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OPTIMAL HEAT KERNEL ESTIMATES FOR SCHRÖDINGER OPERATORS WITH MAGNETIC FIELDS IN TWO DIMENSIONS

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Abstract

Sharp smoothing estimates are proven for magnetic Schrödinger semigroups in two dimensions under the assumption that the magnetic field is bounded below by some positive constant B_0. As a consequence the L^∞ norm of the associated integral kernel is bounded by the L^∞ norm of the Mehler kernel of the Schrödinger semigroup with the constant magnetic field B_0.

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1. INTRODUCTION

A recurring problem in the theory of semigroups is to prove smoothing estimates, i.e., estimates of the semigroup acting as an operator from $L^p \to L^q$, $q \geq p$. (see [S82] for a general review on semigroups) In rare cases it is possible to render these estimates in their sharp form. Among the examples are Nelson's hypercontractive estimate [N66] and the sharp smoothing properties of the heat kernel (which follow from the sharp form of Young's inequality [B75], [BL76]). A more recent example is furnished by the Mehler kernel associated with the Schrödinger operator of a charged particle in a constant magnetic field which essentially is a result of Lieb [L90] although not explicitly stated there (see Theorem 1.1).

An especially flexible approach to such problems, due to Gross [G76] (see also Federbush [F69]) and relevant for our paper, is the use of logarithmic Sobolev inequalities. In fact there is complete equivalence between Gross' logarithmic Sobolev inequality and Nelson's hypercontractive estimate. In a similar fashion, Weissler [W78] proved the equivalence of the sharp smoothing estimates for the heat semigroup and logarithmic Sobolev inequalities.

On a more abstract level it was realized in Simon and Davies [DS84] (see also [D89]) that the technique of logarithmic Sobolev inequalities can be used to prove ultracontractivity for Markov semigroups.

It is the aim of this note to implement Gross' method for magnetic Schrödinger operators and to prove sharp smoothing estimates for the associated semigroup. The idea is similar to the one in [CL96] where it was realized that Gross' logarithmic Sobolev inequality when viewed as a family of sharp inequalities on $\mathbb{R}^n$ can be used to obtain sharp smoothing estimates of solutions of diffusion equations with a volume preserving drift. An example is the two dimensional Navier–Stokes equations in the vorticity formulation.

Consider a single charged quantum particle in a magnetic field, i.e., consider the Hamiltonian

$$H(B) = \frac{1}{2}(\nabla + \mathrm{i}A)^2 \ , \tag{1.1}$$

where A is a vector potential for the magnetic field B, i.e., $\mathrm{curl}\, A = B$.

Note that the semigroup generated by (1.1) is not Markovian, in fact not even positivity preserving. Usually, this difficulty is overcome by the important diamagnetic inequality ([K72], [S77,79], [HSU77], see also [AHS77] and [CFKS87])

$$\big|\big(e^{tH}u\big)(x)\big| \le \big(e^{t\Delta/2}|u|\big)(x) \ , \tag{1.2}$$

which relates estimates on magnetic Schrödinger operator semigroup to estimates on the heat semigroup.

The obvious disadvantage of using (1.2) is that all effects due to the magnetic field are completely eliminated. How, then, does the magnetic field affect the behavior of the semigroup? The first paper that addressed this question is presumably [M86]. It is proved in [M86] that for a magnetic Schrödinger operator in $\mathbb{R}^3$

$$\lim_{t\to\infty} \frac{1}{t} \ln \|e^{tH(B)}\|_{L^1\to L^\infty} \le -\Phi(C)B_0 \ , \tag{1.3}$$

for all curl-free magnetic fields B that satisfy $0 < B_0 \le |B| \le CB_0$, Here $|B| = \sqrt{B_1^2+B_2^2+B_3^2}$ with B_1, B_2, B_3 being the components and the function Φ is explicitly given in [M86]. It is noteworthy that the estimates were obtained by probabilistic techniques.

This work was subsequently improved in [E94]. The estimate (1.3) holds in any dimension and without the condition that B be curl-free. In particular the following stronger estimate is true

$$\lim_{t\to\infty} \frac{1}{t} \ln \|e^{tH(LB)}\|_{L^1\to L^\infty} \leq -C_L \min_{x\in\mathbb{R}^n} |B(x)|L \ .$$

Here the constant $C_L = 1 + O(L^\alpha \ln L)$ as $L \to \infty$ for some explicitly given $\alpha < 0$. Thus, this result is in a certain sense optimal since by considering a constant magnetic field in two dimensions one cannot improve the *exponent* $\min_{x\in\mathbb{R}^n} |B(x)|$. Again the results in [E94] were obtained by probabilistic techniques. Similar results can be found in [U94] where techniques from differential geometry were used.

In this paper we consider the simpler problem for a magnetic field of constant direction, or what amounts to the same, a magnetic field in two dimensions. However, we try to retain as much information as possible about the magnetic field in the estimate on the semigroup.

A magnetic field in two dimensions is a scalar function B which can be expressed in terms of a (non-unique) vector potential A,

$$B(x) = \frac{\partial A_2(x)}{\partial x_1} - \frac{\partial A_1(x)}{\partial x_2} \ , \qquad A(x) = \big(A_1(x), A_2(x)\big) \ .$$

For example, if $B(x) = B_0$ is a constant magnetic field then

$$A(x) = \frac{B_0}{2} \big(-x_2, x_1\big) \ . \tag{1.4}$$

We want to estimate (for suitable p and q with $1 \leq p \leq q \leq \infty$)

$$C(t; p, q) := \sup_{u\in L^p} \frac{\|u(t)\|_q}{\|u\|_p} \qquad \text{where} \quad u(t) = e^{tH}\, u \ ,$$

which is just the norm of the Schrödinger semigroup $\exp(tH)$ as a mapping from $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$. This norm is finite, because (1.2) implies that

$$\|e^{tH}\|_{L^p\to L^q} \leq \|e^{t\Delta/2}\|_{L^p\to L^q} < \infty \ .$$

In case of a constant magnetic field $B(x) = B_0 > 0$ these calculations can be done using the explicitly known integral kernel of the time evolution operator. If the vector potential A is chosen as in Eq. (1.4) above, the time evolution is given by

$$u(t,x) = \int G(t,x,y)\, u(y)\, d^2y \ ,$$

where $G(t,x,y)$ is the Mehler kernel [S79, p. 168]

$$G(t,x,y) = \frac{B_0}{4} \, \frac{1}{\pi \sinh\!\left(\frac{B_0 t}{2}\right)} \, \exp\left\{ -\frac{B_0}{4} \coth\!\left(\frac{B_0 t}{2}\right) (x-y)^2 + \mathrm{i}\, \frac{B_0}{2} \,(x_1 y_2 - x_2 y_1) \right\} \ ,$$

which is a complex, degenerate, centered Gaussian kernel (in the terminology of [L90]). In particular, we need the time evolution of the initial function

$$u_0(x) = N_0 \, \exp\left\{ -\frac{B_0}{4a_0}\, x^2 \right\}, \qquad a_0 > 0 \ . \tag{1.5}$$

The solution at time t is again a Gaussian which can be written as

$$u_0(t,x) = N(t) \, \exp\left\{ -\frac{B_0}{4a(t)}\, x^2 \right\} \ ,$$

where

$$N(t) = \frac{N_0\, a_0}{a_0 \, \cosh\left(\frac{B_0 t}{2}\right) + \sinh\left(\frac{B_0 t}{2}\right)} \ , \qquad a(t) = \frac{a_0 \, \cosh\left(\frac{B_0 t}{2}\right) + \sinh\left(\frac{B_0 t}{2}\right)}{a_0 \, \sinh\left(\frac{B_0 t}{2}\right) + \cosh\left(\frac{B_0 t}{2}\right)} \ .$$

The function $a(t)$ is the unique solution of the initial value problem

$$\dot a(t) = \frac{B_0}{2} \left(1 - a(t)^2 \right) \ , \qquad a(0) = a_0 \ .$$

Writing $a_0 = \tanh(\theta)$ (for $a_0 < 1$), or $a_0 = \coth(\theta)$ (for $a_0 > 1$), we obtain

$$a(t) = \begin{cases} \tanh\left(\theta + \frac{B_0 t}{2}\right) & \text{for } a_0 < 1, \\ 1 & \text{for } a_0 = 1, \\ \coth\left(\theta + \frac{B_0 t}{2}\right) & \text{for } a_0 > 1. \end{cases} \tag{1.6}$$

For $a_0 = 1$, the function $u_0(x) = N_0 \exp(-B_0 x^2/4)$ is the eigenfunction of H corresponding to the eigenvalue $-B_0/2$.

The following theorem sets the stage for our investigation. It is essentially a corollary of a theorem in [L90]. We shall give the details in Sect. 2.

Theorem 1.1: *Let $B(x) = B_0 > 0$, all $x \in \mathbb{R}^2$. For $1 < p \le q < \infty$, there is a centered Gaussian function u_0 which maximizes the expression $\|u(t)\|_q / \|u\|_p$ over all $u \in L^p(\mathbb{R}^2)$. The function u_0 is of the form (1.5), where a_0 depends on p, q, t, and B_0. More explicitly, with*

$$a_0 = d_{t,p,q} + \sqrt{d_{t,p,q}^2 + p - 1}\ , \tag{1.7}$$

$$d_{t,p,q} = \frac{1}{2c_t s_t}\left(\frac{p}{q} - 1 + s_t^2(p-2)\right)\ ,$$

$$c_t = \cosh\left\{\frac{B_0 t}{2}\right\}\ , \qquad s_t = \sinh\left\{\frac{B_0 t}{2}\right\}\ .$$

we have

$$\begin{aligned} C_0(t;p,q) \equiv \sup_{u \in L^p} \frac{\|e^{tH(B_0)} u\|_q}{\|u\|_p} &= \frac{\|u_0(t)\|_q}{\|u_0\|_p} \\ &= \frac{N(t)}{N_0}\left(\frac{B_0}{4a_0}\frac{p}{\pi}\right)^{1/p}\left(\frac{B_0}{4a(t)}\frac{q}{\pi}\right)^{-1/q}\ . \end{aligned} \tag{1.8}$$

Remark 1: The condition $p \le q$ in Theorem 1.1 is necessary as the following example shows. Consider the function

$$u(x) = N_0 \sum_{a \in G} \exp(-B_0(x-a)^2/4)\ ,$$

where G is a finite square lattice in $\mathbb{R}^2$ whose lattice spacing is large compared to $1/B_0$. Thus the Gaussian functions in the sum have essentially disjoint support and hence

$$\|u\|_p \sim N_0 \left(\frac{4\pi}{B_0 p}\right)^{1/p} |G|^{1/p}\ ,$$

where $|G|$ denotes the number of lattice points. The function u is an eigenfuction of the magnetic semigroup with eigenvalue $e^{-B_0 t/2}$. Thus,

$$\|u(t)\|_q \sim N_0 e^{-B_0 t/2} \left(\frac{4\pi}{B_0 q}\right)^{1/q} |G|^{1/q} \ ,$$

and

$$\frac{\|u(t)\|_q}{\|u\|_p} \sim e^{-B_0 t/2} \left(\frac{B_0}{4\pi}\right)^{1/p-1/q} \frac{p^{1/p}}{q^{1/q}} |G|^{1/q-1/p} \ .$$

The last expression tends to infinity with $|G|$ if $p > q$.

Remark 2: Instead of using the results of [L90] one might approach Theorem 1.1 a la Gross [G76],i.e., trying to reduce the problem to an integral over infinitesimal time steps, and estimating the semigroup over these time steps by employing logarithmic Sobolev inequalities. It is clear that one would get some smoothing estimates but can one obtain them in the sharp form? That there are some obstructions to reach this goal by this method can be seen as follows.

Eqs. (1.5) and (1.7) determine the Gaussian function u_0 that yields the norm of the magnetic heat kernel as an operator from L^p to L^q. With this Gaussian we may write

$$C_0(t;p,q) = \frac{\|u_0(t)\|_q}{\|u_0\|_p} = \frac{\|u_0(s)\|_r}{\|u_0\|_p} \frac{\|u_0(t)\|_q}{\|u_0(s)\|_r} \ ,$$

for some r and s with $p \le r \le q$, $0 \le s \le t$. Hence it is obvious that

$$C_0(t;p,q) \le C_0(s;p,r)\, C_0(t-s,r,q) \ .$$

Clearly, the proposed method can only work, if there exists a number r with $p < r < q$ such that the above formula is an equality, i.e.,

$$C_0(t;p,q) = \min_{p \le r \le q} \{C_0(s;p,r)\, C_0(t-s,r,q)\} \ . \tag{1.9}$$

However, it is easy to see that Eq. (1.9) cannot hold in general. For example, it is wrong in the specific case $q = p > 2$, $0 < s < t$. From the explicit form of $C_0(t; p, q)$ given in Theorem 1.1 we find that

$$C_0(t; p, q) = C_0(s; p, r(s))\, C_0(t - s; r(s), q) \tag{1.10}$$

holds for some $r(s) \in [p, q]$ if and only if the optimizer u_0 giving $C_0(t; p, q)$ is also the optimizer for $C_0(s; p, r(s))$ (i.e., $C_0(s; p, r(s)) = \|u(s)\|_{r(s)}/\|u_0\|_p$).

Eq. (1.10) can only hold for all $s \in (0, t)$ if the function $r(s)$ is monotonic nondecreasing, with $r(0) = p$ and $r(t) = q$. Given u_0 (and hence a_0) as the optimizer for $C_0(t; p, q)$, we can easily determine $r(s)$ using Theorem 1.1 by solving

$$a_0 = d_{s,p,r(s)} + \sqrt{d^2_{s,p,r(s)} + p - 1}\ .$$

This gives the function

$$r(s) = \frac{p}{[a_0 c_s + (1 - p)s_s][c_s/a_0 + s_s]}\ , \tag{1.11}$$

which is monotonic nondecreasing in the following cases:

$$\frac{q - p}{q(p - 2)} \geq (s_t)^2 \qquad \text{if } 2 \leq p \leq q\ ,$$

$$\frac{q - p}{p(2 - q)} \geq (s_t)^2 \qquad \text{if } 2 \geq q \geq p\ .$$

If $p \leq 2 \leq q$ the function is always monotonic increasing. Thus, the formula (1.9) holds whenever the above inequalities are satisfied and in these cases Eq. (1.10) holds for all $s \in [0, t]$ with $r(s)$ given as above. The identity

$$a(s) = d_{t-s,r(s),q} + \sqrt{d^2_{t-s,r(s),q} + r(s) - 1}$$

(with $a(s)$ as in (1.6)) shows that the optimizer u_0 for $C_0(s;p,r(s))$ is mapped by the time evolution onto the optimizer $u_0(s)$ for $C_0(t-s;r(s),q)$, if $r(s)$ is chosen as in Eq. (1.11).

The approach discussed at the beginning of Remark 2 has the great advantage of being quite flexible and we shall use it to obtain smoothing estimates in the cases where the magnetic field is not constant. Certainly, one cannot expect to get such detailed information as in Theorem 1.1 in this case, however, we have the following result which is proved in Sect. 4.

Theorem 1.2:

Assume that $B(x) \geq B_0 > 0$ is continuous. Then the estimate

$$C(t;p,q) = \|e^{tH(B)}\|_{L^p \to L^q} \leq C_0(t;p,q)$$

(with $C_0(t;p,q)$ given by Eq. (1.8)) holds for

a) *all $t > 0$, if $p \leq 2$, $q \geq 2$,*

b) *$t > 0$ and $2 < p < q \leq \infty$, if*

$$\frac{q-p}{q(p-2)} \geq \left(\sinh\left\{\frac{B_0 t}{2}\right\}\right)^2, \tag{1.12}$$

c) *$t > 0$ and $1 \leq p < q < 2$, if*

$$\frac{q-p}{p(2-q)} \geq \left(\sinh\left\{\frac{B_0 t}{2}\right\}\right)^2. \tag{1.13}$$

Remark 1: In the case where neither a), b) nor c) holds but $p \leq q$ we get a nontrivial bound but it is not sharp.

Remark 2: A simple consequence of Theorem 1.2 is the pointwise bound on the heat kernel

$$\|e^{tH(B)}\|_{L^1\to L^\infty} \le \frac{B_0}{4\pi\sinh(\frac{B_0 t}{2})} \ ,$$

which is best possible. In fact there is equality if $B(x) \equiv B_0$ as can be seen from the Mehler kernel.

Finally, also in Sect. 4, we extend our result on the "diagonal" of the magnetic heat kernel to the "off-diagonal" using a technique due to Davies [D89].

Theorem 1.3: *Assume that $B(x) \ge B_0 > 0$ is continuous. Then the magnetic heat kernel satisfies the bound*

$$|e^{tH}(x,y)| \le \frac{B_0}{4\pi\sinh(\frac{B_0 t}{2})} e^{-\frac{(x-y)^2}{2t}} \ . \tag{1.14}$$

Remark: The Gaussian decay on the right side of (1.14) is the one of the heat kernel, which is considerably weaker than the decay of the Mehler kernel. Is it true that $|e^{tH}(x,y)|$ is bounded by the Mehler kernel $|G(t,x,y)|$? The truth of this estimate would reveal a robust dependence of the magnetic heat kernel on the magnetic field. This is an open problem.

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2. PROOF OF THEOREM 1.1

Let G_t be the integral operator with the Mehler kernel $G(t,x,y)$. We want to determine

$$\sup_u \frac{\|G_t u\|_q}{\|u\|_p} = C_0(t;p,q),$$

which is either finite or infinite. By a result of E. Lieb (see [L90], Theorem 4.1) it is sufficient to take the supremum over centered Gaussian functions of the form $\exp(-x\cdot J_\varepsilon x)$, where J_ε is a complex-valued matrix, symmetric with respect to the scalar product in $\mathbb{R}^2$, and with a strictly positive real part. Even for this case the calculation is still somewhat tedious. Instead it is convenient to rework the proof of Theorem 4.1 in [L90].

Let $t > 0$ be arbitrary. We approximate G_t by the operator G_t^ε with kernel

$$G_t^\varepsilon(x,y) = \mathrm{e}^{-\varepsilon x^2}\, G(t,x,y),$$

which for $\varepsilon > 0$ is a non-degenerate, centered Gaussian kernel. According to [L90], Theorem 3.4, there is a unique (up to a multiplicative constant) centered Gaussian function u_ε which is the maximum of $\|G_t^\varepsilon u\|_q/\|u\|_p$ over all $u \in L^p(\mathbb{R}^2)$. The function u_ε is of the form $u_\varepsilon(x) = \exp(-x\cdot J_\varepsilon x)$, where J_ε is a (possibly complex-valued) matrix which is symmetric with respect to the scalar product in $\mathbb{R}^2$ and has a strictly positive real part. But since the integral operator G_t^ε commutes with rotations, the unique maximum u_ϵ must also be rotationally invariant. Hence $J_\varepsilon = \alpha_\varepsilon \mathbf{1} + \mathrm{i}\beta_\varepsilon \mathbf{1}$, where $\alpha_\varepsilon > 0$, and β_ε is real. Since the integrals of Gaussian functions can be evaluated explicitly, we can evaluate the quotient $\|G_t^\varepsilon u\|_q/\|u\|_p$ for $u(x) = \exp[-(\alpha + \mathrm{i}\beta)\, x^2]$ and maximize this expression over all α and β. The maximum is obtained for $(\alpha,\beta) = (\alpha_\varepsilon,\beta_\varepsilon)$, with $\alpha_\varepsilon > 0$ and $\beta_\varepsilon = 0$.

Since $\exp(-\varepsilon x^2) \leq 1$ we find for any Gaussian function u

$$\|G_t^\varepsilon u\|_q \leq \|G_t u\|_q$$

and hence

$$C^\varepsilon = \frac{\|G_t^\varepsilon u_\varepsilon\|_q}{\|u_\varepsilon\|_p} \leq C_0(t;p,q),$$

where $u_\varepsilon = \exp(-\alpha_\varepsilon x^2)$ is the unique Gaussian maximizer for G_t^ε.

By an explicit calculation one sees that

$$\lim_{\varepsilon\to 0} C^\varepsilon \equiv C^0 = \frac{\|G_t u_0\|_q}{\|u_0\|_p},$$

with $u_0(x) = \exp(-\alpha x^2)$, $\alpha = \lim_{\varepsilon\to 0}\alpha_\varepsilon$. Of course, we have $C^0 \le C_0(t;p,q)$.

Finally, for any Gaussian function u, the limit

$$\lim_{\varepsilon\to 0}\frac{\|G_t^\varepsilon u\|_q}{\|u\|_p} = \frac{\|G_t u\|_q}{\|u\|_p}$$

exists by Lebesgue's dominated convergence, and from

$$\frac{\|G_t^\varepsilon u\|_q}{\|u\|_p} \le C^\varepsilon \quad \text{for all } \varepsilon > 0$$

we find immediately that

$$\frac{\|G_t u\|_q}{\|u\|_p} \le \lim_{\varepsilon\to 0} C^\varepsilon = C^0 \quad \text{for all Gaussian functions } u.$$

Hence, also $\sup \|G_t u\|_q/\|u\|_p = C_0(t;p,q) \le C^0$. This proves $C^0 = C_0(t;p,q)$. A little calculation easily gives the explicit value of this constant.

3. A DIFFERENTIAL INEQUALITY

In the following we assume that the magnetic field B is given by a differentiable vector field A as in Eq. (1.1) and satisfies $B(x) \geq B_0 > 0$, for all $x \in \mathbb{R}^2$. In particular we choose the vector potential to be in $L^4_{\text{loc}}(\mathbb{R}^2)$ and then by the Leinfelder–Simader Theorem [LS81] the formal expression (1.1) defines a selfadjoint operator on some domain $D(H)$ with $C_0^\infty(\mathbb{R}^2)$ as a core.

If we set $u(s) = e^{Hs}u_0$ then $u(s)$ is a solution of

$$\frac{d}{ds}\, u = H\, u = \frac{1}{2}(\nabla + \mathrm{i}A)^2\, u\ . \tag{3.1}$$

If $u_0 \in D(H) \cap L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ then we have that $u(s) \in D(H) \cap L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. This follows from the diamagnetic inequality (1.2) for each $s \in (0,t]$ and the explicit form of the heat kernel.

Theorem 3.1: *Let $r : (0,t] \to \mathbb{R}$ be a twice differentiable function with $r(s) \geq 2$ and $\dot r(s) \geq 0$ for all $s \in (0,t]$. Then*

$$\frac{d}{ds}\, \ln \|u(s)\|_{r(s)} \leq -L(r(s), \dot r(s))\ , \tag{3.2}$$

for all $s \in (0,t]$, where

$$L(r,\dot r) = \frac{\dot r}{r^2}\left\{2 + \ln\frac{4\pi a(r,\dot r)}{B_0 r}\right\} + \frac{a(r,\dot r)B_0}{r}\ . \tag{3.3}$$

and

$$a(r,\dot r) = \frac{1}{B_0 r}\big(\sqrt{\dot r^2 + B_0^2 r^2 (r-1)} - \dot r\big).$$

Equality holds in Eq (3.2) for $B(x) = B_0$ and

$$u(s,x) = \exp\left(-\frac{B_0}{4a\big(r(s),\dot r(s)\big)}\, x^2\right)\ .$$

Proof: Pick $u_0 \in D(H) \cap L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Since $\frac{d}{d\tau} \ln k\|u(\tau)\|_{r(\tau)}$ does not depend on $k > 0$, we assume during the following calculation, without loss of generality, that at the time $\tau = s > 0$ the solution is normalized such that $\|u(s)\|_{r(s)} = 1$. For the derivative at $\tau = s$ we obtain therefore

$$\begin{aligned} \frac{d}{ds} \ln \|u(s)\|_{r(s)} = &\frac{\dot{r}(s)}{r(s)^2} \int |u(s,x)|^{r(s)} \ln |u(s,x)|^{r(s)} \, d^2x \\ &+ \frac{1}{2} \int |u(s,x)|^{(r(s)-2)} \frac{d}{ds} |u(s,x)|^2 \, d^2x \ . \end{aligned} \tag{3.4}$$

The integrals have to be taken over $\mathbb{R}^2$. The formal computation can be easily justified by an approximation argument. For simplicity, the arguments s and x in the integrand on the right side will be omitted from now on.

Using (3.1) we obtain, after a partial integration

$$\begin{aligned} \frac{1}{2} \int |u|^{(r-2)} \frac{d}{ds} |u|^2 = &- \frac{1}{2}(r-2) \int |u|^{(r-2)} \, (\nabla |u|)^2 \\ &- \frac{1}{2} \int |u|^{(r-2)} \, |(\nabla + \mathrm{i}A) \, u|^2 \ . \end{aligned} \tag{3.5}$$

The integration by parts can be justified as follows. Since $u \in D(H)$, by the Leinfelder–Simader Theorem [LS81] we can pick a sequence $u_n \in C_0^\infty(\mathbb{R}^2)$ such that $u_n \to u, Hu_n \to Hu$ in L^2. Inspecting the proof of the Leinfelder–Simader Theorem one sees that the sequence u_n can be chosen to converge to u in L^1 and to have a uniform bound on the L^∞ norm. Thus u_n converges to u in L^p for all $1 \le p < \infty$. In particular, all the following computations can be justified in the same fashion and we can assume without restriction that $u \in C_0^\infty(\mathbb{R}^2)$.

If we set $u = f + \mathrm{i}g$ then $|u| = \sqrt{f^2 + g^2}$. We find

$$|(\nabla + \mathrm{i}A) \, u|^2 = (\nabla |u|)^2 + |A + \nabla S|^2 \, |u|^2 = X^2 + Y^2 \tag{3.6}$$

where we have introduced two real vector fields X and Y over $\mathbb{R}^2$,

$$X = \nabla^{\perp}|u| = \Big(\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1}\Big)|u|,$$

$$Y = (A + \nabla S)\,|u|.$$

Here the symbol ∇S denotes the expression

$$\frac{f\nabla g - g\nabla f}{f^2 + g^2},$$

which is defined wherever $f^2 + g^2 > 0$.

For any $c > 0$ we may estimate

$$c^2 X^2 + Y^2 \geq 2c\, X \cdot Y \tag{3.7}$$

with equality if and only if $cX = Y$. If $B = B_0$ is a constant magnetic field we can choose a Gaussian function

$$u = N \exp\left\{-\frac{B_0}{4c}\, x^2\right\} \tag{3.8}$$

such that equality holds everywhere in (3.7), because (with A as in Eq. (1.2))

$$cX = c\,\nabla^{\perp} u = \frac{B_0}{2}\,(-x_2, x_1)\, u = Au = Y\ .$$

Let us now insert (3.6) and (3.7) into (3.5). Since $r - 1 > 0$ we can add and subtract a positive constant c^2 with $0 < c < \sqrt{r-1}$ in order to obtain

$$\begin{aligned}
\frac{1}{2}\int |u|^{(r-2)}\,\frac{d}{ds}|u|^2 &= -\frac{1}{2}(r-1-c^2)\int |u|^{(r-2)}\,(\nabla|u|)^2 \\
&\quad -\frac{1}{2}\int |u|^{(r-2)}\,c^2(\nabla|u|)^2 - \frac{1}{2}\int |u|^{(r-2)}\,(A+\nabla S)^2\,|u|^2 \\
&\leq -\frac{1}{2}(r-1-c^2)\int |u|^{(r-2)}\,(\nabla|u|)^2 - c\int |u|^{(r-2)}\,(\nabla^{\perp}|u|)\cdot(A+\nabla S)\,|u|
\end{aligned}$$

The last term can be rewritten as

$$\begin{aligned} -\frac{c}{r}\int(\nabla^\perp|u|^r)\cdot(A+\nabla S) \\ &= -\frac{c}{r}\int\{\nabla^\perp\cdot(|u|^r(A+\nabla S))+B|u|^r\} \\ &= -\frac{c}{r}\int B|u|^r. \end{aligned}$$

The only thing to observe is that $\nabla^\perp\cdot\nabla S=0$ wherever $|u|>0$.

Using $B(x)\geq B_0$ and $\int|u|^r=1$ these estimates lead to the following family of inequalities for each $c\in(0,\sqrt{r-1})$

$$\begin{aligned} \frac{d}{ds}\ln\|u\|_r \leq \frac{\dot r}{r^2}\int|u|^r\ln|u|^r \\ -\frac{1}{2}(r-1-c^2)\int|u|^{(r-2)}(\nabla|u|)^2-\frac{cB_0}{r}\ . \end{aligned} \tag{3.9}$$

Equality holds in case of a constant magnetic field $B=B_0$ and u a Gaussian function given by Eq. (3.8). We also note that in the special case $\dot r(s)=0$ the above calculation can be simplified to give

$$\frac{d}{ds}\ln\|u\|_r\leq-\sqrt{r-1}\,\frac{B_0}{r}\ . \tag{3.10}$$

Now we apply the following family of logarithmic Sobolev inequalities

$$\int g^2\ln g^2-\frac{\lambda}{\pi}\int(\nabla g)^2\leq-2-\ln\lambda\ ,\qquad \text{all } \lambda>0, \tag{3.11}$$

where we assumed that $\int g^2=1$. Equality holds if g is a normalized Gaussian,

$$g(x)=\frac{1}{\sqrt{\lambda}}\exp\left\{-\frac{\pi}{2\lambda}\,x^2\right\}\ .$$

This family is essentially Gross' logarithmic Sobolev inequality rewritten as an inequality on $\mathbb{R}^2$ (see, e.g., [CL96]). We insert (3.11) into (3.9) with

$$\lambda=\frac{2\pi}{\dot r}(r-1-c^2),\qquad g=|u|^{r/2}\ . \tag{3.12}$$

Here we have to assume that $\dot{r} > 0$. With this choice of λ we obtain

$$\frac{d}{ds} \ln \|u\|_r \leq -\frac{\dot{r}}{r^2} \left\{ 2 + \ln \frac{2\pi}{\dot{r}} (r - 1 - c^2) \right\} - \frac{cB_0}{r} \ . \tag{3.13}$$

This result holds for all $0 < c < \sqrt{r-1}$. The expression on the right side has a minimum for $c = a$ where

$$a = \frac{1}{B_0 r} \left(\sqrt{\dot{r}^2 + B_0^2 r^2 (r-1)} - \dot{r} \right) = a(r, \dot{r}) \ . \tag{3.14}$$

Note that for $\dot{r} = 0$ this formula is consistent with Eq. (3.10). In order to determine the case of equality in (3.13) we determine λ according to (3.12) for the optimal value a of c. This gives $\lambda = 4\pi a / B_0 r$ so that the Gaussian function which optimizes the logarithmic Sobolev inequality is given by

$$g = u^{r/2} = N \exp \left\{ -\frac{B_0 r}{8a} \, x^2 \right\} .$$

As shown above, the corresponding function u at the same time optimizes Eq. (3.7) in case of a constant magnetic field. Hence we find that equality holds in the estimate (3.13) (with $c = a$) provided that $B = B_0$ is constant and u is the Gaussian function defined above.

For the optimal choice of c the inequality (3.13) becomes

$$\frac{d}{ds} \ln \|u(s)\|_{r(s)} \leq -L(r(s), \dot{r}(s)) \ ,$$

where (with $a = a(r, \dot{r})$ as defined in (3.14))

$$L(r, \dot{r}) = \frac{\dot{r}}{r^2} \left\{ 2 + \ln \frac{4\pi a}{B_0 r} \right\} + \frac{aB_0}{r} \ .$$

This estimate remains true for $\dot{r}(s) = 0$ and $a = \sqrt{r-1}$ with $L(r, 0) = aB_0/r$, cf. Eq (3.10).

Remark: We note that if for some $s \in [0, t]$, the derivative $\dot{r}$ is negative, then $d/ds \ \ln \|u\|_r$ is unbounded. This can be seen by choosing a constant magnetic field $A(x, y) = (-y, x)$

and

$$u(x,y) = \mathrm{e}^{\mathrm{i}xy}\, g(x)\, h(y) \ ,$$

where g and h are positive with $\int g^r = \int h^r = 1$. The integrals in (3.4) and (3.5) can be evaluated explicitly for Gaussian functions. Setting

$$g(x) = \left(\frac{1}{\pi}\right)^{1/2r} \exp\left(-\frac{x^2}{r}\right) \ , \qquad h(y) = \left(\frac{\alpha}{\pi}\right)^{1/2r} \exp\left(-\alpha\frac{y^2}{r}\right) \ ,$$

we find

$$\frac{d}{ds} \ln \|u\|_r = \frac{\dot{r}}{2r^2}\,(\ln\alpha - 2\ln\pi - 2) - \frac{r-1}{r^2}(1+\alpha) - 1 \ ,$$

which for $\dot{r} < 0$ tends to $+\infty$, as $\alpha \to 0$.

4. PROOFS OF THEOREM 1.2 AND 1.3

Proof of Theorem 1.2: By duality, the semigroup $e^{Ht} : L^p \to L^q$ has the same norm as the one from $L^{q'} \to L^{p'}$ where p', q' are the dual indices. Thus c) follows from b).

To prove b) for $q < \infty$ we integrate the inequality (3.2) and obtain

$$\frac{\|u(t)\|_q}{\|u(0)\|_p} \leq \exp\left\{-\int_0^t L(r(s), \dot{r}(s))\, ds\right\} . \tag{4.1}$$

Recall that the only assumption on $r(s)$ is that $r(0) = p$, $r(t) = q$ and that $\dot{r}(s)$ be nonnegative. Thus, we can optimize the right side of (4.1) over $r(s)$ which leads to an interesting problem in the calculus of variations. The associated differential equation is

$$\ddot{r} = 2\,\frac{\dot{r}^2}{r} + \frac{1}{2} B_0^2 r(r-2). \tag{4.2}$$

It is easily verified that $r(s)$ defined by Eq (1.11) is a solution of (4.2) and satisfies $r(0) = p$. Moreover, if we choose a_0 as in (1.7) we see that $r(t) = q$. Using the function $r(s)$ as a trial function in (4.1) we obtain b) after an exceedingly tedious but straightforward computation.

Next we consider b) with $q = \infty$. We remark that the definition (1.11) of the trial function $r(s)$ makes still sense in this case, the condition for monotonicity follows from (1.12). We have $\lim_{s \to t} r(s) = \infty$ and $2 < r(s) < \infty$ for $s < t$. Choose T with $0 < T < t$ and write $u(T) = e^{TH} u_0$. Then

$$\frac{\|e^{tH} u_0\|_\infty}{\|u_0\|_p} = \frac{\|e^{TH} u_0\|_{r(T)}}{\|u_0\|_p}\,\frac{\|e^{(t-T)H} u(T)\|_\infty}{\|u(T)\|_{r(T)}} .$$

Here the second factor is bounded by some constant $K(T)$ which tends to 1, as $T \to t$, because the semigroup e^{sH} is bounded as an operator from L^r to L^∞ for all $r \geq 1$. The first factor can be estimated as above by integrating (3.2), so that for all $0 < T < t$,

$$\frac{\|e^{tH} u_0\|_\infty}{\|u_0\|_p} \leq K(T)\, \exp\left\{-\int_0^T L\big(r(s), \dot{r}(s)\big)\, ds\right\} .$$

For the trial function $r(s)$ with $q = \infty$ one can see that $\dot{r}(s)/r(s)^2$ as well as $a(r(s), \dot{r}(s))$ remain bounded for $s \to t$. Therefore the function $L(r(s), \dot{r}(s))$ given by (3.3) has only a logarithmic and hence integrable singularity, as $s \to t$. Hence the estimate above makes sense for $T \to t$ and we find that (4.1) remains valid even for $q = \infty$.

To prove a) we again consider the function $r(s)$ defined by Eq (1.11). Again $r(0) = p \leq 2$ and if we choose a_0 as in (1.7), $r(t) = q \geq 2$. Denote by θ the time where $r(\theta) = 2$. Such a time exists since $r(s)$ is strictly increasing. Now

$$\frac{\|u(t)\|_q}{\|u(0)\|_p} = \frac{\|u(t)\|_q}{\|u(\theta)\|_2} \frac{\|u(\theta)\|_2}{\|u(0)\|_p} \ ,$$

and the second factor on the right can be estimated according to part b) by the constant $C(\theta; p, 2)$ whose value is given by the right side of (1.8) with $q = 2$ and $t = \theta$.

The first factor is estimated by writing

$$\frac{\|u(t)\|_q}{\|u(\theta)\|_2} = \frac{\|e^{H(t-\theta)}u(\theta)\|_q}{\|u(\theta)\|_2} \leq \sup \frac{\|u(t-\theta)\|_q}{\|u(0)\|_2}$$

which is bounded by $C(t - \theta; 2, q)$. Combining these estimates and using formula (1.10) yields the result.

Proof of Theorem 1.3: Following [D89] we consider the semigroup given by the kernel

$$Q_t(x, y) = e^{-\alpha x} e^{tH(B)}(x, y) e^{\alpha y} \ ,$$

which is generated by the operator

$$\frac{1}{2}(\nabla + \alpha + \mathrm{i}A)^2 \ .$$

Retracing the steps of the proof of Theorem 3.1 for $u(t)(\cdot) = \int Q_t(\cdot, y) u_0(y) \mathrm{d}y$ yields the estimate

$$\frac{d}{ds} \ln \|u(s)\|_{r(s)} \leq -L(r(s), \dot{r}(s)) + \alpha^2 \ ,$$

with $L(r(s),\dot r(s))$ given by (3.3). From the proof of Theorem 1.2 it now follows that

$$|Q_t(x,y)| \le \frac{B_0}{4\pi\sinh(\frac{B_0 t}{2})}\, e^{\alpha^2 t} \ ,$$

or

$$|e^{H(B)t}(x,y)| \le \frac{B_0}{4\pi\sinh(\frac{B_0 t}{2})}\, e^{\alpha(x-y)+\alpha^2 t}.$$

Optimizing with respect to α yields the result.

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