DIRAC PARTICLES IN MAGNETIC FIELDS

B. THALLER
University of Graz
Institute of Mathematics
Heinrichstraße 36
A-8010 Graz
Austria

ABSTRACT. We give a review of spectral and scattering theory for spin-1/2 particles in an external magnetic field. The supersymmetric point of view is strongly emphasized. Recent results on Foldy-Wouthuysen transformations, properties of the resolvent, threshold eigenvalues and scattering theory are presented.

1. Magnetic fields

In any space dimension $\nu \geq 2$ the magnetic field strength B is given by a 2-form

$$B(x) = \sum_{\substack{i,k=1 \\ i<k}}^{\nu} F_{ik}(x) dx_i \wedge dx_k \tag{1}$$

satisfying $dB = 0$ (exterior derivative). We can write $B = dA$ with a magnetic "vector" potential (1-form) A and obtain

$$F_{ik}(x) = \frac{\partial A_i(x)}{\partial x_k} - \frac{\partial A_k(x)}{\partial x_i}, \qquad A(x) = \sum_{i=1}^{\nu} A_i(x) dx_i. \tag{2}$$

We want to stress that the vector potential is not directly observable. Therefore, one should formulate all results under assumptions on the field strengths. Eventually, we shall use the gauge freedom (see below) to make the description as simple as possible. Throughout this lecture we assume that each component of B is a smooth function in $\mathcal{C}^\infty(\mathbb{R}^\nu)$. We refer to the cited literature for possible generalizations (e.g., [1]). We are only interested in results for dimensions $\nu = 2, 3$. For $\nu = 3$ we identify B and A with vector fields $\mathbf{B}(\mathbf{x}) = \big(F_{23}(\mathbf{x}), F_{31}(\mathbf{x}), F_{12}(\mathbf{x})\big)$ and $\mathbf{A}(\mathbf{x}) =$

$\big(A_1(\mathbf{x}), A_2(\mathbf{x}), A_3(\mathbf{x})\big)$ satisfying $\mathbf{B} = \mathrm{rot}\mathbf{A}$, i.e., $\mathrm{div}\mathbf{B} = 0$. In two dimensions, $B = \partial_1 A_2 - \partial_2 A_1$ is simply a scalar field. This corresponds to the three dimensional situation $\mathbf{B}(\mathbf{x}) = \big(0, 0, B(x_1, x_2)\big)$.

2. Dirac and Pauli operators

We recall the Dirac operator with a magnetic field,

$$H(A) \equiv \begin{cases} c\sum_{i=1}^{3} \alpha_i(p_i - A_i) + \beta mc^2 & \text{if } \nu = 3, \\ \\ c\sum_{i=1}^{2} \sigma_i(p_i - A_i) + \sigma_3 mc^2 & \text{if } \nu = 2, \end{cases} \tag{3}$$

where $p_i = -i\partial/\partial x_i$, and where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and β are the famous 4×4 "Dirac matrices", which in the standard representation are defined by

$$\beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix}, \quad i = 1, 2, 3. \tag{4}$$

Here σ_i denote the 2×2 "Pauli matrices"

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

The Dirac operator (3) is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^3)^4$, resp. $\mathcal{C}_0^\infty(\mathbb{R}^2)^2$, even without restriction on the growth of B or A at infinity [2].

In the standard representation $H(A)$ has the abstract "supersymmetric" form

$$H(A) = \begin{pmatrix} mc^2 & cD^* \\ cD & -mc^2 \end{pmatrix} = cQ + mc^2\tau, \tag{6}$$

where

$$Q = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7}$$

and

$$D \equiv \begin{cases} \sum_{i=1}^{3} \sigma_i(p_i - A_i) & \text{if } \nu = 3, \\ \\ (p_1 - A_1) + i(p_2 - A_2) & \text{if } \nu = 2, \end{cases}. \tag{8}$$

Note that $D \neq D^*$ in two dimensions. The Pauli operator

$$H_P(A) \equiv \frac{1}{2m} D^* D = \begin{cases} H_S(\mathbf{A})\mathbf{1} - (1/2m)\vec{\sigma}\cdot\mathbf{B}, & \text{if } \nu = 3, \\ \\ H_S(A) - (1/2m)B, & \text{if } \nu = 2 \end{cases} \tag{9}$$

can be obtained as the nonrelativistic limit of $H(A)$, see Refs. [3], [4], [5]. Here, $H_S(A) = \frac{1}{2m}(-i\nabla - A)^2$ is the nonrelativistic Schrödinger operator for a spinless particle. See Chapter 6 of Ref. [6] for a review of its properties and references.

3. Vector potentials and gauge freedom

The 1-form A is not uniquely determined by (1). Replacing A by $A + dg$, where $g \in \mathcal{C}^\infty(\mathbb{R}^\nu)$ (see [7] for more general g's), does not change the magnetic field strength and leads to an equivalent mathematical description of this physical system ("gauge invariance"). Indeed, if $A^{(2)} = A^{(1)} + \nabla g$, then the unitary transformation $\Psi(\mathbf{x}) \to e^{ig(\mathbf{x})}\Psi(\mathbf{x})$ transforms $H(A^{(1)})$ into $H(A^{(2)})$

$$e^{ig}\, H(A^{(1)})\, e^{-ig} = H(A^{(2)}) \tag{10}$$

(similar for H_P and H_S).

The use of the vector potential in quantum mechanics is sometimes counterintuitive, because the vector potential is usually nonzero in regions where the magnetic field strength vanishes. Consider, e.g., a magnetic field $B(x)$ in two dimensions with compact support and nonvanishing flux $\int B(x) d^2x$. Clearly in this case we expect the particle to move freely once it has left the support of B. But using Stokes law $\oint A ds = \int B d^2x$ we see that $A(x)$ cannot decay faster than $|x|^{-1}$, as $|x| \to \infty$. Hence the vector potential keeps influencing the wavefunction of the particle also at large distances from the support of B. There have been attempts to formulate quantum mechanics entirely in terms of B [8].

4.Supersymmetry

Now we turn to the proof of some spectral properties of $H(A)$. Any operator H of the form (6) will be called a "Dirac operator with supersymmetry".

Theorem. (Thaller, [9]). *Let $H = cQ + mc^2\tau$ be a Dirac operator with supersymmetry. Then there is a unitary transformation U, which brings H to the diagonal form*

$$UHU^{-1} = \begin{pmatrix} \sqrt{c^2D^*D + m^2c^4} & 0 \\ 0 & -\sqrt{c^2DD^* + m^2c^4} \end{pmatrix} = \tau|H|. \tag{11}$$

The unitary transformation is given by

$$U = a_+ + \tau(\mathrm{sgn}Q)a_-, \quad a_\pm = \frac{1}{\sqrt{2}}\sqrt{1 \pm \frac{mc^2}{|H|}}. \tag{12}$$

Proof: For simplicity, we set $c = 1$. It is easy to verify the following formulas for the bounded operators $a_\pm$

$$a_+^2 + a_-^2 = 1, \quad a_+^2 - a_-^2 = m|H|^{-1}, \quad 2a_+a_- = |Q|\,|H|^{-1}. \tag{13}$$

Furthermore we note that $|H| = (Q^2 + m^2)^{1/2}$ commutes with τ and Q, and the following commutation relations hold on $\mathcal{D}(H) = \mathcal{D}(Q)$

$$[H, a_\pm] = [Q, a_\pm], \quad H\tau\,(\mathrm{sgn\,Q}) = -\tau\,(\mathrm{sgn\,Q})\,\mathrm{H}. \tag{14}$$

Now we can verify Eq. (14) in the following way

$$\begin{aligned} UHU^{-1} &= (a_+ + \tau\,(\mathrm{sgn\,Q})\,\mathrm{a}_-)\mathrm{H}(\mathrm{a}_+ - \tau\,(\mathrm{sgn\,Q})\,\mathrm{a}_-) = \\ &= (a_+^2 + 2\tau\,(\mathrm{sgn\,Q})\,\mathrm{a}_+\mathrm{a}_- - \mathrm{a}_-^1)\mathrm{H} \\ &= (m|H|^{-1} + \tau\,(\mathrm{sgn\,Q})\,|\mathrm{Q}|\,|\mathrm{H}|^{-1})\mathrm{H} \\ &= (m+\tau Q)|H|^{-1}H = \tau(m\tau + Q)|H|^{-1}H \\ &= \tau H^2|H|^{-1} = \tau|H| \end{aligned} \tag{15}$$

The matrix form of $\tau|H|$ immediately follows from (12), (13) and $|H| = \sqrt{H^2}$. *QED.*

A generalization of this result has been obtained recently in Ref. [5].

Corollary. *The spectrum of $H(A)$ is symmetric with respect to 0 except possibly at $\pm mc^2$. The open interval $(-mc^2, +mc^2)$ does not belong to the spectrum.*

Proof: We use (11) and the fact that $\sigma(D^*D) = \sigma(DD^*)$ except at 0. The Theorem now follows immediately from the spectral mapping theorem. *QED.*

We have $H(\mathbf{A})\Psi = mc^2\Psi$ if and only if $\Psi = \binom{\psi_1}{0}$ with $D^*D\psi_1 = 0$, or equivalently $D\psi_1 = 0$. On the other hand we see that $H(\mathbf{A})\Psi = -mc^2\Psi$ if and only if $\Psi = \binom{0}{\psi_2}$ with $DD^*\psi_2 = 0$, or equivalently $D^*\psi_2 = 0$. Hence in three dimensions, where $D^* = D$, the spectrum is always symmetric, even at 0. The situation is completely different in two dimensions, as we learn from the following theorem.

6. Eigenvectors belonging to $\pm mc^2$

Theorem. (Aharonov-Casher, [10]). *In two dimensions, let $B(x)$ be a magnetic field with compact support, and denote*

$$F = \frac{1}{2\pi}\int_{\mathbb{R}^2} B(x)d^2x \tag{16}$$

a) If $F = n + \epsilon$ (where n is a positive integer, and $0 \le \epsilon < 1$), then $+mc^2$ (but not $-mc^2$) is an eigenvalue of the Dirac operator $H(A)$ defined in (3).
b) If $F = -n - \epsilon$, then $-mc^2$ (but not $+mc^2$) is an eigenvalue of $H(A)$.
In both cases the multiplicity of the eigenvalue is n, if $\epsilon > 0$, and $n-1$, if $\epsilon = 0$.

Proof: The Green function of $\triangle$ in two dimensions is $\frac{1}{2\pi}\ln|x|$, therefore

$$\phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|x-y|B(y)d^2y \tag{17}$$

satisfies $\triangle\phi(x) = B(x)$, and

$$\phi(x) - F\ln|x| = O\big(\frac{1}{|x|}\big), \quad \text{as } |x| \to \infty. \tag{18}$$

We choose the vector potential $A = (-\partial_2\phi, \partial_1\phi)$, and look for a solution of

$$c\vec{\sigma}\cdot(p-A)\psi = 0, \quad \vec{\sigma} = (\sigma_1, \sigma_2) \tag{19}$$

Writing

$$\omega = e^{\sigma_3\phi}\psi \tag{20}$$

we find that (19) is equivalent to

$$\vec{\sigma}\cdot p\omega = 0 \quad \text{or} \quad \begin{array}{l} \left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)\omega_1(x) = 0, \\ \left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)\omega_2(x) = 0. \end{array} \tag{21}$$

These equations are equivalent to the Cauchy Riemann equations. Hence ω_1 (resp. ω_2) has to be an entire analytic function in the variable $z = x_1 + ix_2$ (resp. $\overline{z} = x_1 - ix_2$). For large $|z| = |x|$ these functions behave as

$$\omega_1(x) \approx e^{+F\ln|x|}\psi_1(x) = |x|^{+F}\psi_1(x), \tag{22}$$

$$\omega_2(x) \approx e^{-F\ln|x|}\psi_2(x) = |x|^{-F}\psi_2(x). \tag{23}$$

If $F > 0$ then ω_2 is square integrabel at infinity and hence zero, because an analytic function cannot vanish in all directions, as $|z| \to \infty$. This shows that $\psi_2 = 0$ and therefore only $+mc^2$ can be an eigenvalue of $H(A)$. But for this we have to fulfill the condition

$$\psi_1 = e^{-\phi}\omega_1 \in L^2(\mathbb{R}^2) \tag{24}$$

which requires that ω_1 should not increase faster than $|x|^{F-1-\delta}$, for some $\delta > 0$. Since ω_1 is an entire function, it must be a polynomial in $x_1 + ix_2$ of degree $\leq n-1$ (resp. $n-2$, if $\epsilon = 0$). Hence there are n linearly independent solutions ψ_1 of $D\psi_1 = 0$, namely (for $\epsilon \neq 0$)

$$e^{-\phi},\ e^{-\phi}(x_1+ix_2),\ e^{-\phi}(x_1+ix_2)^2,\ \ldots\ ,\ e^{-\phi}(x_1+ix_2)^{n-1}. \tag{25}$$

An analogous reasoning applies to the case $F < 0$. *QED.*

The proof of the Aharonov-Casher Theorem shows that if one can find a solution ϕ of $\triangle\phi(x) = B(x)$, such that $e^{-\phi}$ (or $e^{+\phi}$) is a rapidly decreasing function in $\mathcal{S}(\mathbb{R}^3)$, then the eigenvalue $+mc^2$ (or $-mc^2$) is infinitely degenerate and hence in $\sigma_{ess}(H(A))$ (see also Ref. [11]). This is indeed the case, e.g., for a homogeneous magnetic field.

In case of a spherically symmetric magnetic field in two dimensions ($B(x) = B(r)$, $r = |x|$) a solution of $\triangle\phi(r) = \big(\partial^2/\partial r^2 + (1/r)\partial/\partial r\big)\phi(r) = B(r)$ is given by

$$\phi(r) = \int_0^r ds\frac{1}{s}\int_0^s dtB(t)t \tag{26}$$

Hence for a magnetic field with infinite flux like

$$B(r) \approx r^{\delta-2}, \quad \text{for some } \delta > 0,\ r \text{ large}, \tag{27}$$

we find

$$\phi(r) \approx \frac{1}{\delta} r^{\delta}, \quad \text{for } r \text{ large}, \tag{28}$$

i.e., $e^{-\phi}$ decreases faster than any polynomial in $|x|$. Therefore $+mc^2$ is infinitely degenerate in this case. If even $B(r) \to \infty$, as $r \to \infty$, then the next Theorem shows, that $+mc^2$ is the only possible element in the essential spectrum of the Dirac operator.

Theorem. (Helffer-Nourrigat-Wang, [2]). *If in two dimensions $B(x) \to \infty$ (resp. $B(x) \to -\infty$), as $|x| \to \infty$, then λ with $\lambda \neq +mc^2$ (resp. $\lambda \neq -mc^2$) is not in the essential spectrum of the Dirac operator.*

Proof: We assume $B(x) \to +\infty$, the other case can be treated analogously. We show that λ with $\lambda \neq mc^2$ is not in $\sigma_{ess}(H(A))$, because for all $\Psi = \binom{\psi_1}{\psi_2} \in \mathcal{C}_0^\infty(\mathbb{R}^2)^2$ with support outside a ball with sufficiently large radius R there is a constant $C(\lambda)$ such that

$$\big\| \big(H(A) - \lambda\big)\Psi \big\| \geq C(\lambda) \|\Psi\| \tag{29}$$

In order to prove this, we choose R so large, that

$$B(x) \geq \frac{1}{c^2} |\lambda - mc^2| (3 + 2|\lambda + mc^2|), \quad \text{for all } |x| \geq R. \tag{30}$$

Denoting $\Phi = \binom{\phi_1}{\phi_2} = \big(H(A) - \lambda\big)\Psi$, i.e.,

$$\begin{aligned} \phi_1 &= cD^*\psi_2 - (\lambda - mc^2)\psi_1 \\ \phi_2 &= cD\psi_1 - (\lambda + mc^2)\psi_2 \end{aligned} \tag{31}$$

we find

$$\begin{aligned} \|D^*\psi_2\|^2 = (\psi_2, D^*D\psi_2) &= \|(p-A)^2\psi_2\|^2 + (\psi_2, B(x)\psi_2) \\ &\geq \frac{1}{c^2}|\lambda - mc^2|(3 + 2|\lambda + mc^2|)\|\psi_2\|^2, \end{aligned} \tag{32}$$

provided suppψ_2 is outside the ball with radius R. Hence

$$\begin{aligned} (3 + 2|\lambda + mc^2|)\|\psi_2\|^2 &\leq \frac{1}{|\lambda - mc^2|}\|\phi_1 + (\lambda - mc^2)\psi_1\|^2 \\ &= \frac{1}{|\lambda - mc^2|}\|\phi_1\|^2 + 2(\text{sgn}\lambda)\text{Re}(\phi_1, \psi_1) + |\lambda - mc^2|\|\psi_1\|^2. \end{aligned} \tag{33}$$

Since

$$\begin{aligned}(\text{sgn}\lambda)\text{Re}(\phi_1, \psi_1) &= (\text{sgn}\lambda)\text{Re}(cD^*\psi_2, \psi_1) - |\lambda - mc^2| \|\psi_1\|^2 \\ &\leq \|\psi_2\| \|cD\psi_1\| - |\lambda - mc^2| \|\psi_1\|^2 \\ &\leq \|\psi_2\| \|\phi_2\| + |\lambda + mc^2| \|\psi_2\|^2 - |\lambda - mc^2| \|\psi_1\|^2 \end{aligned} \tag{34}$$

we find

$$\begin{aligned}&3\|\psi_2\|^2 + |\lambda - mc^2| \|\psi_1\|^2 \\ &\leq \frac{1}{|\lambda - mc^2|} \|\phi_1\|^2 + 2\|\psi_2\| \|\phi_2\| \end{aligned} \tag{35}$$

Now we have either $\|\psi_2\| \leq \|\phi_2\|$ or $\|\psi_2\| \geq \|\phi_2\|$. In each case

$$\|\Psi\|^2 = \|\psi_1\|^2 + \|\psi_2\|^2 \leq 2 \max \left\{ 1, \frac{1}{(\lambda - mc^2)^2} \right\} \|\Phi\|^2, \tag{36}$$

which proves the Theorem. *QED.*

7. Three space dimensions

There is no analogue of the Aharonov-Casher result for $\nu = 3$. Also the theorem of Helffer, Nourrigat, and Wang is very specific to two dimensions. Concerning eigenvalues at $\pm mc^2$ in three dimensions, so far only some examples are known.

Example. (Loss-Yau, [12]). If, for some real valued λ, we had a solution of

$$\vec{\sigma} \cdot \mathbf{p}\Psi(\mathbf{x}) = \lambda(\mathbf{x})\Psi(\mathbf{x}), \tag{37}$$

which satisfies

$$\langle \Psi, \Psi \rangle(\mathbf{x}) \equiv \sum_{i=1}^{2} \overline{\psi_i(\mathbf{x})} \psi_i(\mathbf{x}) \neq 0, \tag{38}$$

then we can find a solution of $\vec{\sigma} \cdot (\mathbf{p} - \mathbf{A})\Psi = 0$. First note that

$$\langle \Psi, \vec{\sigma}\Psi \rangle \langle \Psi, \vec{\sigma}\Psi \rangle = \langle \Psi, \Psi \rangle^2, \tag{39}$$

implies

$$\vec{\sigma} \cdot \frac{\langle \Psi, \vec{\sigma}\Psi \rangle}{\langle \Psi, \Psi \rangle} \Psi = \Psi, \tag{40}$$

and hence

$$\vec{\sigma} \cdot \mathbf{A}(\mathbf{x})\Psi(\mathbf{x}) = \lambda(\mathbf{x})\Psi(\mathbf{x}), \tag{41}$$

if we choose

$$\mathbf{A}(\mathbf{x}) = \lambda(\mathbf{x}) \frac{\langle \Psi, \vec{\sigma}\Psi \rangle}{\langle \Psi, \Psi \rangle}. \tag{42}$$

But a solution of (37) is easy to find. Choose, for example,

$$\Psi(\mathbf{x}) = \frac{1 + i\vec{\sigma}\cdot\mathbf{x}}{(1+x^2)^{3/2}}\Phi_0, \tag{43}$$

where $\Phi_0 \in \mathbb{C}^2$, with $\langle\Phi_0, \Phi_0\rangle = 1$. Note that

$$0 \neq \langle\Psi, \vec{\sigma}\Psi\rangle(\mathbf{x}) = \frac{1}{(1+x^2)^3}\{(1-x^2)\mathbf{w} + 2(\mathbf{w}\cdot\mathbf{x})\mathbf{x} + 2\mathbf{w}\wedge\mathbf{x}\}, \tag{44}$$

where $w = \langle\Phi_0, \vec{\sigma}\Phi_0\rangle$ is a unit vector in $\mathbb{R}^3$. We obtain

$$\vec{\sigma}\cdot\mathbf{p}\Psi(\mathbf{x}) = \frac{3}{1+x^2}\Psi(\mathbf{x}), \tag{45}$$

and finally

$$\mathbf{A}(\mathbf{x}) = 3(1+x^2)\langle\Psi, \vec{\sigma}\Psi\rangle, \qquad \mathbf{B}(\mathbf{x}) = 12\langle\Psi, \vec{\sigma}\Psi\rangle. \tag{46}$$

The vector field A can be obtained by stereographic projection from a parallel basis vector field on the three dimensional sphere. Hence the flow lines are circles on the Hopf tori.

A characterisation of the essential spectrum in two or three dimensions is given by the next Theorem. This result is not so typical for Dirac operators, because similar statements are true for the nonrelativistic Schrödinger operator without spin (see Leinfelder [6], Miller and Simon [13], [14]).

Theorem. (Leinfelder-Miller-Simon). *In two or three dimensions, if* $|B(x)| \to 0$, *as* $|x| \to \infty$, *then*

$$\sigma_{ess}(H(A)) = (-\infty, -mc^2] \cup [mc^2, \infty). \tag{47}$$

If $B(x)$ *is bounded, then the distance from an arbitrary* $\lambda \notin (-mc^2, mc^2)$ *to* $\sigma_{ess}(H(A))$ *is less than* $2\sqrt{12}\sup\sqrt{|B(x)|}$.

Proof: It is sufficient to consider the essential spectrum of the operator

$$D_\nu = c\sum_{i=1}^{\nu}\sigma_i(p_i - A_i) \equiv c\vec{\sigma}\cdot(p - A) \tag{48}$$

in dimensions $\nu = 2, 3$, because $H(A)$ is unitarily equivalent to

$$\sigma_3\sqrt{(D_2)^2 + m^2c^4} \quad \text{for } \nu = 2, \tag{49}$$

and

$$\begin{pmatrix} \sqrt{(D_3)^2 + m^2c^4} & 0 \\ 0 & -\sqrt{(D_3)^2 + m^2c^4} \end{pmatrix} \quad \text{for } \nu = 3. \tag{50}$$

In order to prove $k \in \sigma_{ess}(D_\nu)$ it is sufficient to find an orthonormal sequence of vectors $\Psi^{(n)}$ in the domain of D_ν, such that

$$\lim_{n\to\infty} \|(D_\nu - k)\Psi^{(n)}\| = 0 \tag{51}$$

(Weyl's criterion). Moreover, the distance between k and $\sigma_{ess}(D_\nu)$ is less than d, if for a suitable orthonormal sequence $\Psi^{(n)}$

$$\|(D_\nu - k)\Psi^{(n)}\| \le d. \tag{52}$$

We are going to construct suitable vectors $\Psi^{(n)}$ as follows. Let $B_n = B_{\rho_n}(x^{(n)})$ be a sequence of disjoint balls with centers $x^{(n)}$ and radii ρ_n. Any two L^2-functions with support in different balls are orthogonal. We use the gauge freedom to define within these balls vector potentials $A^{(n)}$ which are determined by the local properties of B in that region (unlike the original A-field). For each n we define

$$A^{(n)}(x) = \int_0^1 B\big(x^{(n)} + (x - x^{(n)})s\big) \wedge (x - x^{(n)})s ds, \tag{53}$$

or, written in components

$$A_i^{(n)}(x) = \int_0^1 \sum_{i=1}^{\nu} F_{ki}\big(x^{(n)} + (x - x^{(n)})s\big)(x_k - x_k^{(n)})s ds, \quad i = 1, \ldots, \nu. \tag{54}$$

It is easy to see that

$$\sup_{x\in B_n} |A^{(n)}(x)| \le \rho_n \sup_{x\in B_n} |B(x)|. \tag{55}$$

Furthermore, if A is the vector potential we started with, then

$$A - A^{(n)} = \nabla g^{(n)}, \quad \text{with } g^{(n)} \in \mathcal{C}^\infty(\mathrm{I\!R}^\nu). \tag{56}$$

Finally, we choose

$$\Psi^{(n)}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\rho_n^{\nu/2}} \; j\left(\frac{x - x^{(n)}}{\rho_n}\right) \exp\big(ig^{(n)}(x) - ikx_1\big), \tag{57}$$

where j is a localization function with the following properties:

$$j \in \mathcal{C}_0^\infty(\mathrm{I\!R}^\nu), \quad \mathrm{supp} j \subset \{x \mid |x| \le 1\}, \quad \int |j(x)|^2 d^\nu x = 1. \tag{58}$$

It is easily verified that $\text{supp}\Psi^{(n)} \subset B_n$, and $\|\Psi^{(n)}\| = 1$. A little calculation gives for all $k \in \mathbb{R}$

$$\begin{aligned}
&\{\vec{\sigma}\cdot\big(p - A(x)\big) - k\}\Psi^{(n)}(x) \\
&= -i\frac{1}{\rho_n}\vec{\sigma}\cdot(\nabla j)\left(\frac{x - x^{(n)}}{\rho_n}\right)\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\rho_n^{\nu/2}}\exp\big(ig^{(n)}(x) - ikx_1\big) \\
&- \vec{\sigma}\cdot A^{(n)}(x)\Psi^{(n)}(x).
\end{aligned} \tag{59}$$

Using (55) we obtain the estimate

$$\|\{\vec{\sigma}\cdot\big(p - A\big) - k\}\Psi^{(n)}\| \le \frac{1}{\rho_n}\int |\nabla j(x)|^2 d^\nu x + \rho_n \sup_{x\in B_n}|B(x)| \tag{60}$$

Assume first, that $|B(x)| \le M$ for all x. Choosing $\rho_n = \rho$, all n, we find that (60) is bounded, i.e., $\sigma_{ess}(D_\nu)$ is not empty. Moreover, if we choose $j(x) = \text{const.}\cos^2(\pi|x|/2)$, then it is easy to see that $\int|\nabla j(x)|^2 d^3x \approx 11.62$. Setting $\rho = (12/M)^{1/2}$ we obtain the bound $2(12M)^{1/2}$ for (60). Hence the distance of an arbitrary $k \in \mathbb{R}$ to $\sigma_{ess}(D_\nu)$ is less than this constant. The distance from an arbitrary $\lambda \notin (-mc^2, mc^2)$ to the next point in $\sigma_{ess}(H(A))$ is bounded by the same constant.

If $|B(x)| \to 0$, as $|x| \to \infty$, we can proceed as follows. There is a sequence of disjoint balls B_n with increasing radius ρ_n, such that

$$\sup_{x\in B_n}|B(x)| \le \frac{1}{\rho_n^2} \tag{61}$$

But then (60) is bounded by constant$/\rho_n \to 0$, as $n \to \infty$. Hence any $k \in \mathbb{R}^\nu$ is in the essential spectrum of D_ν. *QED.*

In case of electric or scalar potentials which decay at infinity, the essential spectrum mainly consists of a continuous spectrum associated to scattering states. This is not necessarily the case for magnetic fields. This can be seen most clearly by looking at spherically symmetric examples.

8. Cylindrical symmetry

In two dimensions, if the magnetic field strength is cylindrically symmetric we can pass to coordinates $r = |x|$, $\phi = \arctan(x_2/x_1)$. We denote the coordinate unit vectors by $\mathrm{e}_r = \frac{1}{r}(x_1, x_2)$, $\mathrm{e}_\phi = \frac{1}{r}(-x_2, x_1)$, write $B(x) \equiv B(r)$, and choose $A(x) = A_\phi(r)\mathrm{e}_\phi$, where

$$A_\phi(r) = \frac{1}{r}\int_0^r B(s)s\,ds, \quad B(r) = \left(\frac{d}{dr} + \frac{1}{r}\right)A_\phi(r) = \frac{1}{r}\frac{d}{dr}\big(A_\phi(r)r\big). \tag{62}$$

In this notation the flux of B is given by $F = \lim_{r\to\infty}\big(A_\phi(r)r\big)$. The Dirac operator in cylindrical coordinates can be written as

$$
\begin{aligned}
H(A) &\equiv c\vec{\sigma}\cdot(p-A) + \sigma_3 mc^2 \\
&= c(\vec{\sigma}\cdot \mathrm{e}_r)\,\mathrm{e}_r\cdot(p-A) + c(\vec{\sigma}\cdot\mathrm{e}_\phi)\,\mathrm{e}_\phi\cdot(p-A) + \sigma_3 mc^2 \\
&= c(\vec{\sigma}\cdot\mathrm{e}_r)\left\{-i\left(\frac{\partial}{\partial r}+\frac{1}{2r}\right) + i\frac{1}{r}\sigma_3 J_3 - i\sigma_3 A_\phi(r)\right\} + \sigma_3 mc^2. \qquad (63)
\end{aligned}
$$

The angular momentum operator $J_3 = L_3 + \sigma_3/2$ commutes with $H(A)$ and the spinors

$$
\chi_{m_j} = \begin{pmatrix} a\, e^{i(m_j-1/2)\phi} \\ b\, e^{i(m_j+1/2)\phi} \end{pmatrix}, \quad a,b\in\mathbb{C}, \quad m_j = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \dots \qquad (64)
$$

form a complete set of orthogonal eigenvectors of J_3 in $L^2(S^1)^2$ with the properties

$$
J_3\chi_{m_j} = m_j\chi_{m_j}, \qquad (65)
$$

$$
(\vec{\sigma}\cdot\mathrm{e}_r)\chi_{m_j} = \begin{pmatrix} b\, e^{i(m_j-1/2)\phi} \\ a\, e^{i(m_j+1/2)\phi} \end{pmatrix}. \qquad (66)
$$

Any function $\Psi(r,\phi)$ in $L^2(\mathbb{R}^2)^2$ can be written as a sum

$$
\Psi(r,\phi) = \sum_{m_j} \begin{pmatrix} \dfrac{1}{\sqrt{r}} f_{m_j}(r) e^{i(m_j-1/2)\phi} \\ -i\dfrac{1}{\sqrt{r}} g_{m_j}(r) e^{i(m_j+1/2)\phi} \end{pmatrix}, \qquad (67)
$$

with suitable functions f_{m_j} and g_{m_j} in $L^2([0,\infty), dr)$. The action of $H(A)$ on Ψ can be described on each angular momentum subspace as the action of a "radial Dirac operator" h_{m_j} defined in $L^2([0,\infty), dr)^2$

$$
h_{m_j}\begin{pmatrix} f_{m_j} \\ g_{m_j} \end{pmatrix} = \begin{pmatrix} mc^2 & cD^* \\ cD & -mc^2 \end{pmatrix}\begin{pmatrix} f_{m_j} \\ g_{m_j} \end{pmatrix} \qquad (68)
$$

with

$$
D = \frac{d}{dr} - \frac{m_j}{r} + A_\phi(r), \qquad (69)
$$

and $H(A)$ is unitarily equivalent to a direct sum of the operators h_{m_j}. A little calculation shows

$$
\left.\begin{matrix} D^*D \\ DD^* \end{matrix}\right\} = -\frac{d^2}{dr^2} + \frac{\left(m_j \mp \frac{1}{2}\right)^2 - \frac{1}{4}}{r^2} - 2\frac{m_j\mp\frac{1}{2}}{r}A_\phi(r) + A_\phi^2(r) \mp B(r). \qquad (70)
$$

From (62) we see that if $B(r)r \to \infty$ then also $A_\phi(r)\to\infty$, as $r\to\infty$. In this case the term A_ϕ^2 dominates in (70) the interaction at large values of r and clearly the Schrödinger operator D^*D (resp. DD^*) has a pure point spectrum. By (11) the same is true for the Dirac operator on each angular momentum subspace and hence for $H(A)$. Let us summarize these observations in the following Theorem.

Theorem. (Miller-Simon [13], [14]). *In two dimensions, if B is spherically symmetric and $B(r)r \to \infty$, as $r \to \infty$, then the Dirac operator $H(A)$ has a pure point spectrum.*

In addition, if $B(r) \to 0$, then $B(r)r \to \infty$ implies that there is a complete set of orthonormal eigenvectors of $H(A)$ belonging to eigenvalues which are dense in the the union of the intervals $(-\infty, -mc^2]$ and $[mc^2, \infty)$.

9. A review of further results

It is immediately clear from the proof of the Theorem of Leinfelder-Miller-Simon that the condition $|B(x)| \to 0$ can be weakened considerably. It is sufficient to require that there is a sequence of balls with increasing radius on which B tends to zero. These balls can be widely separated and it does not matter how B behaves elsewhere [7,12]. It is even sufficient to require a similar behaviour of the derivatives of B [2]. More, precisely, one defines functions

$$\epsilon_r(x) = \frac{\sum_{|\alpha|=r} |D^\alpha B|}{1 + \sum_{|\alpha|<r} |D^\alpha B|}, \quad \text{if } r \geq 1, \text{ and } \epsilon_0(x) = |B(x)|. \tag{71}$$

and introduces the assumption

(A_r): There exist a sequence of disjoint balls B_n of radii r_n with $r_n \to \infty$ such that the function $\epsilon_r(x)$ restricted to the union of these balls tends to zero at infinity.

It can be seen that if (A_r) holds for some $r \geq 2$, then one of the assumptions (A_0) or (A_1) is true [2]. If the components of B are polynomials of degree r, then (A_{r+1}) holds. The following Theorem applies to this case.

Theorem. (Helffer-Nourrigat-Wang, [2]). *In three dimensions, if (A_r) holds for some $r \geq 0$, then $\sigma_{ess}(H(A)) = (-\infty, -mc^2] \cup [mc^2, \infty)$.*

This result is remarkable, because magnetic fields increasing like polynomials are known to yield Schrödinger operators with compact resolvent [15]. It is clear from Section 6 this result is very specific to three dimensions. Since the essential spectrum may contain embedded eigenvalues it is interesting to know criteria for the absence of eigenvalues as in the next Theorem.

Theorem. (Kalf-Berthier-Georgescu, [1,16]). *If $B(x) \to 0$ and $x \wedge B(x) \to 0$, as $|x| \to \infty$, then the Dirac operator has no eigenvalues λ with $|\lambda| > mc^2$.*

10. Scattering Theory

One of the basic problems in scattering theory is proving asymptotic completeness of the wave operators

$$\Omega_\pm(H, H_0) \equiv \operatorname*{s-lim}_{t\to\pm\infty} e^{iHt} e^{-iH_0 t}\, P_{cont}(H_0), \tag{72}$$

Supersymmetry implies a relation between the wave operators $\Omega_\pm(H_P(A), H_P(0))$ of the nonrelativistic theory and the relativistic wave operators $\Omega_\pm(H(A), H(0))$. Unfortunately, the Dirac operator is not an "admissible" function of the Pauli operator, therefore we cannot apply directly the invariance principle in order to conclude existence of wave operators in the relativistic case from the existence of the nonrelativistic wave operators. Nevertheless we have the following theorem, where F denotes the projection operator to the subspace belonging to the indicated region of the spectrum of the self-adjoint operator.

Theorem. (Thaller, [9, 17]). *Let $H = Q + m\tau$, $H_0 = Q_0 + m\tau$ be two Dirac operators with supersymmetry. Assume that for all $0 < a < b < \infty$ and for Ψ in some dense subset of $F(a < Q_0^2 < b)\mathcal{H}_{a.c.}(Q_0^2)$ the following condition is satisfied with $k = 1{,}2$*

$$\|(Q^k - Q_0^k)\, e^{-iQ_0^2 t}\, \Psi\| \leq \text{const.}(1 + |\text{t}|)^{1-\text{k}-\delta}. \tag{73$_k$}$$

Then the wave operators $\Omega_\pm(Q^2, Q_0^2)$ and $\Omega_\pm(H, H_0)$ exist, and

$$\Omega_\pm(H, H_0) = \Omega_\pm(Q^2, Q_0^2)\, F(H_0 > 0) + \Omega_\mp(Q^2, Q_0^2)\, F(H_0 < 0) \tag{74}$$

A proof of this theorem is given in Ref. [17]. Now we apply it to the case of magnetic fields. Note that $Q^2 = H_P(A)$ is just the nonrelativistic Pauli operator. Assume that the magnetic field strength B decays at infinity, such that we have, for some $\delta > 0$,

$$B(x) \leq \text{const.}(1 + |\text{x}|)^{-3/2-\delta} \tag{75}$$

Choose the transversal (or Poincaré) gauge

$$A(x) = \int_0^1 s\, B(xs) \wedge x\, ds. \tag{76}$$

A is uniquely characterized by $A(x) \cdot x = 0$, and $A(x)$ decays like $|x|^{-1/2-\delta}$, as $|x| \to \infty$. Hence the expressions div$A$, A^2, $\vec{\Sigma} \cdot B$ occurring in $Q^2 - Q_0^2$ are all of short-range. The remaining long-range term is $A(x) \cdot p$. It can be written as $A(x) \cdot p = G(x) \cdot (x \wedge p)$, where

$$G(x) = \int_0^1 s\, B(xs)\, ds \tag{77}$$

satisfies,

$$|G(x)| \leq \text{const.}(1 + |\text{x}|)^{-3/2-\delta)} \tag{78}$$

and since the angular momentum $L = x \wedge p$ remains constant under the nonrelativistic free time evolution $\exp(-iQ_0^2 t)$, we easily obtain by a stationary phase argument for Ψ in a suitable dense set (see also [22])

$$\|A(x) \cdot p\, e^{-ip^2 t}\, \Psi\| \leq \text{const.}(1 + |\text{t}|)^{-3/2-\delta}. \tag{79}$$

The condition $(73)_1$ simply becomes

$$\|A(x)\, e^{-ip^2 t}\, \Psi\| \leq \text{const.}(1+|\text{t}|)^{-\delta} \tag{80}$$

and is trivially satisfied. Hence we have proven existence of the nonrelativistic and relativistic wave operators. But also asymptotic completeness is true.

Theorem. (Loss-Thaller, [18]). *Let $H(A)$ and $H(0)$ be given as in (2) and assume that the magnetic field strength B satisfies*

$$D^\gamma B(x) \leq \text{const.}(1+|\text{x}|)^{-3/2-\delta-\gamma}, \tag{81}$$

for some $\delta > 0$ and multiindices γ with $|\gamma| = 0, 1, 2$. Then the nonrelativistic and relativistic wave operators in the transversal gauge are asymptotically complete.

Existence of wave operators is usually expected to hold for "short-range potentials", where (each component of) the potential matrix V satisfies

$$|V(x)| \leq \text{const}(1+|\text{t}|)^{-1-\delta}. \tag{82}$$

A famous counter example is the electrostatic Coulomb potential, where $|V(x)|$ decays like $|x|^{-1}$. In this case the wave operators do not exist [19] and one has to introduce modifications of the asymptotic time evolution. For the magnetic fields in the theorem above the potential matrix $-\vec{\alpha}\cdot A$ has a much slower decay. Indeed, previous results in the literature have been obtained only by introducing modifications of the wave operators (see, e.g., Ref. [20] and the references therein). Asymptotic completeness is due to the transversality of A. In another gauge A is not transversal and if ∇g is long-range, then the unmodified wave operators (72) would not exist. Instead, asymptotic completeness holds for $\Omega_\pm(H(A), H(\nabla g))$. These remarks might be of importance, because physicists use almost exclusively the Coulomb gauge instead of the transversal gauge, which is best adapted to scattering theory. Note that although the wave and scattering operators depend on the choosen gauge, the physically observable quantities like scattering cross sections are gauge independent.

In situations like the Aharonov Bohm effect one has used the free asymptotics (e.g., plane waves for the asymptotic description of stationary scattering states), together with the Coulomb gauge, although the vector potential is long-range. But in this case the calculations are justified, because in two dimensions and for rotationally symmetric fields the Coulomb gauge coincides with the transversal gauge (see also Ref. [21], for a discussion).

Under weaker decay conditions on the magnetic field strength the wave operators would not exist in that form, because then the term A^2 occurring in $Q^2 - Q_0^2$ would become long-range. In this case one really needs modified wave operators, similar to the Coulomb case.

The scattering problem is nontrivial even in classical mechanics. From special examples we know that classical paths of particles in magnetic fields satisfying our requirements do not have asymptotes. It is easy to see that the velocity of the particles is asymptotically constant. But if we compare the asymptotic motion of a particle in a magnetic field with a free motion, one would have to add a correction which is transversal to the asymptotic velocity and which increases for $\delta < 1/2$ like $|t|^{1/2-\delta}$. Thus the situation seems to be worse than in the Coulomb problem. There the interacting particles also cannot be asymptotically approximated by free particles, but at least the classical paths do have asymptotes. (The correction in the Coulomb problem increases like $\ln|t|$ and is parallel to the asymptotic velocity). A discussion of these effects in the classical scattering theory with magnetic fields is given in Ref. [22].

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