A CRITERION FOR ESSENTIAL SELF-ADJOINTNESS

B. THALLER

Erwin Schrödinger International Institute for Mathematical Physics, Pasteurgasse 6/7, A-1090 Vienna, Austria

Abstract. We prove some simple facts on the essential self-adjointness of a symmetric operator T in a Hilbert space $\mathfrak{H}$. The main tools are bounded commutators of T with a suitable sequence of bounded self-adjoint operators. A special case is a Theorem of Jörgens and Chernoff stating that the essential self-adjointness of the Dirac operator in an external field depends only on the local properties of the potential.

Permanent address: Institut für Mathematik, Universität Graz, Heinrichstraße 36, A-8010 Graz, Austria
E-mail address: thaller@bkfug.kfunigraz.ac.at

1. Assumptions and Main Results

Let $\mathfrak{H}$ be a separable Hilbert space. $\{A_n\}_{n=0}^{\infty}$ denotes a sequence of bounded and self-adjoint operators on $\mathfrak{H}$. We consider a linear operator T in $\mathfrak{H}$, densely defined and symmetric on $\mathfrak{D}(\mathfrak{T}) = \mathfrak{D}_\circ \subset \mathfrak{H}$.

Notation . If a linear operator S is defined and closeable on $\mathfrak{D}_\circ$ we denote its adjoint by S^* and its closure S^{**} by S^c. The restriction to $\mathfrak{D}_\circ$ of an operator B defined on some larger domain is occasionally denoted by $B|_{\mathfrak{D}_\circ}$.

For the operators A_n and T we formulate the following conditions:

$\mathbf{A_1}$: For all n there exist k, m with

$$A_k A_n = A_n, \qquad A_m A_k = A_k. \tag{1.1}$$

$\mathbf{A_2}$: For all n, $[T, A_n] \equiv TA_n - A_n T$ is defined on $\mathfrak{D}_\circ$ and bounded.

$\mathbf{A_3}$: For all n the operator $A_n T A_n|_{\mathfrak{D}_\circ}$ is essentially self-adjoint.

$\mathbf{A_4}$: For all $\psi \in \mathfrak{D}(\mathfrak{T}^*)$, there is a subsequence $\{A_{n_k}\}_{k=0}^{\infty}$ converging weakly to $\mathbf{1}$ such that

$$\lim_{k\to\infty} (T^* A_{n_k}\psi, \psi - A_{n_k}\psi) = 0. \tag{1.2}$$

One of the main results which can be proved under the conditions above (or under similar conditions discussed below) is the following:

Theorem 1.1. *Assume* $\mathbf{A_1}$–$\mathbf{A_4}$. *Then* T *is essentially self-adjoint on* $\mathfrak{D}_\circ$.

As an illustration consider a Hilbert space which is an infinite orthogonal sum of closed subspaces. Define

$$\mathfrak{H} = \bigoplus_{\mathfrak{k}=\mathfrak{o}}^{\infty} \mathfrak{H}^{(\mathfrak{k})}, \qquad \mathfrak{H}_\mathfrak{n} = \bigoplus_{\mathfrak{k}=\mathfrak{o}}^{\mathfrak{n}} \mathfrak{H}^{(\mathfrak{k})}, \tag{1.3}$$

and let A_n be the projection onto $\mathfrak{H}_\mathfrak{n}$. Hence A_n converges (strongly) to $\mathbf{1}$ and Assumption $\mathbf{A_1}$ is trivially satisfied. Let T be symmetric on some dense subset $\mathfrak{D}_\circ$. Then Theorem 1.1 gives conditions which assure that the essential self-adjointness of the restrictions $A_n T A_n$ of T is equivalent to the essential self-adjointness of T (the equivalence follows with the help of Theorem 3.1 and Remark 3.3 below).

Another example is provided by the Dirac operator in the Hilbert space $L^2(\mathbb{R}^{\not\models})^{\not\trianglerighteq}$. The theorem contains as a special case a famous result of Chernoff [1, 2] and Jörgens [3] stating that the essential self-adjointness of the Dirac operator on $\mathcal{C}_0^\infty(\mathbb{R}^3)^4$ depends only on the local properties of the potential. In this case the A_n are multiplication operators with suitable $\mathcal{C}_0^\infty$-functions. See Section 4 for a discussion and a simplified proof of this result.

We conclude this section with some remarks on the assumptions.

Remark 1.1. Assumption $\mathbf{A_1}$ is satisfied, e.g., if the A_n form an increasing sequence: $A_m A_n = A_n$ for all $m > n$. One might think of a sequence of projections, but we do not require $A_n^2 = A_n$. From $A_n^* = A_n$ for all n we easily conclude that the operators A_n, A_m, and A_k in $\mathbf{A_1}$ commute.

Remark 1.2. The commutator $[T, A_n]$ is assumed to be defined on its natural domain. Since A_n is defined on all of $\mathfrak{H}$, we have $\mathfrak{D}([\mathfrak{T}, \mathfrak{A}_\mathfrak{n}]) = \mathfrak{D}(\mathfrak{T}\mathfrak{A}_\mathfrak{n}) \cap \mathfrak{D}(\mathfrak{T})$. Assumption $\mathbf{A_2}$ requires that $\mathfrak{D}([\mathfrak{T}, \mathfrak{A}_\mathfrak{n}]) = \mathfrak{D}_\circ = \mathfrak{D}(\mathfrak{T})$ which means

$$A_n \mathfrak{D}_\circ \subset \mathfrak{D}_\circ. \tag{1.4}$$

Boundedness of $[T, A_n]$ on a dense domain implies the existence of a unique bounded extension $[T, A_n]^c$ to all of $\mathfrak{H}$.

Remark 1.3. Theorem 1.1 can be formulated as a perturbation theoretic result. Let $T = H_0 + V$, where H_0 is essentially self-adjoint and V is symmetric on $\mathfrak{D}_\circ$. If we replace $\mathbf{A_3}$ by the assumption that $H_0 + A_n V A_n$ be essentially self-adjoint on $\mathfrak{D}_\circ$, then we may conclude the essential self-adjointness of T. See Section 5 for details.

Remark 1.4. The expression $T^* A_{n_k} \psi$ occurring in $\mathbf{A_4}$ is well defined for all $\psi \in \mathfrak{D}(\mathfrak{T}^*)$, provided $\mathbf{A_2}$ holds (see Lemma 2.1 below). If $\mathbf{A_4}$ holds, then

$$\|A_{n_k}\| \leq K, \tag{1.5}$$

where the constant K is independent of k (any weakly convergent sequence of operators is bounded). If the commutators $[T, A_{n_k}]^c$ are also bounded uniformly in k, then we can replace $\mathbf{A_4}$ by a more convenient assumption:

Lemma 1.1. *Assume $\mathbf{A_2}$. Then $\mathbf{A_4}$ is implied by*

$\mathbf{A_4'}$: For all $\psi \in \mathfrak{H}$ there is a subsequence $\{A_{n_k}\}_{k=0}^\infty$ converging strongly to $\mathbf{1}$ such that

$$\| [T, A_{n_k}]^c \psi \| \leq C(\psi),$$

where $C(\psi) > 0$ is independent of k.

By the uniform boundedness principle, Assumption $\mathbf{A_4'}$ implies $\| [T, A_{n_k}]^c \| \leq K'$, uniformly in k.

2. Lemmas and Proofs

We start by proving some elementary technical lemmas. As always we assume that A_n is self-adjoint and bounded and that T is symmetric on $\mathfrak{D}_\circ$.

Lemma 2.1. *Assume* $\mathbf{A_2}$*. Then* $A_n\mathfrak{D}(\mathfrak{T}^*) \subset \mathfrak{D}(\mathfrak{T}^*)$ *and*

$$[T, A_n]^c\psi = T^*A_n\psi - A_nT^*\psi \qquad \textit{for all } \psi \in \mathfrak{D}(\mathfrak{T}^*). \tag{2.1}$$

Proof. Let $\psi \in \mathfrak{D}(\mathfrak{T}^*)$. For all $\phi \in \mathfrak{D}_\circ$

$$(A_n\psi, T\phi) = (\psi, (TA_n - [T, A_n])\phi) = ((A_nT^* - [T, A_n]^*)\psi, \phi). \tag{2.2}$$

Hence, by the definition of the adjoint operator, $A_n\psi \in \mathfrak{D}(\mathfrak{T}^*)$ and

$$T^*A_n\psi = (A_nT^* - [T, A_n]^*)\psi. \tag{2.3}$$

Since $\mathrm{i}[T, A_n]$ is obviously symmetric on $\mathfrak{D}_\circ$, its bounded extension is self-adjoint, i.e.,

$$[T, A_n]^* = -[T, A_n]^c. \tag{2.4}$$

Combining this with Eq. (2.3) completes the proof of the Lemma 2.1. □

Lemma 2.2. *Assume* $\mathbf{A_1}$*–*$\mathbf{A_3}$*. Then* $A_n\mathfrak{D}(\mathfrak{T}^*) \subset \mathfrak{D}(\mathfrak{T}^c)$ *and*

$$T^cA_n\psi = (A_mTA_m)^cA_n\psi + [T, A_m]^cA_n\psi \qquad \textit{for all } \psi \in \mathfrak{D}(\mathfrak{T}^*), \tag{2.5}$$

where m is chosen according to Assumption $\mathbf{A_1}$*.*

Proof. Let $\psi \in \mathfrak{D}(\mathfrak{T}^*)$. By Lemma 2.1, $A_n\psi \in \mathfrak{D}(\mathfrak{T}^*)$. Choose m, k according to $\mathbf{A_1}$ and note that $A_mA_n = A_mA_kA_n = A_kA_n = A_n$. Hence for $\phi \in \mathfrak{D}_\circ$,

$$(A_n\psi, A_mTA_m\phi) = (A_mT^*A_n\psi, \phi). \tag{2.6}$$

Since by Assumption $\mathbf{A_3}$

$$(A_mTA_m|_{\mathfrak{D}_\circ})^* = (A_mTA_m|_{\mathfrak{D}_\circ})^c, \tag{2.7}$$

Eq. (2.6) implies

$$A_n\psi \in \mathfrak{D}\Big((\mathfrak{A}_\mathfrak{m}\mathfrak{T}\mathfrak{A}_\mathfrak{m}|_{\mathfrak{D}_\circ})^\mathfrak{c}\Big). \tag{2.8}$$

Hence there is a sequence $\{\chi_j\}_{j=0}^{\infty}$ with $\chi_j \in \mathfrak{D}_\circ$ for all j, $\lim \chi_j = A_n\psi$, such that $\{A_m T A_m \chi_j\}_{j=0}^{\infty}$ is convergent. But then the sequence $\{\xi_j\}_{j=0}^{\infty}$ with $\xi_j = A_k\chi_j$ has the same properties: $\xi_j \in \mathfrak{D}_\circ$ and (by continuity of A_n) $\lim \xi_j$ exists. Furthermore

$$A_m T A_m \xi_j = A_k A_m T A_m \chi_j + A_m [T, A_k] A_m \chi_j, \tag{2.9}$$

which converges, as $j \to \infty$. Therefore

$$T\xi_j = T A_m \xi_j = A_m T A_m \xi_j + [T, A_m]\xi_j \tag{2.10}$$

again converges in $\mathfrak{H}$, as $j \to \infty$. Hence $A_n\psi = \lim \xi_j \in \mathfrak{D}(\mathfrak{T}^{\mathfrak{c}})$ and

$$T^c A_n \psi = \lim_{j\to\infty} T\xi_j, \tag{2.11}$$

which is just what we wanted to prove. □

Proof of Theorem 1.1. Let $\psi \in \mathfrak{D}(\mathfrak{T}^*)$. By Lemma 2.2, $A_n\psi \in \mathfrak{D}(\mathfrak{T}^{\mathfrak{c}})$, and

$$(\psi, T^*\psi) = (T^c A_{n_k}\psi, A_{n_k}\psi) + (T^c A_{n_k}\psi, \psi - A_{n_k}\psi) + (\psi - A_{n_k}\psi, T^*\psi). \tag{2.12}$$

Using $\mathbf{A_4}$ we find

$$|(T^c A_{n_k}\psi, \psi - A_{n_k}\psi)| + |(\psi - A_{n_k}\psi, T^*\psi)| \xrightarrow[n\to\infty]{} 0. \tag{2.13}$$

Note that

$$(T^c A_{n_k}\psi, A_{n_k}\psi) = (A_{n_k}\psi, T^c A_{n_k}\psi) \tag{2.14}$$

is real, because the closure of a symmetric operator is always symmetric. We conclude that

$$(\psi, T^*\psi) = \lim_{k\to\infty} (A_{n_k}\psi, T^c A_{n_k}\psi) \tag{2.15}$$

is real as a limit of real numbers. Hence (by the polarization identity) T^* is symmetric which proves that T is essentially self-adjoint. □

Proof of Lemma 1.1. Assuming $\mathbf{A_2}$ we find with the help of Lemma 2.1 that for all $\psi \in \mathfrak{D}(\mathfrak{T}^*)$

$$\|T^* A_{n_k}\psi\| \le \|A_{n_k}\| \, \|T^*\psi\| + \| \, [T, A_{n_k}]^c \, \psi\|. \tag{2.16}$$

Using Eq (1.5) and $\mathbf{A_4'}$ we find

$$\|T^* A_{n_k}\psi\| \le K\|T^*\psi\| + C(\psi) \equiv C_1(\psi). \tag{2.17}$$

Hence we obtain

$$|(T^c A_{n_k}\psi, \psi - A_{n_k}\psi)| \le C_1(\psi)\|\psi - A_{n_k}\psi\|, \tag{2.18}$$

which tends to zero because the sequence A_{n_k} is assumed to converge strongly to $\mathbf{1}$. This proves Lemma 1.1. □

3. Some Remarks and Further Results

Remark 3.1. A slight modification of the proof of Theorem 1.1 allows to replace the condition $\mathbf{A_4}$ resp. $\mathbf{A_4'}$ with, e.g.,

$\mathbf{B_4}$: For some subsequence $\{A_{n_k}\}_{k=0}^{\infty}$ and for all $\psi \in \mathfrak{H}$,

$$\sum_{k=1}^{\infty} A_{n_k} = \mathbf{1},$$

where the series converges in the strong sense, and

$$\| \sum_{k=1}^{j} [T, A_{n_k}]^c \psi \| \leq C(\psi),$$

where $C(\psi) > 0$ is independent of j.

Repeat the calculation Eqs. (2.12)–(2.18) with A_n replaced by $\tilde{A}_j \equiv \sum_{k=1}^{j} A_{n_k}$ to arrive at the conclusion of Theorem 1.1.

Remark 3.2. An immediate consequence of the essential self-adjointness of T on $\mathfrak{D}_\circ$ and the boundedness of the commutator is the following.

Corollary 3.1. *Assume* $\mathbf{A_2}$ *and let* T *be essentially self-adjoint on* $\mathfrak{D}_\circ$*. Denote*

$$\psi(t) \equiv \exp(-\mathrm{i}tT^c)\,\psi, \quad \text{for all } \psi \in \mathfrak{H}. \tag{3.1}$$

Then for all ψ, $\phi \in \mathfrak{H}$ *the function* $t \to (\psi(t), A_n\phi(t))$ *is continuously differentiable with*

$$\frac{d}{dt}(\psi(t), A_n\phi(t)) = (\psi(t), \mathrm{i}[T, A_n]^c\phi(t)). \tag{3.2}$$

Proof. For convenience we give a proof of this simple fact: For ψ, $\phi \in \mathfrak{H}$ choose $\{\psi_j\}$, $\{\phi_j\}$ in $\mathfrak{D}_\circ$ with $\psi = \lim \psi_j$, $\phi = \lim \phi_j$. By the strong continuity of the unitary group, $f_j(t) \equiv (\psi_j(t), A_n\phi_j(t))$ is continuous, and by $\mathbf{A_2}$ even continuously differentiable with $f_j'(t) \equiv (\psi_j(t), \mathrm{i}[T, A_n]\phi_j(t))$. Define $g(t) \equiv (\psi(t), \mathrm{i}[T, A_n]^c\phi(t))$. Then

$$|f_j'(t) - g(t)| \leq \|[T, A_n]^c\|(\|\phi_j\|\|\psi_j - \psi\| + \|\psi\|\|\phi_j - \phi\|) < \varepsilon, \tag{3.3}$$

where ε can be chosen independently of t and is arbitrarily small for j large. Hence we can exchange the differentiation and the limit to conclude that $f(t) = \lim f_j(t)$ is differentiable with $f'(t) = \lim f_j'(t)$. □

The following theorem is a converse of Theorem 1.1. We give conditions for the essential self-adjointness of A_nTA_n as a consequence of the self-adjointness of T^c. First we state a lemma similar to Lemma 2.1.

Lemma 3.1. *Assume* $\mathbf{A_2}$*. Then* $A_n\mathfrak{D}(\mathfrak{T}^{\mathfrak{c}}) \subset \mathfrak{D}(\mathfrak{T}^{\mathfrak{c}})$ *and*

$$[T, A_n]^c\psi = T^cA_n\psi - A_nT^c\psi \qquad \text{for all } \psi \in \mathfrak{D}(\mathfrak{T}^{\mathfrak{c}}). \tag{3.4}$$

Proof. By the definition of closure, for any $\psi \in \mathfrak{D}(\mathfrak{T}^{\mathfrak{c}})$ there is a sequence $\{\psi_k\}_{k=0}^{\infty}$ with $\psi_k \in \mathfrak{D}_\circ$, $\lim \psi_k = \psi$, and $\lim T\psi_k = T^c\psi$. Using the continuity of A_n and $[T, A_n]$ we find

$$TA_n\psi_k = A_nT\psi_k + [T, A_n]\psi_k \xrightarrow[k\to\infty]{} A_nT^c\psi + [T, A_n]^c\psi \tag{3.5}$$

Hence the sequences $\{A_n\psi_k\}$ and $\{TA_n\psi_k\}$ both converge. This implies

$$\lim_{k\to\infty} A_n\psi_k = A_n\psi \in \mathfrak{D}(\mathfrak{T}^{\mathfrak{c}}), \qquad \mathfrak{T}^{\mathfrak{c}}\mathfrak{A}_{\mathfrak{n}}\psi = \lim_{\mathfrak{k}\to\infty} \mathfrak{T}\mathfrak{A}_{\mathfrak{n}}\psi_{\mathfrak{k}}. \tag{3.6}$$

Now Eq. (3.4) follows immediately. □

Theorem 3.1. *Assume* $\mathbf{A_2}$*. Let* T *be essentially self-adjoint on* $\mathfrak{D}_\circ$*. In addition we assume for all* ψ *that* $A_n^2\psi \in \mathfrak{D}(\mathfrak{T}^{\mathfrak{c}})$ *implies* $A_n\psi \in \mathfrak{D}(\mathfrak{T}^{\mathfrak{c}})$*. Then* $A_nTA_n|_{\mathfrak{D}_\circ}$ *is essentially self-adjoint.*

Proof. Denote $B = A_nTA_n|_{\mathfrak{D}_\circ}$. Let $\psi \in \mathfrak{D}(\mathfrak{B}^*)$, $\phi \in \mathfrak{D}_\circ$. Then

$$(B^*\psi, \phi) = (\psi, A_nTA_n\phi) = (A_n^2\psi, T\phi) + ([T, A_n]^*A_n\psi, \phi) \tag{3.7}$$

shows that $A_n^2\psi \in \mathfrak{D}(\mathfrak{T}^*) = \mathfrak{D}(\mathfrak{T}^{\mathfrak{c}})$ and

$$T^cA_n^2\psi = B^*\psi - [T, A_n]^*A_n\psi. \tag{3.8}$$

The additional assumption implies $A_n\psi \in \mathfrak{D}(\mathfrak{T}^{\mathfrak{c}})$ and with Lemma 3.1 we obtain

$$T^cA_n^2\psi = A_nT^cA_n\psi + [T, A_n]^cA_n\psi. \tag{3.9}$$

Using Eq. (2.4) we find that

$$B^*\psi = A_nT^cA_n\psi \quad \text{for all } \psi \in \mathfrak{D}(\mathfrak{B}^*). \tag{3.10}$$

Hence B^* is symmetric which is equivalent to the essential self-adjointness of B. □

Remark 3.3. The additional assumption in Theorem 3.1 is trivially satisfied if the operators A_n are projections.

4. An Example: Dirac Operators

In the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^3)^4$ of $\mathbb{C}^4$-valued square integrable functions on $\mathbb{R}^3$ we define the free Dirac operator as the closure of the operator

$$H_0 = -\mathrm{i}\alpha \cdot \nabla + \beta \quad \text{on} \quad \mathfrak{D}_\circ = \mathcal{C}_\circ^\infty(\mathbb{R}^3 \setminus \{\mathfrak{o}\})^4. \tag{4.1}$$

Here $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β are the Hermitian 4×4 Dirac matrices (see [4] for details). It is well known that H_0 is essentially self-adjoint on $\mathfrak{D}_\circ$, its closure H_0^c is self-adjoint on the first Sobolev space

$$\mathfrak{D}(\mathfrak{H}_\circ^c) = \mathfrak{H}^1(\mathbb{R}^3)^4 \equiv \{\psi \in \mathfrak{L}^2(\mathbb{R}^3)^4 \mid \alpha \cdot \nabla\psi \in \mathfrak{L}^2(\mathbb{R}^3)^4\}\,. \tag{4.2}$$

Here ∇ denotes the distributional derivative.

We denote by g the bounded operator of multiplication by a function $g \in \mathcal{C}_0^\infty(\mathbb{R}^3)$,

$$\psi \to g\psi, \quad (g\psi)(x) = g(x)\psi(x), \quad \text{all } \psi \in \mathfrak{H}, \text{ all } x \in \mathbb{R}^3. \tag{4.3}$$

It is easy to see that

$$[H_0, g]^c = -\mathrm{i}\alpha \cdot (\nabla g) \tag{4.4}$$

is a bounded operator of multiplication with a Hermitian matrix-valued $\mathcal{C}_0^\infty$-function.

Lemma 4.1. *The operator gH_0g is essentially self-adjoint on $\mathfrak{D}_\circ$.*

Proof. Since β is a bounded operator in $\mathfrak{H}$, it is sufficient to consider the operator

$$T_0 \equiv -\mathrm{i}\alpha \cdot \nabla \tag{4.5}$$

when investigating the self-adjointness-properties. Again, T_0 is essentially self-adjoint on $\mathfrak{D}(\mathfrak{T}_\circ) = \mathfrak{D}_\circ$, self-adjoint on $\mathfrak{D}(\mathfrak{T}_\circ^c) = \mathfrak{D}(\mathfrak{H}_\circ^c)$, and $[T_0, g]^c = [H_0, g]^c$. Our proof uses the self-adjointness of T_0^c on $H^1(\mathbb{R}^3)^4$ and the boundedness of $[T_0, g]^c$. Denote $B = gT_0g|_{\mathfrak{D}_\circ}$. As in the calculation leading to Eq. (3.8) we conclude for all $\psi \in \mathfrak{D}(\mathfrak{B}^*)$ that $g^2\psi \in \mathfrak{D}(\mathfrak{T}_\circ^*) = \mathfrak{D}(\mathfrak{T}_\circ^c)$ and

$$B^*\psi = -\mathrm{i}\alpha \cdot \nabla g^2\psi + \mathrm{i}g\alpha \cdot (\nabla g)\psi. \tag{4.6}$$

By the Leibniz rule which holds for the product of a distribution with a smooth function we obtain

$$-\mathrm{i}\alpha \cdot \nabla g^2\psi = -2\mathrm{i}g\alpha \cdot (\nabla g)\psi - \mathrm{i}g^2\alpha \cdot \nabla\psi. \tag{4.7}$$

This shows that for all $\psi \in \mathfrak{D}(\mathfrak{B}^*)$ the distribution $\alpha \cdot \nabla\psi \in H^{-1}(\mathbb{R}^3)^4$ satisfies $g^2\alpha \cdot \nabla\psi \in L^2(\mathbb{R}^3)^4$. Therefore we can perform the following calculation for arbitrary $\psi, \phi \in \mathfrak{D}(\mathfrak{B}^*)$

$$\begin{aligned}(\phi, B^*\psi) &= (\phi, -\mathrm{i}\alpha \cdot \nabla g^2\psi + \mathrm{i}g\alpha \cdot (\nabla g)\psi) \qquad (4.8)\\ &= (-\mathrm{i}g^2\alpha \cdot \nabla\phi, \psi) + (-\mathrm{i}g\alpha \cdot (\nabla g)\phi, \psi) \qquad (4.9)\\ &= (-\mathrm{i}\alpha \cdot \nabla g^2\phi + \mathrm{i}g\alpha \cdot (\nabla g)\phi, \psi) = (B^*\phi, \psi) \qquad (4.10)\end{aligned}$$

Hence B^* is symmetric, i.e., B is essentially self-adjoint. □

Now, let consider a function $f \in \mathcal{C}_0^\infty(\mathbb{R})$ with the properties $f(0) = 1$, and $f(r) = 0$ if $r \geq 1$. Define the sequence of multiplication operators A_n, $n = 1, 2, \dots$ by

$$\Big(A_n\psi\Big)(x) = \begin{cases}\psi(x) & \text{if } |x| \leq n,\\ f(|x| - n)\psi(x) & \text{if } |x| \geq n.\end{cases} \qquad (4.11)$$

The sequence $\{A_n\}$ satisfies $\mathbf{A_1}$ and $\mathbf{A_4'}$, the commutators $[H_0, A_n]^c$ are bounded uniformly in n by $\sup|f'(r)|$. By Lemma 4.1 all the operators $A_nH_0A_n$ are essentially self-adjoint on $\mathfrak{D}_\circ$. Hence the following corollary is an immediate consequence of Theorem 1.1

Corollary 4.1. *Let A_n be defined as above and let V be a symmetric operator on $\mathfrak{D}_\circ$ such that $\mathbf{A_4}$ holds with $T = V$, and assume that $A_n(H_0 + V)A_n$ is essentially self-adjoint on $\mathfrak{D}_\circ$. Then $H_0 + V$ is essentially self-adjoint on $\mathfrak{D}_\circ$.*

The assumptions of the corollary are trivially satisfied, if V is multiplication by a locally bounded Hermitian matrix-valued function (no matter how fast it grows at infinity), because in this case A_nVA_n is a bounded perturbation of the essentially self-adjoint operator $A_nH_0A_n$. As noted by Chernoff [2] this result is in marked contrast to the situation for the second-order Schrödinger operator and is related to the existence of a limiting velocity for the propagation of wavepackets according to the Dirac equation.

Because of the Kato-Rellich theorem only the relative boundedness of A_nVA_n with respect to $A_nH_0A_n$ is needed and one could also consider potentials with local singularities. Other examples include nonlocal potentials. See [3, 4] for details.

A variant of the preceeding proof is obtained by using a partition of unity $\{f_n\}$ on $\mathbb{R}^3$ with $\sup_{x,n} |\nabla f_n(x)| \leq M < \infty$ to define operators A_n satisfying $\mathbf{B_4}$.

5. Perturbation Theory

Let H_0 and V be symmetric operators defined on a dense subset $\mathfrak{D}_\circ$ of some Hilbert space $\mathfrak{H}$. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of bounded self-adjoint operators satisfying $\mathbf{A_1}$. Assume that $\mathbf{A_2}$ holds with T replaced by H_0 and V, respectively. Instead of $\mathbf{A_3}$ let us now assume

$\mathbf{C_3}$: H_0 is essentially self-adjoint on $\mathfrak{D}_\circ$, and for all n the operator $H_0 + A_n V A_n$ is essentially self-adjoint on $\mathfrak{D}_\circ$ with

$$\mathfrak{D}\Big((\mathfrak{H}_\circ + \mathfrak{A}_\mathfrak{n}\mathfrak{V}\mathfrak{A}_\mathfrak{n})^\mathfrak{c}\Big) = \mathfrak{D}(\mathfrak{H}_\circ^\mathfrak{c}). \tag{5.1}$$

The operator $T \equiv H_0 + V$ is well defined and symmetric on $\mathfrak{D}_\circ$ and we assume that it satisfies $\mathbf{A_4}$.

Theorem 5.1. *Under the above assumptions, T is essentially self-adjoint on $\mathfrak{D}_\circ$.*

With the help of the following lemma, the proof of Theorem 5.1 is an easy modification of the proof of Theorem 1.1.

Lemma 5.1. *Assume $\mathbf{A_1}$, $\mathbf{C_3}$, and $\mathbf{A_2}$ with H_0 and V. Then $\psi \in \mathfrak{D}\Big(\mathfrak{T}^*\Big)$ implies that $A_n\psi \in \mathfrak{D}\Big(\mathfrak{T}^\mathfrak{c}\Big)$ and*

$$T^c A_n\psi = H_0^c A_n\psi + V^c A_n\psi, \quad \textit{for all } \psi \in \mathfrak{D}\Big(\mathfrak{T}^*\Big). \tag{5.2}$$

Proof. Let $\psi \in \mathfrak{D}\Big(\mathfrak{T}^*\Big)$. Since T satisfies $\mathbf{A_2}$ we find with Lemma 2.1 that $A_n\psi \in \mathfrak{D}\Big(\mathfrak{T}^*\Big)$. Hence for $\phi \in \mathfrak{D}_\circ$ we obtain using $A_m A_n = A_n$

$$(T^* A_n\psi, \phi) = (A_n\psi, \{T_m - [V, A_m]\}\phi), \qquad T_m \equiv H_0 + A_m V A_m. \tag{5.3}$$

This shows that $A_n\psi \in \mathfrak{D}(\mathfrak{T}_\mathfrak{m}^*) = \mathfrak{D}(\mathfrak{T}_\mathfrak{m}^\mathfrak{c}) = \mathfrak{D}(\mathfrak{H}_\circ^\mathfrak{c})$, where we have used $\mathbf{C_3}$. By definition of closure, there is a sequence $\chi_j \in \mathfrak{D}_\circ$ with $\chi_j \to A_n\psi$ and $H_0^c\chi_j \to H_0^c A_n\psi$, i.e., $\{\chi_j\}$ converges in the Hilbert space $\mathfrak{D}(\mathfrak{H}_\circ)$ equipped with the graph norm $\|\psi\|_0^2 = \|H_0^c\psi\|^2 + \|\psi\|^2$. Since T_m^c is closed on $(\mathfrak{D}(\mathfrak{H}_\circ), \|\cdot\|_\circ)$, it is bounded and hence the sequence $\{T_m^c\chi_j\}$ is again convergent. As in the proof of Lemma 2.2 we can replace χ_j by the sequence $\xi_j = A_k\chi_j$, which has the same properties, if A_k is chosen according to $\mathbf{A_1}$. In particular, the sequences $\{T_m\xi_j\}$ and $\{H_0\xi_j\}$ are convergent. But then

$$V\xi_j = VA_m\xi_j = A_m V A_m\xi_j + [V, A_m]\xi_j = (T_m - H_0)\xi_j + [V, A_m]^c\xi_j \tag{5.4}$$

is convergent, i.e., $\lim \xi_j = A_n\psi \in \mathfrak{D}(\mathfrak{V}^\mathfrak{c})$. Finally,

$$T\xi_j = H_0\xi_j + V\xi_j \to H_0^c A_n\psi + V^c A_n\psi, \tag{5.5}$$

which implies $A_n\psi \in \mathfrak{D}(\mathfrak{T}^c)$ together with Eq. (5.2). $\square$

Theorem 5.2. *Let H_0 be essentially self-adjoint on $\mathfrak{D}_\circ$ and X be self-adjoint on $\mathfrak{D}(\mathfrak{X})$, such that $[H_0^c, X]$ is well defined on $\mathfrak{D}_\circ$ and bounded. Let V be a real-valued function on $\mathbb{R}$, which is locally bounded. Define $V(X) = \int V(\lambda)\, dE_X(\lambda)$ and assume $\mathfrak{D}_\circ \subset \mathfrak{D}(\mathfrak{V}(\mathfrak{X}))$. Then $H_0 + V(X)$ is essentially self-adjoint on $\mathfrak{D}_\circ$.*

Proof. Let g be a real-valued function, which can be written as the Fourier-transform of a function $\tilde{g}$, such that $(1 + |\xi|)\tilde{g}(\xi)$ is integrable:

$$g(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{i}\lambda\xi}\, \tilde{g}(\xi)\, d\xi. \tag{5.6}$$

Define $g(X)$ by the weak integral

$$g(X)\psi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{i}X\xi}\psi\, \tilde{g}(\xi)\, d\xi, \quad \text{for all } \psi \in \mathfrak{H}. \tag{5.7}$$

Then

$$\|[H_0^c, g(X)]^c\| \le \frac{1}{\sqrt{2\pi}}\, \|[H_0^c, X]^c\| \int_{-\infty}^{\infty} |\xi|\, |\tilde{g}(\xi)|\, d\xi. \tag{5.8}$$

Let $f \in \mathcal{C}_0^\infty(\mathbb{R})$ be real-valued, with $f(\lambda) = 1$, if $|\lambda| \le 1/2$ and $f(\lambda) = 0$, if $|\lambda| \ge 1$. Define $A_n := f(X/n) = \int_{-n}^{n} f(\lambda/n)\, dE_X(\lambda)$, where E_X is the spectral measure of X.

Now it is easy to see that $T = H_0$ and A_n satisfy the assumptions $\mathbf{A_1}$–$\mathbf{A_4}$, and even $\mathbf{A_4'}$. By our assumptions, $V(X) = \int V(\lambda)\, dE_X(\lambda)$ is a densely defined self-adjoint operator which commutes with all A_n. Moreover, $A_n V(X) A_n$ is bounded and symmetric on $\mathfrak{D}_\circ$. Hence $H_0 + A_n V(X) A_n$ is essentially self-adjoint on $\mathfrak{D}_\circ$. Essential self-adjointness of $H_0 + V(X)$ now follows immediately from Theorem 5.1. $\square$

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