which in turn shows that $\delta^+ \le \delta$ and $\Delta(\mathfrak{X}_{\ge 0}) \neq \varnothing$. This leads to the desired conclusion, since $\varnothing \neq \Delta(\mathfrak{X}_{\ge 0}) \subseteq \Delta(\mathfrak{X})$ yields $\delta = \inf \Delta(\mathfrak{X}) \le \inf \Delta(\mathfrak{X}_{\ge 0}) = \delta^+$.

(vi) Assume that $\mathfrak{X}_{\ge 0} \neq \mathfrak{X} \neq \mathfrak{X}_{\le 0}$. Then $X_m \cap \mathbf{Z}^- \neq \varnothing$ and $X_n \cap \mathbf{N}^+ \neq \varnothing$ for some $m, n \in \mathbf{N}^+$, and considering that $\Delta(\mathfrak{Y}) = \Delta(-\mathfrak{Y})$ for every sequence $\mathfrak{Y}$ of subsets of $\mathbf{Z}$, it is immediate from (i) and (v) that $\delta(\mathfrak{X}) = \delta(\mathfrak{X}_{\ge 0}) = \delta(\mathfrak{X}_{\ge 0})$. Moreover, we get from the subadditivity of $\mathfrak{X}$ that $nX_m \cup mX_n \subseteq X_{mn}$, with the result that $X_{mnk} \cap \mathbf{Z}^- \neq \varnothing \neq X_{mnk} \cap \mathbf{N}^+$ for every $k \in \mathbf{N}^+$.

In particular, let $a \in X_{mn} \cap \mathbf{Z}^-$ and $b \in X_{mnk} \cap \mathbf{N}^+$, and set $p := |a| + b$. Then $bp + (a-b) \cdot [\![0, p]\!] \subseteq pX_{mn} \subseteq X_{mnp}$ (again by subadditivity of $\mathfrak{X}$), and it is easily checked that

$$ap < a(p-1) + b \le 0 \le b(p-1) + a < bp.$$

Therefore, we conclude that $\varnothing \neq \Delta(X_{mnp} \cap \mathbf{Z}_{\le 0}) \subseteq \Delta(\mathfrak{X}_{\le 0})$ and $\varnothing \neq \Delta(X_{mnp} \cap \mathbf{N}) \subseteq \Delta(\mathfrak{X}_{\ge 0})$. ■

Lemma 4.2. *Let $\mathfrak{X} = (X_k)_{k \ge 1}$ be a subadditive, primitive sequence of subsets of $\mathbf{Z}$ with $X_\ell \cap \mathbf{Z}^- \neq \varnothing$ and $X_m \cap \mathbf{N}^+ \neq \varnothing$ for some $\ell, m \in \mathbf{N}^+$. Then the following hold:*

(i) *There is $N \in \mathbf{N}^+$ such that, for all large k, $X_k \cap [\![-N, 0]\!]$ and $X_k \cap [\![1, N]\!]$ are both non-empty.*

(ii) *$\rho_k \to \infty$ and $\lambda_k \to -\infty$ as $k \to \infty$.*

Proof. (i) Since $\mathfrak{X}$ is primitive, there exists $k_0 \in \mathbf{N}^+$ such that X_k is non-empty for $k \ge k_0$; and since $\mathfrak{X}$ is subadditive and $\mathfrak{X}_{\ge 0} \neq \mathfrak{X} \neq \mathfrak{X}_{\le 0}$ (as we see from the fact that $X_\ell \cap \mathbf{Z}^- \neq \varnothing \neq X_m \cap \mathbf{N}^+$), we have from Lemma 4.1(vi) that $X_n \cap \mathbf{Z}^- \neq \varnothing \neq X_n \cap \mathbf{N}^+$ for some $n \in \mathbf{N}^+$. Pick $a \in X_n \cap \mathbf{Z}^-$ and $b \in X_n \cap \mathbf{N}^+$, and for each $r \in [\![0, n-1]\!]$ let $x_r \in X_{nk_0+r}$. Then fix $k \ge nb^{-1}(\bar{x} + k_0)$, where $\bar{x} := 1 + \max_{0 \le i < n} |x_i|$.

Similarly as in the proof of Lemma 4.1(iii), we have by the division algorithm that $k = n(q + k_0) + r$ for some $q \in \mathbf{N}$ and $r \in [\![0, n-1]\!]$. Thus, we conclude from the subadditivity of $\mathfrak{X}$ that

$$aq + (b-a) \cdot [\![0, q]\!] + x_r \in qX_n + X_{nk_0+r} \subseteq X_{nq} + X_{nk_0+r} \subseteq X_{n(q+k_0)+r} = X_k. \tag{16}$$

On the other hand, it is transparent that $q \ge b^{-1}\bar{x} > b^{-1}|x_r|$, whence

$$0 \le h := \left\lfloor -\frac{aq + x_r}{b-a} \right\rfloor \le \frac{|a|}{b+|a|}q + \frac{|x_r|}{b+|a|} < q.$$

So $h \in [\![0, q-1]\!]$, and we see from (16) that $y := aq + (b-a)h + x_r$ and $y + b - a$ are both in X_k. It follows that $X_k \cap [\![a-b, 0]\!] \neq \varnothing \neq X_k \cap [\![0, b-a]\!]$, for it easily checked that $y \le 0 < y + b - a \le b - a$. This proves the claim with $N := b + |a|$, since k was an arbitrary integer $\ge nb^{-1}(\bar{x} + k_0)$.

(ii) It is evident that $X_k \cap \mathbf{N}^+ \ne \varnothing \ne X_h \cap \mathbf{Z}^-$, for some $h, k \in \mathbf{N}^+$, if and only if $(-X_k) \cap \mathbf{Z}^- \ne \varnothing \ne (-X_h) \cap \mathbf{N}^+$. In the light of Lemma 3.5(i), it is therefore enough to show that $\rho_k \to \infty$ as $k \to \infty$.

To this end, we get from $\mathfrak{X}$ being primitive that there exists $k_0 \in \mathbf{N}^+$ such that $X_k \ne \varnothing$ for $k \ge k_0$. Accordingly, let $k \ge 2nk_0$. By the division algorithm, $k = nq + r$ for some $q \in \mathbf{N}$ and $r \in [\![0, n-1]\!]$, and it is clear that $q \ge 2k_0$. So, using that $\mathfrak{X}$ is subadditive, we obtain from Lemma 2.11(iii) that

$$\rho_k = \rho_{nq+r} \ge \rho_{n(q-k_0)} + \rho_{nk_0+r} \ge (q-k_0)\rho_n + \rho_{nk_0+r} \ge \frac{k - nk_0 - r}{n}\rho_n + \tilde{\rho},$$

where $\tilde{\rho} := \min_{0 \le i < n} \rho_{nk_0+i}$ and we have used that $\rho_n \ge 1$ (since $X_n \cap \mathbf{N}^+ \ne \varnothing$). It follows that $\rho_k \to \infty$ as $k \to \infty$, because $X_{nk_0+i} \ne \varnothing$ for every $i \in [\![0, n-1]\!]$ and hence $\tilde{\rho} \in \mathbf{Z} \cup \{\infty\}$. ■

Theorem 4.3. *Let $\mathfrak{X} = (X_k)_{k \ge 1}$ be a subadditive, primitive sequence of subsets of $\mathbf{Z}$. Then are equivalent:*

(a) *$\mathfrak{X}$ is an* AAPS.

(b) *$\mathfrak{X}_{\ge 0}$ and $\mathfrak{X}_{\le 0}$ are* AAPS*s.*

Proof. (a) ⇒ (b): By hypothesis, there are $d \in \mathbf{N}^+$, $M \in \mathbf{N}$, and $x_1, x_2, \ldots \in \mathbf{Z}$ such that

$$\mathfrak{P}_d(x_k) \cap [\![\lambda_k(\mathfrak{X}) + M, \rho_k(\mathfrak{X}) - M]\!] \subseteq X_k \subseteq \mathfrak{P}_d(x_k), \quad \text{for all but finitely many } k; \tag{17}$$

If $\mathfrak{X} = \mathfrak{X}_{\ge 0}$ (the case when $\mathfrak{X} = \mathfrak{X}_{\le 0}$ is essentially the same), then $X_k \cap \mathbf{Z}_{\le 0} \subseteq \{0\}$ and $X_k \cap \mathbf{N} = X_k$ for every k; so it is straightforward from (17) that each of $\mathfrak{X}_{\ge 0}$ and $\mathfrak{X}_{\le 0}$ is an AAPS. Otherwise, it is clear that $X_m \cap \mathbf{Z}^- \ne \varnothing \ne X_n \cap \mathbf{N}^+$ for some $m, n \in \mathbf{N}^+$; so we derive from Lemma 4.2(ii) that

$$\lambda_k(\mathfrak{X}) + M \le 0 \le \rho_k(\mathfrak{X}) - M, \quad \lambda_k(\mathfrak{X}) = \lambda_k(\mathfrak{X}_{\le 0}), \quad \text{and} \quad \rho_k(\mathfrak{X}) = \rho_k(\mathfrak{X}_{\ge 0}),$$

for all large k. This finishes the proof of the "only if" part, for it implies by (17) that, from some k on,

$$\mathfrak{P}_d(x_k) \cap [\![\lambda_k(\mathfrak{X}_{\le 0}) + M, 0]\!] \subseteq X_k \cap \mathbf{Z}_{\le 0} \subseteq \mathfrak{P}_d(x_k)$$

and

$$\mathfrak{P}_d(x_k) \cap [\![0, \rho_k(\mathfrak{X}_{\ge 0}) - M]\!] \subseteq X_k \cap \mathbf{N} \subseteq \mathfrak{P}_d(x_k).$$

(b) ⇒ (a): If $\mathfrak{X} = \mathfrak{X}_{\ge 0}$ or $\mathfrak{X} = \mathfrak{X}_{\le 0}$, the conclusion is obvious. If not, we have from Lemma 4.1(vi) that

$$\Delta(\mathfrak{X}_{\ge 0}) \ne \varnothing \ne \Delta(\mathfrak{X}_{\le 0}) \quad \text{and} \quad \delta := \delta(\mathfrak{X}) = \delta(\mathfrak{X}_{\ge 0}) = \delta(\mathfrak{X}_{\ge 0}). \tag{18}$$

By Lemma 4.2(i), it follows that there exists $N \in \mathbf{N}^+$ such that, for all large k,

$$\rho_k(\mathfrak{X}_{\ge 0}) = \rho_k(\mathfrak{X}), \quad \lambda_k(\mathfrak{X}) = \lambda_k(\mathfrak{X}_{\le 0}), \quad \text{and} \quad -N \le \rho_k(\mathfrak{X}_{\le 0}) \le 0 \le \lambda_k(\mathfrak{X}_{\ge 0}) \le N. \tag{19}$$

Furthermore, we know from Lemma 4.1(ii) that $\mathfrak{X}_{\ge 0}$ and $\mathfrak{X}_{\le 0}$ are subadditive sequences; so taking into account that they are also AAPSs, we see from (18), (19), and Corollary 3.3 (applied first to $\mathfrak{X}_{\ge 0}$ and then to $\mathfrak{X}_{\le 0}$) that there can be found $k_0 \in \mathbf{N}^+$, $K \in \mathbf{N}_{\ge N}$, and $y_1, y_2, \ldots, z_1, z_2, \ldots \in \mathbf{Z}$ such that, for every $k \ge k_0$,

$$\mathfrak{P}_\delta(y_k) \cap [\![\lambda_k(\mathfrak{X}) + K, -K]\!] \subseteq X_k \cap \mathbf{Z}_{\le 0} \subseteq \mathfrak{P}_\delta(y_k) \tag{20}$$

and

$$\mathfrak{P}_\delta(z_k) \cap [\![K, \rho_k(\mathfrak{X}) - K]\!] \subseteq X_k \cap \mathbf{N} \subseteq \mathfrak{P}_\delta(z_k); \tag{21}$$

in addition, we can guarantee by Lemma 4.2(ii) that the leftmost sides of (20) and (21) are non-empty. In view of Corollary 2.10(i), this in turn entails that, for $k \ge k_0$,

$$X_k \subseteq \mathfrak{P}_\delta(y_k) = \mathfrak{P}_\delta(z_k) \quad \text{and} \quad \mathfrak{P}_\delta(y_k) \cap \big(\llbracket \lambda_k(\mathfrak{X}) + K, -K \rrbracket \cup \llbracket K, \rho_k(\mathfrak{X}) - K \rrbracket\big) \subseteq X_k. \tag{22}$$

Thus, it remains to establish that $\mathfrak{P}_\delta(y_k) \cap \llbracket -K, K \rrbracket \subseteq X_k$ from some k on.

To begin, we have from (18) and Corollary 2.10(ii) that there are $x \in \mathbf{Z}$ and $q \in \mathbf{N}^+$ such that

$$x + \delta \cdot \llbracket -3K, 3K \rrbracket \subseteq X_q; \tag{23}$$

and from Lemma 4.2(ii) that there exists $\kappa_0 \in \mathbf{N}_{\ge k_0}$ such that

$$\lambda_k \le -(2K + |x| + \delta) \quad \text{and} \quad \rho_k \ge 2K + |x| + \delta, \quad \text{for } k \ge \kappa_0. \tag{24}$$

Accordingly, let $k \ge \kappa_0 + q$. We claim that $|\tilde{x} + x| \le K + \delta$ for some $\tilde{x} \in X_{k-q}$. Indeed, it is clear from our definitions that $k - q \ge k_0$. Therefore, we obtain from (22) and (24) that

$$I^- := \mathfrak{P}_\delta(y_k) \cap \big\llbracket \lambda_{k-q}(\mathfrak{X}) + K, -\big(K + |x|\big) \big\rrbracket \quad \text{and} \quad I^+ := \mathfrak{P}_\delta(y_{k-q}) \cap \llbracket K + |x|, \rho_{k-q}(\mathfrak{X}) - K \rrbracket$$

are non-empty subsets of $X_{k-q} \cap \mathbf{Z}^-$ and $X_{k-q} \cap \mathbf{N}$, respectively. Then we take $\tilde{x} := \min I^+$ if $x \le 0$, and $\tilde{x} := \max I^-$ otherwise, with the result that

$$|\tilde{x} + x| = |\tilde{x}| - |x| \le K + |x| + \delta - |x| = K + \delta. \tag{25}$$

With this in hand, it follows from (23) and the subadditivity of $\mathfrak{X}$ that

$$\mathfrak{P}_\delta(\tilde{x} + x) \cap \llbracket -K, K \rrbracket \subseteq \tilde{x} + x + \delta \cdot \llbracket -3K, 3K \rrbracket \subseteq X_{k-q} + X_q \subseteq X_k \tag{26}$$

because $K \ge N \ge 1$ and, hence, it is easy to check that

$$\tilde{x} + x - 3\delta K \overset{(25)}{\le} K + \delta - 3\delta K \le -K \quad \text{and} \quad K \le -(K + \delta) + 3\delta K \overset{(25)}{\le} \tilde{x} + x + 3\delta K.$$

We are left to show that $\mathfrak{P}_\delta(\tilde{x} + x) = \mathfrak{P}_\delta(y_k)$, which is now immediate, since we have from (22) and (26) that $\tilde{x} + x \in X_k \subseteq \mathfrak{P}_\delta(y_k)$. ■

Corollary 4.4. *Let $\mathfrak{X} = (X_k)_{k \ge 1}$ be a subadditive, primitive sequence of subsets of $\mathbf{Z}$. Then are equivalent:*

(a) *$\mathfrak{X}$ is a periodic* AAPS.

(b) *$\mathfrak{X}_{\ge 0}$ and $\mathfrak{X}_{\le 0}$ are periodic* AAPS*s.*

Proof. (a) ⇒ (b): Since $\mathfrak{X}$ is an AAPS (by hypothesis), there exist $d, k_0, \mu \in \mathbf{N}^+$, $M \in \mathbf{N}$, $x_1, x_2, \ldots \in \mathbf{Z}$, and $X'_0, X''_0, \ldots, X'_{\mu-1}, X''_{\mu-1} \subseteq \llbracket 0, M \rrbracket$ such that, for $k \ge k_0$,

$$X_k = \big(\lambda_k(\mathfrak{X}) + X'_{k \bmod \mu}\big) \uplus \mathscr{P}_k \uplus \big(\rho_k(\mathfrak{X}) - X''_{k \bmod \mu}\big) \subseteq \mathfrak{P}_d(x_k), \tag{27}$$

where $\mathscr{P}_k := \mathfrak{P}_d(x_k) \cap \llbracket \lambda_k(\mathfrak{X}) + M, \rho_k(\mathfrak{X}) - M \rrbracket$ is an AP (with difference d).

If $\mathfrak{X} = \mathfrak{X}_{\ge 0}$, then of course $\mathfrak{X}_{\ge 0}$ is also a periodic AAPS. Moreover, $X_k \cap \mathbf{Z}_{\le 0} \subseteq \{0\}$ for every k. So either $X_k \cap \mathbf{Z}_{\le 0} = \varnothing$ for all k; or $p := \wp(\mathfrak{X}_{\le 0})$ is a positive integer and, by Lemma 4.1(ii) and points (iv) and (v) of Lemma 2.12, $X_k = X_{k+p}$ for all large k (more precisely, $X_k = \varnothing$ if $p \nmid k$, and $X_k = \{0\}$ otherwise): In both circumstances, $\mathfrak{X}_{\le 0}$ is an AAPS, too.

The case when $\mathfrak{X} = \mathfrak{X}_{\le 0}$ is essentially the same. Hence we can suppose from now on that $\mathfrak{X}_{\ge 0} \ne \mathfrak{X} \ne \mathfrak{X}_{\le 0}$. Then $X_m \cap \mathbf{Z}^-$ and $X_n \cap \mathbf{N}^+$ are non-empty for some $m, n \in \mathbf{N}^+$, so we get from (ii) and (iii) of

Lemma 4.1 (applied first to $\mathfrak{X}$ and then to $-\mathfrak{X}$) that $\mathfrak{X}_{\ge 0}$ and $\mathfrak{X}_{\le 0}$ are both primitive and subadditive. It follows from Lemma 4.2(ii) that, for all large k,

$$\lambda_k(\mathfrak{X}) = \lambda_k(\mathfrak{X}_{\le 0}), \quad \rho_k(\mathfrak{X}_{\ge 0}) = \rho_k(\mathfrak{X}), \quad \text{and} \quad \lambda_k(\mathfrak{X}) + M \le \rho_k(X_{\le 0}) \le \lambda_k(\mathfrak{X}_{\ge 0}) \le \rho_k(\mathfrak{X}) - M,$$

which, together with (27), implies that

$$X_k \cap \mathbf{N} = \big(\lambda_k(\mathfrak{X}_{\ge 0}) + d \cdot [\![0, (M-1)/d]\!]\big) \uplus \mathscr{P}_k^{\mathsf{up}} \uplus \big(\rho_k(\mathfrak{X}) - X''_{k \bmod \mu}\big) \subseteq \mathfrak{P}_d(x_k) \tag{28}$$

and

$$X_k \cap \mathbf{Z}_{\le 0} = \big(\lambda_k(\mathfrak{X}_{\ge 0}) + X'_{k \bmod \mu}\big) \uplus \mathscr{P}_k^{\mathsf{low}} \uplus \big(\rho_k(\mathfrak{X}_{\le 0}) - d \cdot [\![0, (M-1)/d]\!]\big) \subseteq \mathfrak{P}_d(x_k),$$

where

$$\mathscr{P}_k^{\mathsf{up}} := \mathfrak{P}_d(x_k) \cap [\![\lambda_k(\mathfrak{X}_{\ge 0}) + M, \rho_k(\mathfrak{X}) - M]\!] \quad \text{and} \quad \mathscr{P}_k^{\mathsf{low}} := \mathfrak{P}_d(x_k) \cap [\![\lambda_k(\mathfrak{X}) + M, \rho_k(\mathfrak{X}_{\le 0}) - M]\!].$$

This shows that, again, $\mathfrak{X}_{\le 0}$ and $\mathfrak{X}_{\ge 0}$ are AAPSs, so completing the proof of the "only if" part.

(b) ⇒ (a): Set $\delta := \delta(\mathfrak{X})$. If $\mathfrak{X} = \mathfrak{X}_{\ge 0}$ or $\mathfrak{X} = \mathfrak{X}_{\le 0}$, there is nothing to prove. Otherwise, we have from Theorem 4.3, Lemma 4.1(vi), and Corollary 3.3 that there are $M \in \mathbf{N}$ and $x_1, x_2, \ldots \in \mathbf{Z}$ such that

$$\mathfrak{P}_\delta(x_k) \cap [\![\lambda_k(\mathfrak{X}) + M, \rho_k(\mathfrak{X}) - M]\!] \subseteq X_k \subseteq \mathfrak{P}_\delta(x_k), \quad \text{for all but finitely many } k.$$

Moreover, $\mathfrak{X}_{\ge 0}$ and $\mathfrak{X}_{\le 0}$ being periodic AAPSs implies that there must exist $\mu^+, \mu^- \in \mathbf{N}^+$, $N^+, N^- \in \mathbf{N}$, $x_1^+, x_2^+, \ldots, x_1^-, x_2^-, \ldots \in \mathbf{Z}$, $Y'_0, Y''_0, \ldots, Y'_{\mu-1}, Y''_{\mu-1} \subseteq [\![0, N^+]\!]$, and such that, for all large k,

$$X_k \cap \mathbf{N} = \big(\lambda_k(\mathfrak{X}_{\ge 0}) + Y'_{k \bmod \mu^+}\big) \uplus \mathscr{P}_k^+ \uplus \big(\rho_k(\mathfrak{X}_{\le 0}) - Y''_{k \bmod \mu^+}\big) \subseteq \mathfrak{P}_\delta(x_k^-),$$

and

$$X_k \cap \mathbf{Z}^- = \big(\lambda_k(\mathfrak{X}_{\le 0}) + Y'_{k \bmod \mu^-}\big) \uplus \mathscr{P}_k^- \uplus \big(\rho_k(\mathfrak{X}_{\le 0}) - Y''_{k \bmod \mu^-}\big) \subseteq \mathfrak{P}_\delta(x_k^-),$$

If $\mathfrak{X} = \mathfrak{X}_{\ge 0}$ or

If $\mathfrak{X} = \mathfrak{X}_{\ge 0}$ or $\mathfrak{X} = \mathfrak{X}_{\le 0}$, then the conclusion is trivial. Otherwise, we know from Theorem 4.3 that $\mathfrak{X}$ is an AAPS, namely, there are $d \in \mathbf{N}^+$, $M \in \mathbf{N}$, and $x_1, x_2, \ldots \in \mathbf{Z}$ such that, for all but finitely many k,

$$\mathfrak{P}_d(x_k) \cap [\![\lambda_k(\mathfrak{X}) + M, \rho_k(\mathfrak{X}) - M]\!] \subseteq X_k \subseteq \mathfrak{P}_d(x_k). \tag{29}$$

■

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