

ON SMALL SETS OF INTEGERS

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ABSTRACT. An upper quasi-density on \mathbf{H} (the integers or the non-negative integers) is a subadditive function $\mu^* : \mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R}$ such that $\mu^*(X) \leq \mu^*(\mathbf{H}) = 1$ and $\mu^*(k \cdot X + h) = \frac{1}{k} \mu^*(X)$ for all $X \subseteq \mathbf{H}$ and $h, k \in \mathbf{N}^+$, where $k \cdot X := \{kx : x \in X\}$. Then an upper density (on \mathbf{H}) is a non-decreasing upper quasi-density, and a set $X \subseteq \mathbf{H}$ is uqd-small if $\mu^*(X) = 0$ for every upper quasi-density μ^* on \mathbf{H} .

Main examples of upper densities are given by the upper analytic, upper Banach, upper Buck, and upper Pólya densities, as well as the uncountable family of upper α -densities, where α is a real parameter ≥ -1 (most notably, $\alpha = -1$ corresponds to the upper logarithmic density, and $\alpha = 0$ to the upper asymptotic density).

We show that a subset of \mathbf{H} is uqd-small if and only if it belongs to the zero set of the upper Buck density on \mathbf{Z} . This characterization, combined with a generalization of an inequality of R.C. Buck, allows us to show that many interesting sets are uqd-small, including the integers with no more than k prime factors (counted with multiplicity) for a fixed $k \in \mathbf{N}$, the numbers represented by a binary quadratic form with integer coefficients whose discriminant is not a perfect square, or the image of \mathbf{Z} through a non-linear integral polynomial in one variable.

Overall, we can thus strengthen, extend, and unify several “independent results” previously known for some of the “classical upper densities” studied in the literature.

1. INTRODUCTION

It is not infrequently the case in number theory that we are faced with the problem of determining whether a given set of integers is “small” in a suitable sense, driven by the yoga that “largeness implies structure” (and with the implicit understanding that a set is “small” only if its complement is “large”).

A paradigmatic example is provided by P. Erdős’ conjecture that a set $X \subseteq \mathbf{N}^+$ such that $\sum_{x \in X} \frac{1}{x} = \infty$ must contain arbitrarily long (finite) arithmetic progressions, see e.g. [15, §35.4]. Another example comes from a result of H. Davenport and the same Erdős, according to which, for every set $X \subseteq \mathbf{N}^+$ of positive upper logarithmic density, there exists a (strictly) increasing sequence $(x_n)_{n \geq 1}$ of elements of X such that $x_n \mid x_{n+1}$ for

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each n , see [5, Theorem 2]. (We refer the reader to §1.1 and Remark 2.2 for notation and terminology.)

In both examples, we are not really interested in “quantifying” the largeness of a set X , but only in establishing whether X is “large”. To this end, a classical approach is to let the collection of all “small sets” be an *ideal*, i.e., a family \mathcal{I} of proper subsets of a fixed “universe” \mathcal{U} such that \mathcal{I} is closed under taking subsets and finite unions: E.g., in the case of Erdős’ conjecture, a set is small if it belongs to $\mathcal{S} := \{X \subseteq \mathbf{N}^+ : \sum_{x \in X} \frac{1}{x} < \infty\}$; while in the Davenport-Erdős theorem, a set is small if it belongs to the zero set \mathcal{L} of the logarithmic density on \mathbf{N} . Further, both \mathcal{S} and \mathcal{L} are ideals on \mathbf{N} .

In fact, most natural ideals on \mathbf{N} can be represented as the inverse image of 0 through a “sufficiently nice measure of largeness” $f : \mathcal{P}(\mathbf{N}) \rightarrow \mathbf{R}$, cf. [6, §1]; and perhaps the simplest idea that comes to mind in this regard, is to appeal to the established theory of probability. Such a framework is, however, incompatible with some of our strongest intuitions about numbers, since there is no countably additive probability measure f such that $f(k \cdot \mathbf{N}) = \frac{1}{k}$ for every $k \in \mathbf{N}^+$, see [17, Chapter III.1, Theorem 1]. Added to that is the fact that the existence of finitely additive, shift-invariant probability measures cannot be proved, in the frame of classical mathematics, without (some form of) the axiom of choice, which makes them unwieldy in many situations, see [10, Remark 2.11].

Similar considerations have eventually led to the study of a diversity of (set) functions that, while retaining fundamental features of measures, are better suited than the latter to certain applications. Some of these “surrogate measures”, recently considered by the authors in [10, 11] and called *upper quasi-densities* (Definition 2.1), are also the subject of the present work. More specifically, the plan of the paper is as follows.

In §2, we introduce uqd-small sets (Definition 2.3) and characterize them in terms of the zero set of the upper Buck density (Theorem 2.4). Then, in §3, we prove that each of the following sets is uqd-small: the integers that factor, for a fixed $k \in \mathbf{N}$, into a product of at most k primes (Corollary 3.4); the image of a factorial-like sequence (Corollary 3.5); the perfect powers (Corollary 3.6); the non-negative integers whose base b representation, for a fixed base $b \in \mathbf{Z}_{\geq 2}$, does not contain a given non-empty string of digits (Corollary 3.7); the image of \mathbf{Z} through a non-linear integer polynomial in one variable (Theorem 3.9); and the integers k such that $F(k)$ is prime, where F is a non-constant integral polynomial in one variable (Theorem 3.10). In §4, we generalize to homogeneous quadratic forms the well-known result that the set of (non-negative) integers which can be expressed as a sum of two squares, has asymptotic density zero (Theorem 4.2). Finally, in §5 we make some remarks and pose a few questions.

Overall, we can thus strengthen, extend, and unify several “independent results” previously known for some of the classical upper densities encountered in the literature.

1.1. Generalities. We refer to [10] for most notation, terminology, and conventions used through this paper. In particular, we write \mathbf{R} , \mathbf{Z} , and \mathbf{N} , respectively, for the sets of reals, integers, and non-negative integers; and unless noted otherwise, we reserve the letters h , i , and k (with or without subscripts) for non-negative integers, the letters m and n for positive integers, and the letter s for a real number. Also, we take $\llbracket a, b \rrbracket := [a, b] \cap \mathbf{Z}$ for every $a, b \in \mathbf{R} \cup \{\pm\infty\}$, and define $X^+ := X \cap]0, \infty[$ and $k \cdot X + h := \{kx + h : x \in X\}$ for all $X \subseteq \mathbf{R}$ and $h, k \in \mathbf{R}$. Lastly, we let $\mathbf{P} \subseteq \mathbf{Z}$ be the set of (positive or negative) primes, and for every $k \in \mathbf{Z}$ and $m \in \mathbf{N}^+$ we denote by $k \bmod m$ the smallest non-negative integer r such that $k \equiv r \pmod{m}$.

2. UPPER QUASI-DENSITIES AND UQD-SMALL SETS

Throughout, \mathbf{H} will denote either the integers or the non-negative integers; we regard \mathbf{H} as a sort of parameter allowing for different scenarios and some flexibility.

Definition 2.1. A function $\mu^* : \mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R}$ is an upper density (on \mathbf{H}) if it is

- (F1) normalized, i.e., $\mu^*(\mathbf{H}) = 1$;
- (F2) monotone, i.e., $\mu^*(X) \leq \mu^*(Y)$ for all $X, Y \subseteq \mathbf{H}$ with $X \subseteq Y$;
- (F3) subadditive, i.e., $\mu^*(X \cup Y) \leq \mu^*(X) + \mu^*(Y)$ for every $X, Y \subseteq \mathbf{H}$;
- (F4) (-1) -homogeneous, i.e., $\mu^*(k \cdot X) = \frac{1}{k} \mu^*(X)$ for all $X \subseteq \mathbf{H}$ and $k \in \mathbf{N}^+$;
- (F5) shift-invariant, i.e., $\mu^*(X + h) = \mu^*(X)$ for every $X \subseteq \mathbf{H}$ and $h \in \mathbf{N}$.

In addition, we call μ^* an upper quasi-density (on \mathbf{H}) if $\mu^*(X) \leq 1$ for all $X \subseteq \mathbf{H}$ and μ^* satisfies (F1) and (F3)-(F5).

Every upper density is obviously an upper quasi-density, and the existence of non-monotone upper quasi-densities is guaranteed by [10, Theorem 3.4].

Remark 2.2. One motivation for considering the type of functions introduced in the above is that each of the following is an upper density in the sense of Definition 2.1:

- the upper α -density (on \mathbf{H}), that is, the function

$$\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \limsup_{n \rightarrow \infty} \frac{\sum_{i \in X \cap \llbracket 1, n \rrbracket} i^\alpha}{\sum_{i \in \llbracket 1, n \rrbracket} i^\alpha},$$

where α is a real parameter ≥ -1 (most notably, $\alpha = -1$ corresponds to the upper logarithmic density, and $\alpha = 0$ to the upper asymptotic density);

- the upper Banach (or upper uniform) density, that is, the function

$$\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \lim_{n \rightarrow \infty} \max_{k \geq 0} \frac{|X \cap \llbracket k + 1, k + n \rrbracket|}{n},$$

- the upper analytic density, that is, the function

$$\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \limsup_{s \rightarrow 1^+} \frac{1}{\zeta(s)} \sum_{i \in X^+} \frac{1}{i^s},$$

where ζ is the restriction to the interval $]1, \infty[$ of the Riemann zeta function;

- the *upper Pólya density*, that is, the function

$$\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \lim_{s \rightarrow 1^-} \limsup_{n \rightarrow \infty} \frac{|X \cap [1, n]| - |X \cap [1, ns]|}{(1-s)n};$$

- the *upper Buck density*, that is, the function

$$\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \inf_{A \in \mathcal{A} : X \subseteq A} d^*(A),$$

where d^* is the upper asymptotic density on \mathbf{H} and \mathcal{A} denotes the family of all finite unions of *arithmetic progressions* of \mathbf{H} , that is, sets of the form $k \cdot \mathbf{H} + h$ with $k \in \mathbf{N}^+$ and $h \in \mathbf{N}$.

See [10, §2 and Examples 4.1, 4.4, 4.7, and 6.13] for further details.

As mentioned in the introduction, “densities” are mainly a technical device to formalize the idea that a set is, or is not, “small”. This leads straight to the next definition.

Definition 2.3. *We say that a subset X of \mathbf{H} is uqd-small if $\mu^*(X) = 0$ for every upper quasi-density μ^* on \mathbf{H} .*

Throughout, we will often use the following characterization of uqd-small sets, which is basically a corollary of some of the main results of [10, §§4 and 6].

Theorem 2.4. *A set $X \subseteq \mathbf{H}$ is uqd-small if and only if $\mathfrak{b}^*(X) = 0$, where \mathfrak{b}^* is the upper Buck density on \mathbf{Z} .*

Proof. By [10, Theorem 6.5 and Remark 6.6], \mathfrak{b}^* is the maximum of the family of all upper quasi-densities on \mathbf{H} with respect to the pointwise order induced by the usual ordering of \mathbf{R} on the set of all functions $\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R}$. Therefore, a set $X \subseteq \mathbf{H}$ is uqd-small if and only if $\mathfrak{b}^*(X) = 0$, and this is enough to finish the proof, when considering that the upper Buck density on \mathbf{H} coincides with the upper Buck density on \mathbf{Z} restricted to $\mathcal{P}(\mathbf{H})$, see [10, Example 4.4]. \blacksquare

As a consequence of Theorem 2.4, the property of being uqd-small is independent of the choice of \mathbf{H} . In addition, since the upper Buck density on \mathbf{Z} is monotone and subadditive (as is true for any upper density), we obtain:

Corollary 2.5. *The family of uqd-small subsets of \mathbf{H} is an ideal on \mathbf{H} . In particular, every subset of a uqd-small set is uqd-small.*

Notice that Corollary 2.5 is not obvious a priori, since it is unknown whether the zero set of an upper quasi-density on \mathbf{H} is closed under taking subsets, cf. [10, Question 5].

With this said, we are going to derive an “explicit formula” for (a certain generalization of) the upper Buck density that is perhaps of independent interest in light of Theorem

2.4 and the role played by the upper Buck density in the present work (cf. [14, Theorem 1] for a weaker result along the same lines). The reader may want to review [10, Example 4.4 and, in particular, Proposition 4.6] before reading further.

Proposition 2.6. *Let μ^* be an upper quasi-density on \mathbf{H} and $\mathfrak{b}^*(\mathcal{A}; \mu^*)$ the function*

$$\mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto \inf_{A \in \mathcal{A} : X \subseteq A} \mu^*(A),$$

where \mathcal{A} is family of all sets that can be written as a union of finitely many arithmetic progressions of \mathbf{H} . Moreover, let X be a subset of \mathbf{H} and $(k_n)_{n \geq 1}$ an increasing sequence of positive integers with the property that, for every $m \in \mathbf{N}^+$, k_n is a multiple of m for all but finitely many $n \in \mathbf{N}^+$. Then

$$\mathfrak{b}^*(\mathcal{A}; \mu^*)(X) = \mathfrak{b}^*(X) = \inf_{k \geq 1} \frac{r_k(X)}{k} = \lim_{n \rightarrow \infty} \frac{r_{k_n}(X)}{k_n},$$

where \mathfrak{b}^* is the upper Buck density on \mathbf{Z} , and for each $k \in \mathbf{N}^+$ we denote by $r_k(X)$ the number of residues $h \in \llbracket 0, k-1 \rrbracket$ such that $X \cap (k \cdot \mathbf{H} + h) \neq \emptyset$.

Proof. To begin, let $A \in \mathcal{A}$. By definition, we have that $A = \bigcup_{h \in \mathcal{H}} (k \cdot \mathbf{H} + h)$ for some $k \in \mathbf{N}^+$ and $\mathcal{H} \subseteq \llbracket 0, k-1 \rrbracket$. Accordingly, set

$$\mathcal{H}^\circ := \{h \in \mathcal{H} : X \cap (k \cdot \mathbf{H} + h) \neq \emptyset\} \quad \text{and} \quad A^\circ := \bigcup_{h \in \mathcal{H}^\circ} (k \cdot \mathbf{H} + h).$$

Then $X \subseteq A^\circ \subseteq A$ and $A^\circ \in \mathcal{A}$, and it is clear that $r_k(X) = |\mathcal{H}^\circ|$. Hence, we obtain from [10, Propositions 5.2 and 5.5] that

$$\frac{r_k(X)}{k} = \frac{|\mathcal{H}^\circ|}{k} = \mu^*(A^\circ) \leq \mu^*(A).$$

Conversely, let $k \in \mathbf{N}^+$, and take $A := \bigcup_{h \in \mathcal{H}} (k \cdot \mathbf{H} + h)$, where \mathcal{H} denotes the set of all residues $h \in \llbracket 0, k-1 \rrbracket$ for which $X \cap (k \cdot \mathbf{H} + h)$ is non-empty. Then $A \in \mathcal{A}$ and $X \subseteq A$, and similarly as in the previous paragraph, $r_k(X) = |\mathcal{H}|$ and $\mu^*(A) = r_k(X)/k$.

So putting it all together, we can readily conclude from the above that

$$\mathfrak{b}^*(\mathcal{A}; \mu^*)(X) = \inf_{k \geq 1} \frac{r_k(X)}{k}. \tag{1}$$

On the other hand, it is straightforward that $r_k(X)/k \leq r_h(X)/h$ for all $h, k \in \mathbf{N}^+$ such that $h \mid k$. Therefore, it follows from our assumptions that, for every $k \in \mathbf{N}^+$, the inequality $r_{k_n}(X)/k_n \leq r_k(X)/k$ holds true for all but finitely many $n \in \mathbf{N}^+$; and this implies by (1) that $r_{k_n}(X)/k_n \rightarrow \mathfrak{b}^*(\mathcal{A}; \mu^*)(X)$ as $n \rightarrow \infty$.

The proof is thus complete, because the preceding conclusions are independent from the choice of the upper quasi-density μ^* , and letting μ^* be the upper asymptotic density on \mathbf{H} yields by [10, Example 4.4] that $\mathfrak{b}^*(\mathcal{A}; \mu^*)(X) = \mathfrak{b}^*(X)$. ■

3. CRITERIA FOR UQD-SMALLNESS AND EXAMPLES

Given an upper [quasi-]density μ^* on \mathbf{H} , we follow [10] and refer to the function

$$\mu_* : \mathcal{P}(\mathbf{H}) \rightarrow \mathbf{R} : X \mapsto 1 - \mu^*(\mathbf{H} \setminus X)$$

as the *lower [quasi-]density* on \mathbf{H} conjugate to μ^* , or simply as the *conjugate* of μ^* . We list below some basic properties of upper and lower [quasi-]densities.

Lemma 3.1. *Let μ^* be an upper quasi-density on \mathbf{H} with conjugate μ_* , and let X be a subset of \mathbf{H} . The following hold:*

- (i) $\mu^*(\mathbf{H}) = \mu_*(\mathbf{H}) = [0, 1]$ and $\mu_*(X) \leq \mu^*(X)$.
- (ii) If X is finite, then $\mu^*(X) = 0$.
- (iii) If $k \cdot \mathbf{H} + h \subseteq X$ for some $k \in \mathbf{N}^+$ and $h \in \mathbf{N}$, then $\mu^*(X) \geq \frac{1}{k}$. Symmetrically, if $X \subseteq k \cdot \mathbf{H} + h$ then $\mu^*(X) \leq \frac{1}{k}$.

Proof. See [10, Theorem 5.7, Proposition 2.5(vi), Proposition 5.1, and Corollary 5.6]. \blacksquare

The next result extends a criterion used in [4, §3, p. 563] to demonstrate that the upper Buck density of the set of squares is zero, see the corollary to [4, Theorem 3].

Proposition 3.2. *Let μ^* be an upper quasi-density on \mathbf{H} with conjugate μ_* , and for each $k \in \mathbf{N}^+$ and $S \subseteq \mathbf{H}$ denote by $w_k(S)$ the cardinality of the set*

$$\mathcal{W}_k(S) := \{h \in \llbracket 0, k-1 \rrbracket : \mu^*(S \cap (k \cdot \mathbf{H} + h)) \neq 0\},$$

and by $r_k(S)$ the number of residues $h \in \llbracket 0, k-1 \rrbracket$ such that $X \cap (k \cdot \mathbf{H} + h) \neq \emptyset$.

In addition, let X be a subset of \mathbf{H} and $(k_n)_{n \geq 1}$ a sequence of pairwise coprime positive integers. Then the following hold:

- (i) $w_k(X) \leq r_k(X)$ for all $k \in \mathbf{N}^+$, and for every $n \in \mathbf{N}^+$ we have

$$\prod_{i=1}^n \left(1 - \frac{w_{k_i}(\mathbf{H} \setminus X)}{k_i}\right) \leq \mu_*(X) \leq \mu^*(X) \leq \prod_{i=1}^n \frac{w_{k_i}(X)}{k_i}. \quad (2)$$

- (ii) If $\sum_{n=1}^{\infty} (1 - w_{k_n}(X)/k_n) = \infty$, then $\mu^*(X) = 0$. In particular, if $\sum_{n=1}^{\infty} k_n^{-1} = \infty$ and $w_{k_n}(X) \leq k_n - 1$ for all $n \in \mathbf{N}^+$, then $\mu^*(X) = 0$.
- (iii) If $\sum_{n=1}^{\infty} k_n^{-1} = \infty$ and $r_{k_n}(X) \leq k_n - 1$ for all $n \in \mathbf{N}^+$, or more generally if $\sum_{n=1}^{\infty} (1 - r_{k_n}(X)/k_n) = \infty$, then X is uqd-small.

Proof. (i) The first inequality is obvious. For the other, notice that the function $\mathbf{N}^+ \rightarrow \mathbf{N} : q \mapsto w_q(X)$ is submultiplicative, that is, $w_{mn}(X) \leq w_m(X)w_n(X)$ for all coprime $m, n \in \mathbf{N}^+$: This is so because, for each $m, n \in \mathbf{N}^+$ with $\gcd(m, n) = 1$, the function

$$\mathcal{W}_{mn}(X) \rightarrow \mathcal{W}_m(X) \times \mathcal{W}_n(X) : h \mapsto (h \bmod m, h \bmod n)$$

is injective. Consequently, we get from [10, Proposition 6.4] that, for every $n \in \mathbf{N}^+$,

$$\mu^*(X) \leq \frac{w_{k_1 \cdots k_n}(X)}{k_1 \cdots k_n} \leq \prod_{i=1}^n \frac{w_{k_i}(X)}{k_i}$$

and

$$\mu_*(X) \geq 1 - \frac{w_{k_1 \cdots k_n}(\mathbf{H} \setminus X)}{k_1 \cdots k_n} \geq 1 - \prod_{i=1}^n \frac{w_{k_i}(\mathbf{H} \setminus X)}{k_i} \geq \prod_{i=1}^n \left(1 - \frac{w_{k_i}(\mathbf{H} \setminus X)}{k_i}\right),$$

where we have used that $1 - a_1 \cdots a_n \geq 1 - a_1 \geq (1 - a_1) \cdots (1 - a_n)$ for all $a_1, \dots, a_n \in [0, 1]$. By Lemma 3.1(i), this suffices to finish the proof.

(ii) If $w_{k_n}(X) = 0$ for some $n \in \mathbf{N}^+$, the claim follows at once from (2). Otherwise, $1 \leq w_{k_n}(X) \leq k_n$ for all $n \in \mathbf{N}^+$. So, recalling that

$$\log x \leq -(1 - x), \quad \text{for } x \in]0, 1],$$

we find that, for every $n \in \mathbf{N}^+$,

$$\prod_{i=1}^n \frac{w_{k_i}(X)}{k_i} = \exp\left(\sum_{i=1}^n \log \frac{w_{k_i}(X)}{k_i}\right) \leq \exp\left(-\sum_{i=1}^n \frac{1 - w_{k_i}(X)}{k_i}\right). \quad (3)$$

But the right-most side of (3) tends to 0 as $n \rightarrow \infty$, since we are assuming $\sum_{n=1}^{\infty} (1 - w_{k_n}(X)/k_n) = \infty$; and this, together with part (i), implies $\mu^*(X) = 0$.

The rest is straightforward, because if $w_{k_n}(X) \leq k_n - 1$ for every $n \in \mathbf{N}^+$, then it is clear that $\sum_{n=1}^{\infty} (1 - w_{k_n}(X)/k_n) \geq \sum_{n=1}^{\infty} k_n^{-1} = \infty$.

(iii) This is straightforward from parts (i) and (ii) and the arbitrariness of μ^* . ■

We continue with a common generalization of [13, Corollary 2] and [3, Lemma 2].

Proposition 3.3. *Let $X \subseteq \mathbf{H}$, and let μ^* be an upper quasi-density on \mathbf{H} . Also, assume that $(k_n)_{n \geq 1}$ is a sequence of pairwise coprime positive integers such that $\sum_{n=1}^{\infty} k_n^{-1} = \infty$, and for each $n \in \mathbf{N}^+$ set $X_n := \{x \in X : k_n \mid x\}$. The following hold:*

- (i) *If there are only finitely many $n \in \mathbf{N}^+$ for which $\mu^*(X_n) > 0$, then $\mu^*(X) = 0$. In particular, the set $\{k_n : n \in \mathbf{N}^+\}$ is uqd-small.*
- (ii) *If $\mu^*(X) > 0$, then there exist infinitely many $n \in \mathbf{N}^+$ for which $|X_n| = \infty$.*

Proof. Part (ii) is immediate by the contrapositive of (i) and Lemma 3.1(ii). Therefore, we will restrict our attention to part (i).

By hypothesis, the series $\sum_{n=1}^{\infty} k_n^{-1}$ diverges to ∞ and the set $I := \{0\} \cup \{n \in \mathbf{N}^+ : \mu^*(X_n) > 0\}$ is finite (and non-empty). Hence, $\sum_{n=n_0}^{\infty} k_n^{-1} = \infty$ and $w_{k_n}(X) \leq k_n - 1$ for all $n \geq n_0$, where $n_0 := 1 + \max I$ and $w_{k_n}(X)$ is the number of residues $h \in \llbracket 0, k_n - 1 \rrbracket$ such that $\mu^*(X \cap (k \cdot \mathbf{H} + h)) \neq 0$. By Proposition 3.2(ii), it follows that $\mu^*(X) = 0$.

In particular, if X is the set $\{k_n : n \in \mathbf{N}^+\}$, then it is clear that $X_n = \{k_n\}$ for all $n \in \mathbf{N}^+$, and we conclude from Lemma 3.1(ii) and the above that $\mu^*(X) = 0$. So X is uqd-small, because μ^* was an arbitrary upper quasi-density on \mathbf{H} . ■

We will repeatedly resort to Propositions 3.2 and 3.3 to show that various subsets of \mathbf{H} are uqd-small. We begin with a generalization of [3, Theorems 1–3] and [4, Theorem 1], and a result on the “density” of a factorial-like sequence (cf. Remark 3.8).

Corollary 3.4. *For every $k \in \mathbf{N}$, the set $X^{(k)}$ (respectively, $Y^{(k)}$) of all integers $x \in \mathbf{H}$ that factor into a product of exactly k (respectively, at most k) primes (counted with multiplicity), is a uqd-small subset of \mathbf{H} .*

Proof. Fix $k \in \mathbf{N}$. By Corollary 2.5, it suffices to show that $X^{(k)}$ is a uqd-small set, since we have that $Y^{(k)} = X^{(0)} \cup \dots \cup X^{(k)}$. For, we proceed by induction on k .

In light of Lemma 3.1(ii), the claim is trivial if $k = 0$, because $X^{(0)} = \mathbf{H} \cap \{\pm 1\}$. Accordingly, assume that k is a positive integer and $X^{(k-1)}$ is uqd-small, and let $p \in \mathbf{P}^+$. Then it is clear that

$$X_p^{(k)} := \{x \in X^{(k)} : p \mid x\} = p \cdot X^{(k-1)},$$

and this implies by (F4) and the inductive hypothesis that $\mu^*(X_p^{(k)}) = 0$ for every upper quasi-density μ^* on \mathbf{H} . Therefore, we can conclude from Proposition 3.3(i), applied with k_n equal to the n -th prime of \mathbf{N}^+ , that also $X^{(k)}$ is a uqd-small set, since it is well known (see, e.g., [1, Theorem 1.13]) that $\sum_{p \in \mathbf{P}^+} \frac{1}{p} = \infty$. ■

Corollary 3.5. *Let $(x_n)_{n \geq 1}$ be a sequence in \mathbf{H} with the property that x_n divides x_{n+1} for each n . Then the set $X := \{x_n : n \in \mathbf{N}^+\}$ is uqd-small.*

Proof. By Lemma 3.1(ii), we can assume without loss of generality that the sequence $(x_n)_{n \geq 1}$ consists of pairwise distinct elements and $|x_n| \leq |x_{n+1}|$ for all $n \in \mathbf{N}^+$. In particular, this ensures that $x_1 \neq 0$ and $|x_{2n}| < |x_{2n+2}|$ for every n .

Accordingly, let $r_k(X)$ be, for each $k \in \mathbf{N}^+$, the number of residues $h \in \llbracket 0, k-1 \rrbracket$ for which $X \cap (k \cdot \mathbf{Z} + h)$ is non-empty. Then $r_{|x_{2n}|}(X) \leq 2n$ for every n , since $x_h \mid x_k$ for all $h, k \in \mathbf{N}^+$ such that $h \mid k$. On the other hand, it is easy to verify (by induction) that $|x_{2n}| \geq 2^{n-1}$ for all n . So, we obtain from Proposition 3.2(i) that

$$\mu^*(X) \leq \inf_{n \geq 1} \frac{r_{|x_{2n}|}(X)}{|x_{2n}|} \leq \liminf_{n \rightarrow \infty} \frac{2n}{2^{n-1}} = 0,$$

for every upper quasi-density μ^* on \mathbf{H} . In other terms, X is uqd-small. ■

In particular, it follows from Corollaries 2.5 and 3.5 that a set $X \subseteq \mathbf{H}$, whose elements are factorials, primorials, or numbers of the form a^k for some fixed base $a \in \mathbf{H}$, is uqd-small. This is further strengthened by the next result, which is also a generalization of the unnumbered corollary after Theorem 3 in [4, p. 565].

Corollary 3.6. *The set $X := \bigcup_{n \geq 2} \{a^n : a \in \mathbf{H}\}$ is uqd-small.*

Proof. Let $p \in \mathbf{P}^+$ and pick an element $x \in X$. It is clear that $p \mid x$ if and only if $p^2 \mid x$. Hence, $r_{p^2}(X) \leq p^2 - p + 1 \leq p^2 - \frac{1}{2}p$, where $r_k(X)$ denotes, for every $k \in \mathbf{N}^+$, the number of residues $h \in \llbracket 0, k-1 \rrbracket$ with $X \cap (k \cdot \mathbf{H} + h)$ is non-empty. It follows (cf. Corollary 3.4) that

$$\sum_{p \in \mathbf{P}^+} \left(1 - \frac{r_{p^2}(X)}{p^2}\right) \geq \frac{1}{2} \sum_{p \in \mathbf{P}^+} \frac{1}{p} = \infty.$$

So we can conclude from Proposition 3.2(iii), applied with k_n equal to the square of the n -th prime of \mathbf{N}^+ , that X is uqd-small. \blacksquare

We conclude our series of corollaries with a result on “digit representations”, herein regarded as words in the free monoid over $\llbracket 0, b-1 \rrbracket$ for a fixed base $b \geq 2$.

Corollary 3.7. *Given $b \in \mathbf{N}_{\geq 2}$, let $\mathfrak{s} = (s_1, \dots, s_k)$ be a non-empty sequence of length $k \in \mathbf{N}^+$ with entries in $\llbracket 0, b-1 \rrbracket$. Then the set X of all $x \in \mathbf{H}$ which do not have the word $s_1 \cdots s_k$ appearing in their base b representation, is a uqd-small subset of \mathbf{H} .*

Proof. Let $n \in \mathbf{N}^+$, and denote by $r_{b^{nk}}(X)$ the number of residues $h \in \llbracket 0, b^{nk} - 1 \rrbracket$ such that $X \cap (b^{nk} \cdot \mathbf{H} + h) \neq \emptyset$. Clearly, $r_{b^{nk}}(X)$ is bounded above by the number of residues $h \in \llbracket 0, b^{nk} - 1 \rrbracket$ whose base b representation does not contain the word $s_1 \cdots s_k$, or equivalently by the number of sequences $(a_0, \dots, a_{nk-1}) \in \llbracket 0, b-1 \rrbracket^{nk}$ such that $(a_i, \dots, a_{i+k-1}) \neq \mathfrak{s}$ for every $i \in \llbracket 0, k-1 \rrbracket$. It follows $r_{b^{nk}} \leq (b^k - 1)^n$, and hence $r_{b^{nk}}(X)/b^{nk} \rightarrow 0$ as $n \rightarrow \infty$. So, by Proposition 3.2(i), X is a uqd-small subset of \mathbf{H} . \blacksquare

Remark 3.8. Based on the above, it might be tempting to speculate that every “sufficiently sparse” set of integers is uqd-small. However, this vague idea is contradicted by the fact that the set $\{k! + k : k \in \mathbf{N}\}$ has upper Buck density 1, which is perhaps surprising at first sight, but can be explained by noticing that, roughly speaking, upper and lower densities do not really quantify the “sparsity” of a set, but rather its “distribution through the residue classes modulo n (in the limit as $n \rightarrow \infty$)”.

The next theorems are about integral polynomials in one variable: It would be interesting to extend them to more general classes of integer-valued functions (see §4 for a first step in this direction).

Theorem 3.9. *Let $F : \mathbf{Z} \rightarrow \mathbf{Z}$ be a polynomial function with coefficients in \mathbf{H} . Then $F(\mathbf{H})$ is a uqd-small subset of \mathbf{H} if and only if $\deg F \neq 1$.*

Proof. If F is constant, then its image is uqd-small, by Lemma 3.1(ii). If, on the other hand, F is of degree 1, then there exist $a, b \in \mathbf{H}$ with $a \neq 0$ such that $F(x) = ax + b$ for all $x \in \mathbf{H}$; and this in turn implies that, for every upper quasi-density μ^* on \mathbf{H} ,

$$\mu^*(F(\mathbf{H})) = \mu^*(a \cdot \mathbf{H} + b) = \frac{1}{a} > 0.$$

Accordingly, assume hereafter that $\deg F \geq 2$. Then a well-known theorem of Frobenius (see, e.g., [16, p. 32]) ensures that the set P_F of primes $p \in \mathbf{P}^+$ for which F has at least two roots modulo p , has non-zero *Dirichlet density*, meaning that the limit

$$\lim_{s \rightarrow 1^+} \frac{\sum_{p \in P_F} 1/p^s}{\sum_{p \in \mathbf{P}^+} 1/p^s}$$

exists and is positive. It follows (by the monotone convergence theorem for series) that $\sum_{p \in P_F} 1/p = \infty$, because $\sum_{p \in \mathbf{P}^+} 1/p = \infty$ (cf. Corollary 3.4); and since $r_p(F(\mathbf{Z})) \leq p-1$ for every $p \in P_F$, we conclude from Proposition 3.2(iii) that $F(\mathbf{H})$ is uqd-small. ■

Theorem 3.10. *Let $F : \mathbf{Z} \rightarrow \mathbf{Z}$ be a non-constant polynomial function with integer coefficients. Then the set $X := \{k \in \mathbf{H} : F(k) \in \mathbf{P}\}$ is a uqd-small subset of \mathbf{H} .*

Proof. Let μ^* be an upper quasi-density on \mathbf{H} , and for each $n \in \mathbf{N}^+$ denote by $w_n(X)$ the number of residues $h \in \llbracket 0, k-1 \rrbracket$ such that $\mu^*(X \cap (k \cdot \mathbf{H} + h)) \neq 0$.

Similarly as in the proof of Theorem 3.9, there is a set $P_F \subseteq \mathbf{P}^+$ such that $\sum_{p \in P_F} 1/p = \infty$ and F has at least one zero modulo p for every $p \in P_F$, that is, $p \mid F(h_p)$ for some $h_p \in \llbracket 0, p-1 \rrbracket$. In particular, it follows that, for each $p \in P_F$, the set $X \cap (p \cdot \mathbf{H} + h_p)$ is finite (otherwise, $|F(pk + h_p)| = p$ for infinitely many $k \in \mathbf{H}$, in contradiction to the fact that F is non-constant), and hence, by Lemma 3.1(ii), $w_p(X) \leq p-1$.

So, putting it all together, we get from Proposition 3.2(i) that $\mu^*(X) = 0$, and this is enough to show that X is a uqd-small subset of \mathbf{H} (since μ^* was arbitrary). ■

4. BINARY QUADRATIC FORMS

It is folklore that the asymptotic density of the set of integers that can be written as a sum of two squares is zero. In the present section, we generalize this to binary quadratic forms, while replacing the asymptotic density with an arbitrary upper quasi-density.

Lemma 4.1. *Let d be an integer, but not a perfect square. Then there exist $m \in \mathbf{N}^+$ and $r \in \mathbf{N}$ with $\gcd(m, r) = 1$ such that, for every prime $p \in \mathbf{P}^+$ with $p \equiv r \pmod{m}$, d is not a quadratic residue modulo p .*

Proof. Write $d = 2^k t^2 u \varepsilon$, where t and u are odd positive integers, u is squarefree, k is a non-negative integer, and ε is the sign of d (i.e., $\varepsilon = 1$ if $d \geq 1$, and $\varepsilon = -1$ otherwise).

If $u = 1$, then it is sufficient to consider that, for every odd prime $p \in \mathbf{P}^+$, we have from [1, Theorems 9.5 and 9.9(d)] that

$$\left(\frac{d}{p}\right) = \left(\frac{2^k}{p}\right) \left(\frac{\varepsilon}{p}\right) = (-1)^{\frac{1}{8}k(p^2-1)} \varepsilon^{\frac{1}{2}(p-1)},$$

where (\cdot) is a Jacobi symbol, see [1, §9.7]; and hence we can take $p \equiv 5 \pmod{8}$ if k is odd, and $p \equiv 3 \pmod{4}$ otherwise (note that, in the latter case, ε must be equal to -1 , or else d would be a perfect square).

Thus, assume from now on that $u \geq 3$. Then $u = q_1 \cdots q_n$, where $q_1, \dots, q_n \in \mathbf{P}^+$ are pairwise distinct odd prime numbers; and it follows by [1, Theorems 5.26 and 9.1] that there exists $r \in \mathbf{N}$ with the property that

$$\left(\frac{r}{q_1}\right) = -1 \quad \text{and} \quad \left(\frac{r}{q_i}\right) = 1 \quad \text{for } i \in \llbracket 2, n \rrbracket.$$

Therefore, if we let $s \in \mathbf{N}$ be such that $8s + 1 \equiv r \pmod{u}$ (which is possible, since u is odd), then we have by [1, Theorems 9.9(c) and 9.9(b)] that

$$\left(\frac{8s+1}{u}\right) = \left(\frac{r}{u}\right) = \prod_{i=1}^n \left(\frac{r}{q_i}\right) = -1, \tag{4}$$

and this implies in particular that $8s + 1$ and u are coprime.

Consequently, if $p \equiv 8s + 1 \pmod{8u}$, then $\gcd(2u, p) = 1$ and $p \equiv 1 \pmod{8}$; and we get from [1, Theorems 9.5, 9.9(a), 9.9(d), 9.10 and 9.11] that

$$\left(\frac{d}{p}\right) = \left(\frac{2^k}{p}\right) \left(\frac{\varepsilon}{p}\right) \left(\frac{u}{p}\right) = (-1)^{\frac{1}{8}k(p^2-1) + \frac{1}{4}(p-1)(u-1)} \varepsilon^{\frac{1}{2}(p-1)} \left(\frac{p}{u}\right) = \left(\frac{8s+1}{u}\right),$$

which, together with (4), yields $\left(\frac{d}{p}\right) = -1$ and completes the proof. ■

Theorem 4.2. *Let μ^* be an upper quasi-density on \mathbf{H} , and set $X := \{ax^2 + bxy + cy^2 : x, y, z \in \mathbf{H}\}$ and $D := b^2 - ac$, where $a, b, c \in \mathbf{H}$ are fixed. The following hold:*

- (i) *If D is not a perfect square or $D = 0$, then X is uqd-small.*
- (ii) *If D is a non-zero perfect square and ($ac = 0$ or $\mathbf{H} = \mathbf{Z}$), then $\mu^*(X) > 0$.*

Proof. Let $w_k(\cdot)$ and $r_k(\cdot)$ be defined as in Proposition 3.2. We have several cases.

CASE 1: D is not a perfect square (and hence $ac \neq 0$). In light of Lemma 4.1, there are $m \in \mathbf{N}^+$ and $r \in \mathbf{N}$ such that, for every $p \in \mathbf{P}^+$ with $p \equiv r \pmod{m}$, D is not a quadratic residue modulo p . Accordingly, let P be the set of all primes $p \equiv r \pmod{m}$ such that $p \geq 2 + \max(|a|, |b|, |c|) \geq 3$.

If $p \in P$, $z \in \mathbf{Z}$, and $p \nmid z$, then $ax^2 + bxy + cy^2 = pz$ for all $x, y \in \mathbf{Z}$: Otherwise, we have that $(2ax + by)^2 \equiv Dy^2 \pmod{p}$, which is only possible if $p \mid x$ or $p \mid y$ (since D is not a quadratic residue modulo p); consequently, p^2 divides z , in contradiction to the assumption that z is not a multiple of p .

Thus, $w_{p^2}(X) \leq r_{p^2}(X) \leq p^2 - p + 1 \leq p^2 - \frac{1}{2}p$ for every $p \in P$, which implies, by Theorem 3.2(ii), that $\mu^*(X) = 0$, since we have from Dirichlet's theorem on primes in arithmetic progressions (see, e.g., [12, Corollary 4.12(c)]) that

$$\sum_{p \in P} \left(1 - \frac{w_{p^2}(X)}{p^2}\right) \geq \frac{1}{2} \sum_{p \in P} \frac{1}{p} = \infty.$$

CASE 2: $D = ac = 0$. Note that $b = 0$, and assume by symmetry that $c = 0$. It follows that $X = \{ax^2 : x \in \mathbf{H}\}$, and Corollary 3.6 yields $\mu^*(X) = 0$.

CASE 3: D is a non-zero perfect square and $ac = 0$. We have $|b| = \sqrt{D} > 0$, and it is immediate that $X \supseteq |b| \cdot \mathbf{H} + a + c$. Hence $\mu^*(X) \geq |b|^{-1} > 0$, by Lemma 3.1(iii).

CASE 4: $D = q^2$ for some $q \in \mathbf{N}$ and $ac \neq 0$. Let ε be the sign of a , and observe that $b - q \neq 0$ and, by axiom (F4), $\mu^*(X) = 4|a| \mu^*(4|a| \cdot X)$. Next, notice that

$$4|a| \cdot X = \{(2ax + (b - q)y)(2ax + (b + q)y)\varepsilon : x, y \in \mathbf{H}\}. \quad (5)$$

Building on these premises, we distinguish two subcases.

Case 4.1: $q = 0$. It is clear from (5) that $4|a| \cdot X \subseteq \{x^2\varepsilon : x \in \mathbf{H}\}$, and we derive from Corollaries 2.5 and 3.6 that $\mu^*(X) = 4|a| \mu^*(4|a| \cdot X) = 0$.

Case 4.2: $q \neq 0$ and $\mathbf{H} = \mathbf{Z}$. Let ε_0 be the sign of $a(b - q)$, and denote by Y the set

$$\{(-2a(b - q)z + 2a(b - q)(z + 1))(-2a(b - q)z + 2a(b + q)(z + 1))\varepsilon : z \in \mathbf{Z}\}.$$

By (5), Y is a subset of $4|a| \cdot X$, and we see that

$$\begin{aligned} Y &= 4a^2 |b - q| \cdot \{2q\varepsilon_0 z + (b + q)\varepsilon_0 : z \in \mathbf{Z}\} \\ &= 8a^2 q |b - q| \cdot \mathbf{Z} + 4a^2 \varepsilon_0 |b^2 - q^2|. \end{aligned}$$

In other words, $4|a| \cdot X$ contains an arithmetic progression of \mathbf{Z} , which, along with Lemma 3.1(iii) and the above, implies $\mu^*(X) = 4|a| \mu^*(4|a| \cdot X) > 0$. \blacksquare

Remark 4.3. Set $X := \{ax^2 + bxy + cy^2 : x, y \in \mathbf{N}\}$, where $a, b, c \in \mathbf{N}$, $ac \neq 0$, and $b^2 - 4ac = q^2$ for some $q \in \mathbf{N}^+$. This case is not covered by Theorem 4.2, and it turns out that $\mu^*(X)$ is zero for some choices of the upper density μ^* and positive for others.

Indeed, let \mathbf{d}^* and \mathbf{b}^* be, respectively, the upper asymptotic and upper Buck densities on \mathbf{N} ; we want to prove that $\mathbf{d}^*(X) = 0 \neq \mathbf{b}^*(X)$. For, it is easy to check that

$$4a \cdot X = \{(2ax + (b - q)y)^2 + 2qy : x, y \in \mathbf{N}\}.$$

Consequently, we have that $2a(b - q) \cdot A \subseteq 4a \cdot X \subseteq B$, where

$$A := \{2a(b - q)(u + v)^2 + 4aqv : u, v \in \mathbf{N}\} \subseteq \mathbf{N}$$

and

$$B := \{xy : x, y \in \mathbf{N} \text{ and } x \leq y \leq (2q + 1)x\} \subseteq \mathbf{N}.$$

Since \mathbf{d}^* and \mathbf{b}^* are upper densities (and hence satisfy (F2) and (F4)), it follows that

$$\mathbf{b}^*(X) = 4a \mathbf{b}^*(4a \cdot X) \geq 2(b - q)^{-1} \mathbf{b}^*(A) \quad \text{and} \quad \mathbf{d}^*(X) \leq \mathbf{d}^*(B).$$

Thus, it suffices to show that $\mathbf{b}^*(A) \neq 0 = \mathbf{d}^*(B)$. To this end, fix $k \in \mathbf{N}^+$, and let $r_k(\cdot)$ be defined as in Proposition 3.2. It is clear that

$$\{2a(b - q)(ku + 1)^2 + 4aq((k - 1)u + 1) : u \in \mathbf{N}\} \subseteq A,$$

which means that $r_k(A)$ is larger than or equal to the number of residues $h \in \llbracket 0, k-1 \rrbracket$ such that $2a(b-q) + 4aq - 4aqu \equiv h \pmod k$ for some $u \in \mathbf{N}$. Then $r_k(A) \geq \lfloor (4aq)^{-1}k \rfloor$, and we conclude from Proposition 2.6 that $\mathfrak{b}^*(A) \geq (4aq)^{-1} > 0$.

As for the rest, let $n \in \mathbf{N}^+$ and pick $z \in B \cap \llbracket 1, n \rrbracket$. By construction, $z = xy$ for some $x, y \in \mathbf{N}^+$ with $x \leq y \leq (2q+1)x$. Therefore, we have $y^2 \leq (2q+1)xy \leq (2q+1)n$, and hence $x \leq y \leq \sqrt{(2q+1)n}$. In other words, $B \cap \llbracket 1, n \rrbracket$ is a subset of the multiplication table for positive integers $\leq \sqrt{(2q+1)n}$. So, we get from a classical result of Erdős [7, Part 3] that $|B \cap \llbracket 1, n \rrbracket| = o(n)$ as $n \rightarrow \infty$, which yields $\mathfrak{d}^*(B) = 0$.

5. CLOSING REMARKS AND OPEN QUESTIONS

We conclude the paper with a couple of open questions naturally stemming from the results of §§2 and 4 that we have not been able to settle.

Question 5.1. By Corollary 2.5 and Lemma 3.1(ii), the family \mathcal{S} of all uqd-small subsets of \mathbf{H} is an ideal containing all finite subsets of \mathbf{H} ; and by Theorem 2.4, it coincides with the intersection of \mathbf{H} and the zero set of the upper Buck density on \mathbf{Z} . One may wonder if \mathcal{S} is also “closed under products”, meaning that

$$XY := \{xy : (x, y) \in X \times Y\} \in \mathcal{S}, \quad \text{for all } X, Y \in \mathcal{S}.$$

The answer is no. Indeed, let μ^* be an upper quasi-density on \mathbf{H} , and let A (respectively, B) be the set of all $x \in \mathbf{H}$ whose positive prime divisors are all equal to 1 (respectively, 3) modulo 4. Then $\{x \in A : p \mid x\}$ is empty for every $p \in \mathbf{P}^+$ with $p \equiv 3 \pmod 4$; and in a similar way, $\{y \in B : q \mid y\}$ is empty for every $q \in \mathbf{P}^+$ with $q \equiv 1 \pmod 4$ (observe that $0 \notin A \cup B$, because $k \mid 0$ for all $k \in \mathbf{N}$). Therefore, we get from Dirichlet’s theorem on primes in arithmetic progressions, Lemma 3.1(ii), and Proposition 3.3(i) that both A and B are uqd-small. But $AB = 2 \cdot \mathbf{H} + 1$ (note that $\mathbf{H} \cap \{\pm 1\} \subseteq A \cap B$), and we have by (F4) that $\mu^*(2 \cdot \mathbf{H} + 1) = \frac{1}{2}$.

Likewise, \mathcal{S} is not “closed under sums” either, in the sense that there is $X \in \mathcal{S}$ such that $X + X := \{x + y : x, y \in X\} \notin \mathcal{S}$. In fact, let \mathfrak{b}^* be the upper Buck density on \mathbf{H} . By Theorem 4.2, $\mathcal{Q} := \{x^2 + y^2 : x, y \in \mathbf{H}\} \subseteq \mathbf{H}$ is uqd-small. But $\mathcal{Q} + \mathcal{Q} = \mathbf{N}$ (by Lagrange’s four square theorem), and hence $\mathfrak{b}^*(\mathcal{Q} + \mathcal{Q}) = 1$.

So, we may ask whether, for a given $\alpha \in [0, 1]$, there exists $X \in \mathcal{S}$ such that $\mu_*(XX) = \mu^*(XX) = \alpha$ (respectively, $\mu_*(X + X) = \mu^*(X + X) = \alpha$) for every upper quasi-density μ^* on \mathbf{H} , where μ_* is the conjugate of μ^* (as defined in the first lines of §3).

Questions in the same vein have been recently addressed by various authors in the special case of the upper asymptotic density, see [8, Theorem 1.3 and Question Q4], [9, §§2 and 4], and [2, Theorems 1.1.a), 1.4, and 1.9.a) and c)].

Question 5.2. Let $k \in \mathbf{N}$. By Corollary 3.4, the set $X^{(k)}$ of all $x \in \mathbf{H}$ that factor into a product of at most k primes (counted with multiplicity), is uqd-small. Does the same

hold true for the inverse image of $\llbracket 0, k \rrbracket$ under the function $\omega : \mathbf{H} \rightarrow \mathbf{N} \cup \{\infty\}$ that maps 0 to ∞ and a non-zero integer $x \in \mathbf{H}$ to the number of primes $p \in \mathbf{P}^+$ such that $p \mid x$? If yes, this would be a stronger result than Corollary 3.4, because every subset of a uqd-small set is uqd-small (Corollary 2.5) and it is clear that $X^{(k)} \subseteq \omega^{-1}(\llbracket 0, k \rrbracket)$.

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