FACTORIZATION OF DEGENERATE CAUCHY PROBLEMS: THE LINEAR CASE

B. THALLER AND S. THALLER *

Institut für Mathematik, Universität Graz, Heinrichstr. 36, A-8010 Graz, Austria

Abstract. We investigate an abstract degenerate Cauchy problem with a non-invertible operator M at the derivative. The problem is formulated in a Hilbert space $\mathfrak{H}$ which can be written as an orthogonal direct sum of $\operatorname{Ker} M$ and $\operatorname{Ran} M^*$. Under certain conditions it is possible to reduce the problem to an equivalent non-degenerate Cauchy problem in the factor space $\mathfrak{H}/\operatorname{Ker} M$. The explicit form of the generator for the restricted problem is investigated. As an example we discuss the Dirac equation, where our theory leads to a new interpretation of the nonrelativistic limit. We show that this limit can be understood in terms of a degenerate Cauchy problem where the generator of the restricted problem is a Schrödinger operator. Finally, we describe some consequences for the treatment of degenerate control systems. In particular, we introduce dual methods of factorization in order to investigate the compatibility of factorization and transition to the dual system, which is essential for the definition of basic notions like observability and detectability.

* Supported by SFB F003 "Optimierung und Kontrolle"

1. Introduction

We are going to study the abstract system

$$\frac{d}{dt} M\, z(t) = A\, z(t) + f(t), \tag{1.1}$$

where M and A are linear operators acting between Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$. These systems are called "degenerate" or "singular", if the operator M is not invertible. Under certain assumptions it is possible to restrict the whole problem to the orthogonal complement $(\operatorname{Ker} M)^\perp$ of the kernel of M. Here we investigate the structure of the restricted problem without assuming that M or A be self-adjoint. Our approach is particularly useful for systems of differential equations, as will be demonstrated in Section 3.

Degenerate Cauchy problems have received considerable interest during the past few decades. For finite dimensional linear systems the problem is understood completely, because it is possible to transform the matrices M and A to a common normal form which allows to classify the cases in which Eq. (1.1) has a unique solution for each initial value. This is discussed in the book by L. Dai [2], where one can find many examples, applications to control theory, and references to the earlier literature.

In the infinite dimensional case we mention the book by Carroll and Showalter [1] which also treats the nonlinear and nonautonomous case. Showalter further considers the linear problem in Refs. [16], [17]. In his setting the operator M has to be self-adjoint and non-negative. The operators M and A are initially given as quadratic forms on some vector space E. The Hilbert space of the system is then defined to be the completion $\mathfrak{H}_M$ of the factor space $E/\operatorname{Ker} M$ in the seminorm defined by M. In this Hilbert space M becomes the identity operator (Riesz map). In order to obtain in $\mathfrak{H}_M$ a single-valued operator A_0 which corresponds to A one has to assume, e.g., that A defines a sectorial quadratic form ([17], cf. also Remark 2.2). This assumption also implies that A_0 generates an analytic semigroup for the factorized system. Being very elegant, this approach has nevertheless the disadvantage that the whole setup depends on the operator M, and it becomes difficult to investigate what happens when M is perturbed or approximated. Our approach uses different assumptions allowing to express the factorization in terms of orthogonal projection operators whose behavior under perturbations is easier to describe. Moreover, we do not require M to be non-negative or self-adjoint. Likewise the assumption of parabolicity, although convenient because it leads to a bounded evolution operator, is not essential for the process of factorization.

In an interesting series of papers A. Favini investigates degenerate Cauchy problems of parabolic type in Banach spaces. The linear system is considered in [3] using Laplace transforms of the resolvent. In [5] the problem is treated also under the assumption that the Banach space of the system can be written as a direct sum of suitable subspaces (this is the main reason for our use of Hilbert spaces in the present paper). Most recently, in [11] the authors discuss in particular regularity properties of solutions. The paper [4] treats the controllability of degenerate systems, based on results in [3]. Results concerning existence, uniqueness and regularity of solutions have also been obtained in more general situations. [6], [7] treat the case of time dependent operators $M(t)$ and $A(t)$, nonlinear operators are considered in [8] and [11]. See [9] and [11] for applications and [10] for results concerning multivalued operators.

Among the older literature we further mention [13], [14], [15].

In the present paper there is, of course, some overlap with the previously mentioned work. But the emphasis of this paper is on the possibility of factorization and the relation of the factorized problem with the original degenerate system – without assuming parabolicity: Making use of the decomposition of the Hilbert space into a direct sum of $\operatorname{Ker} M$ and $\operatorname{Ran} M^*$ (resp. $\operatorname{Ker} M^*$ and $\operatorname{Ran} M$) we formulate the conditions which allow us to obtain an equivalent but nondegenerate Cauchy problem in the factor space $\mathfrak{H}/(\operatorname{Ker} M) = \operatorname{Ran} M^*$ (resp. in $\operatorname{Ran} M$) and give the explicit form of generators of the factorized problems. The crucial assumption is that the restriction of A to a mapping from $\operatorname{Ker} M$ to $\operatorname{Ker} M^*$ is well defined and invertible (Assumption 2.2). This allows to define a factorization operator Z_A which maps the solutions of the factorized system to solutions of the original degenerate system. It is interesting that this operator is generally unbounded.

We treat the basic concepts and the homogeneous equation ($f(t) = 0$) in Section 2, the inhomogeneous equation is discussed in Section 4. As an example illustrating the application of our method to systems of differential equations we consider the (time dependent) Dirac equation of relativistic quantum mechanics which in the nonrelativistic limit can be written as a degenerate Cauchy problem. This degenerate Cauchy problem is by our method of factorization easily seen to be equivalent (in a certain sense) to the Schrödinger equation.

In Section 5 we make some remarks concerning the setting of control theory under simplifying assumptions. In particular the conceptionally important relation between a control system and its dual is compared to the corresponding relations of the factorized systems, which leads to the notion of dual methods of factorization.

2. The homogeneous equation

Notation . For a linear operator T we denote by $\mathfrak{D}(T)$, $\operatorname{Ker} T$, and $\operatorname{Ran} T$ its domain, kernel, and range, respectively. If $\mathfrak{D}(T)$ is dense, the adjoint operator is well defined and will be denoted by T^*. The restriction to $\mathfrak{D}$ of an operator T defined on some larger domain will be written as $T \restriction \mathfrak{D}$.

Let us consider the degenerate Cauchy problem

$$\frac{d}{dt} M\, z(t) = A\, z(t), \qquad z(0) = z_0, \tag{2.1}$$

where throughout this paper we make the following assumption:

Assumption 2.1. *Let A and M be closed linear operators which are densely defined in some Hilbert space $\mathfrak{H}$ and map into a Hilbert space $\mathfrak{K}$.*

Here M need not be invertible. Since M is closed, $\operatorname{Ker} M$ is a closed subspace of $\mathfrak{H}$. The orthogonal projection onto $\operatorname{Ker} M$ will be denoted by P, and $P^\perp = \mathbf{1} - P$ projects onto the orthogonal complement of the kernel. Since the adjoint M^* of a closed operator is again densely defined in $\mathfrak{K}$, we have the relations

$$P\mathfrak{H} = \operatorname{Ker} M, \qquad P^\perp \mathfrak{H} = (\operatorname{Ran} M^*)^c.$$

(The symbol ^{c} denotes the closure). For the projection Q onto $\operatorname{Ker} M^*$ we obtain the analogous relations

$$Q\mathfrak{K} = \operatorname{Ker} M^*, \qquad Q^\perp \mathfrak{K} = (\operatorname{Ran} M)^c.$$

We start with an obvious definition and a few observations.

Definition 2.1. A strict solution of the degenerate Cauchy problem is a continuous function $z : [0, \infty) \longrightarrow \mathfrak{H}$ such that $z(t) \in \mathfrak{D}(A) \cap \mathfrak{D}(M)$ for all $t \geq 0$, Mz is continuously differentiable, and Eq. (2.1) holds.

Any strict solution z of the degenerate Cauchy problem Eq. (2.1) clearly satisfies $z(t) \in \mathfrak{D}_A$ for all $t \geq 0$, where

$$\mathfrak{D}_A = \{z \in \mathfrak{D}(A) \mid Az \in (\operatorname{Ran} M)^c\}.$$

Obviously, we have

$$\operatorname{Ker} A \subset \mathfrak{D}_A. \tag{2.2}$$

Lemma 2.1. *Under the Assumption 2.1 the operator $A \restriction \mathfrak{D}_A$ is closed.*

Proof. If z_n is a sequence in $\mathfrak{D}_A \subset \mathfrak{D}(A)$ such that $u = \lim z_n$ and $v = \lim Az_n$ both exist, then (since A is closed) $u \in \mathfrak{D}(A)$ and $Au = v$. But each Az_n is in the closed set $(\operatorname{Ran} M)^c$, hence the limit v is again in $(\operatorname{Ran} M)^c$. Thus, u is in $\mathfrak{D}_A$. □

In order to restrict the degenerate Cauchy problem Eq. (2.1) to the subspace $P^\perp \mathfrak{H} = (\operatorname{Ker} M)^\perp$ we define the operator

$$M^\perp = M \restriction P^\perp \mathfrak{D}(M) \tag{2.3}$$

which is invertible from $P^{\perp}\mathfrak{D}(M) = (\operatorname{Ker} M)^{\perp} \cap \mathfrak{D}(M)$ into $(\operatorname{Ker} M^*)^{\perp}$. The operator A, however, will become a multi-valued operator A_0 on $(\operatorname{Ker} M)^{\perp}$ which can be defined as

$$\mathfrak{D}(A_0) = \{x \in (\operatorname{Ker} M)^{\perp} \mid (P^{\perp})^{-1}\{x\} \cap \mathfrak{D}_A \neq \emptyset\}, \tag{2.4}$$

$$A_0 x = A\{(P^{\perp})^{-1}\{x\} \cap \mathfrak{D}_A\} \subset (\operatorname{Ran} M)^c, \quad \text{for all } x \in \mathfrak{D}(A_0). \tag{2.5}$$

Here $(P^{\perp})^{-1}\{x\} = \{x + y \mid y \in \operatorname{Ker} M\}$, $x \in (\operatorname{Ker} M)^{\perp}$, is the inverse image of x under the projection $P^{\perp}$. Instead of the degenerate Cauchy problem Eq. (2.1) we could investigate

$$\frac{d}{dt} M^{\perp} x(t) \in A_0\, x(t), \qquad x(0) = P^{\perp} z_0, \tag{2.6}$$

because for any family of vectors $x(t)$ satisfying this relation there exists a family $z(t)$ in $\mathfrak{D}_A$ satisfying (2.1). In general, the family $z(t)$ is neither unique nor continuous: If $k(t)$ is an arbitrary family of vectors in $\operatorname{Ker} M \cap \operatorname{Ker} A$ then $z(t) + k(t)$ also satisfies (2.1). In the following typical (non exclusive) cases, the operator A_0 will be single-valued:

(A) : $\operatorname{Ker} A \supset \operatorname{Ker} M \cap \mathfrak{D}_A$,

(B) : $(P^{\perp})^{-1}\{x\} \cap \mathfrak{D}_A$ consists of precisely one element.

Remark 2.1. Let $\mathfrak{H} = \mathfrak{K}$, assume that A, M are self-adjoint, and $P\mathfrak{D}(A) \subset \mathfrak{D}(A)$. Then $\operatorname{Ker} A \supset \operatorname{Ker} M \cap \mathfrak{D}(A)$ is equivalent to $\mathfrak{D}_A$ dense in $\mathfrak{H}$. If this is true, then $\mathfrak{D}_A = \mathfrak{D}(A)$.

Proof. Assume that $\mathfrak{D}_A$ is dense. M self-adjoint means $\operatorname{Ker} M = (\operatorname{Ran} M)^{\perp}$. Hence, for $y \in \operatorname{Ker} M \cap \mathfrak{D}(A)$

$$0 = (Ax, y) = (x, Ay), \quad \text{all } x \in \mathfrak{D}_A.$$

This implies $y \in \operatorname{Ker} A$.

Conversely, assume $\operatorname{Ker} A \supset B \equiv \operatorname{Ker} M \cap \mathfrak{D}(A)$. For all $y \in B$ we have $Ay = 0$ and hence $(x, Ay) = (Ax, y) = 0$ for all $x \in \mathfrak{D}(A)$. This implies $\operatorname{Ran} A \subset B^{\perp}$. But B is dense in $\operatorname{Ker} M$, because $P\mathfrak{D}(A)$ is dense in $\operatorname{Ker} M$ and a subset of B. Hence $B^{\perp} = (\operatorname{Ker} M)^{\perp} = (\operatorname{Ran} M)^c$. Finally, $\operatorname{Ran} A \subset (\operatorname{Ran} M)^c$ is equivalent to $\mathfrak{D}(A) = \mathfrak{D}_A$. Hence $\mathfrak{D}_A$ is dense by Assumption 2.1. □

Remark 2.2. (cf. [17]). Let M be self-adjoint and A be sectorial (with vertex 0 and semi-angle θ, cf. [12], Sect. VI.1.2). Then Case (A) holds.

Proof. Let $x \in \operatorname{Ker} M \cap \mathfrak{D}_A$. Since M is self-adjoint, we have $A\mathfrak{D}_A \perp \operatorname{Ker} M$, hence $(x, Ax) = 0$. But A is sectorial, and therefore

$$|(y, Ax)|^2 \leq (1 + \tan\theta)^2 \operatorname{Re}(y, Ay) \operatorname{Re}(x, Ax) = 0$$

for all $y \in \mathfrak{D}(A)$ (see [12], loc.cit.). Since $\mathfrak{D}(A)$ is dense by Assumption 2.1, we have $Ax = 0$, i.e., $x \in \operatorname{Ker} A$. □

Showalter uses this result as an important ingredient in Ref. [17]. In his setup M and A are defined in the sense of quadratic forms. M must be non-negative in order to define a semi-scalar product. (See also [1], [15] and [16]).

The degenerate Cauchy problem Eq. (2.1) is expected to have unique solutions only if $\operatorname{Ker} M \cap \operatorname{Ker} A = \{0\}$, and it is this case which we want to investigate further.

Example 2.1. Let $\mathfrak{H} = \mathfrak{K} = L^2(\mathbb{R})$, $A = d^2/dx^2$ on its natural domain

$$\mathfrak{D}(A) = \{z \in L^2 \mid z, z' \text{ absolutely continuous}, z'' \in L^2\} \equiv W^{2,2}(\mathbb{R}).$$

Let I be the open interval $(-1, 1)$ and M be multiplication by the characteristic function of $\mathbb{R} \setminus I$. Clearly, $\operatorname{Ran} M$ is closed and consists of all L^2-functions vanishing on I. Hence $\mathfrak{D}_A$ consists of those functions $z \in \mathfrak{D}(A)$ which are linear on I and thus satisfy

$$z'(+1) - z'(-1) = 0, \qquad z(+1) - z(-1) = 2z'(-1). \tag{2.7}$$

The set $P^\perp \mathfrak{D}_A \subset \operatorname{Ran} M$ consists of all functions $u \in L^2$ vanishing on I, which together with their derivative u' are absolutely continuous on $\mathbb{R} \setminus I$, such that u'' is in $L^2(\mathbb{R} \setminus I)$, and such that the boundary conditions (2.7) are satisfied for u. On this domain we define the operator

$$(A_0 u)(x) = \begin{cases} 0 & \text{if } x \in I, \\ u''(x) & \text{if } x \notin I, \end{cases} \qquad \text{for all } u \in \mathfrak{D}(A_0) \equiv P^\perp \mathfrak{D}_A. \tag{2.8}$$

$\mathfrak{D}(A_0)$ is the set of all $u \in \operatorname{Ran} M$ for which there exists a unique $v \in \operatorname{Ker} M$ such that $z = u + v \in \mathfrak{D}_A$. (The function v interpolates linearly on I between the boundary values of u and vanishes outside I). Moreover, $A_0 u = Az$. This is the situation described in case (B) above. Hence the degenerate Cauchy problem Eq. (2.1) is equivalent to the ordinary Cauchy problem

$$\frac{d}{dt}\, u(t) = A_0\, u(t), \qquad u(0) = P^\perp z_0 \tag{2.9}$$

in the Hilbert space $L^2(\mathbb{R} \setminus I)$. It is easy to see that the operator $A_0 = d^2/dx^2$ with the above boundary conditions is self-adjoint and negative. Hence for each $z_0 \in \mathfrak{D}_A$, Eq. (2.9) has a unique solution $u(t) \in \mathfrak{D}(A_0)$ $(t \geq 0)$ from which it is easy to obtain the unique strict solution of Eq. (2.1).

The restriction to an ordinary Cauchy problem in $(\operatorname{Ker} M)^\perp$ is sometimes even possible, if the operator A does not leave this subspace invariant. In order to investigate this case further we make the following assumption.

Assumption 2.2. *$P\mathfrak{D}_A \subset \mathfrak{D}(A)$ and the operator $(QAP) \restriction P\mathfrak{D}_A$ has a bounded inverse (which is an operator from $Q\mathfrak{K}$ to $P\mathfrak{D}_A$).*

Example 2.2. Let $\mathfrak{H} = L^2(\mathbb{R})^2 = L^2(\mathbb{R}, \mathbb{C}^2)$, and let

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & a \end{pmatrix}.$$

The matrix differential operator A is defined on its natural domain, the Sobolev space $W^{1,2}(\mathbb{R})^2$. Assumption 2.2 implies $a \neq 0$. We find that

$$\mathfrak{D}_A = \left\{ \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in W^{1,2}(\mathbb{R})^2 \;\middle|\; g(x) = -\tfrac{1}{a}\, f'(x) \right\}.$$

We see that $z = \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \mathfrak{D}_A$ implies $f \in W^{2,2}(\mathbb{R})$. Moreover, $z \in \mathfrak{D}_A$ is uniquely determined by its part $f = P^\perp z$ in $(\operatorname{Ker} M)^\perp$. This is also the content of the next lemma.

Lemma 2.2. *Under the Assumptions 2.1 and 2.2 a vector $z \in \mathfrak{H}$ is in the subspace $\mathfrak{D}_A$ if and only if*

$$z \in \mathfrak{D}(A) \;\textit{ and }\; Pz = -(QAP)^{-1} QAP^\perp z.$$

Proof. By definition, $z \in \mathfrak{D}_A$ iff $z \in \mathfrak{D}(A)$ and $QAz = 0$. By Assumption 2.2 the vectors Pz and $P^\perp z$ are in $\mathfrak{D}(A)$. Writing $z = Pz + P^\perp z$ gives $QAPz + QAP^\perp z = 0$ from which the result follows immediately, since QAP is invertible on the range of the projection Q. ☐

Remark 2.3. According to Lemma 2.2, any $x \in P^\perp \mathfrak{D}_A \subset (\operatorname{Ker} M)^\perp$ uniquely determines $z \in \mathfrak{D}_A$ such that $x = P^\perp z$:

$$z = (\mathbf{1} - (QAP)^{-1} QA)x. \tag{2.10}$$

Hence the set $(P^\perp)^{-1}\{x\} \cap \mathfrak{D}_A$ contains precisely one element, which is just the case (B) described above.

Remark 2.4. From Lemma 2.2 above we conclude for $z \in \operatorname{Ker} M \cap \mathfrak{D}_A$ that $x = P^\perp z = 0$ and hence $z = 0$ by Eq. (2.10). Therefore $\operatorname{Ker} M \cap \mathfrak{D}_A = \{0\}$, and the condition (A) described above is satisfied in a trivial way. In particular, because of Eq. (2.2), we have

$$\operatorname{Ker} M \cap \operatorname{Ker} A = \{0\}.$$

Remark 2.5. Example 2.1 is a counter-example for Assumption 2.2, because in this case $P\mathfrak{D}_A \not\subset \mathfrak{D}(A)$. (Note, however, that M and $-A$ are non-negative and thus Remark 2.2 applies).

According to the Remark 2.3 we define the operator Z_A by

$$Z_A = P^\perp - P(QAP)^{-1} QAP^\perp. \tag{2.11}$$

This operator is defined on $\mathfrak{D}(Z_A) \supset P^\perp \mathfrak{D}_A$. The restriction $Z_A \restriction P^\perp \mathfrak{D}_A$, given by $\mathbf{1} - (QAP)^{-1} QA$ on $P^\perp \mathfrak{D}_A$, is the inverse of the projection $P^\perp \restriction \mathfrak{D}_A$:

$$Z_A P^\perp = \mathbf{1} \quad \text{on } \mathfrak{D}_A, \qquad P^\perp Z_A = \mathbf{1} \quad \text{on } P^\perp \mathfrak{D}_A. \tag{2.12}$$

Now we simply set

$$A_0 = AZ_A \quad \text{on } \mathfrak{D}(A_0) = P^{\perp}\mathfrak{D}_A \tag{2.13}$$

and for all $z \in \mathfrak{D}_A$ we find, writing $x = P^{\perp}z$,

$$A_0 x = Az. \tag{2.14}$$

Since $Az = Q^{\perp}Az$ for all $z \in \mathfrak{D}_A$, the operator A_0 can be written in the more symmetric form

$$A_0 = Q^{\perp}AP^{\perp} - Q^{\perp}AP\,(QAP)^{-1}\,QAP^{\perp} \quad \text{on } \mathfrak{D}(A_0). \tag{2.15}$$

Obviously, we can also factorize A_0 with the help of the operator

$$Y_A = Q^{\perp} - Q^{\perp}AP\,(QAP)^{-1}\,Q. \tag{2.16}$$

From $Y_A AP = 0$ we find $Y_A AP^{\perp} = Y_A A$ and

$$A_0 = Y_A A \quad \text{on } \mathfrak{D}(A_0) = P^{\perp}\mathfrak{D}_A. \tag{2.17}$$

Remark 2.6. For the Example 2.2 above A_0 can be identified with $-\frac{1}{a}\partial^2/\partial x^2$ on $W^{2,2}(\mathbb{R})$ and $M^{\perp} = \mathbf{1}$ on $L^2(\mathbb{R})$. Hence for $a < 0$ the factorized system defines a well-posed (non-degenerate) Cauchy problem (in the Hilbert space $L^2(\mathbb{R})$).

Remark 2.7. In the finite dimensional case, it is well known that Eq. (2.1) has only unique solutions if the matrix pencil (A, M) is similar to $(A_1 \oplus \mathbf{1}, \mathbf{1} \oplus N)$, where N is nilpotent (see [2]). This is equivalent to (A, M) being a regular pencil, which means that there exists a $\lambda_0 \in \mathbb{C}$ such that $A - \lambda_0 M$ is invertible. We are going to assume this type of regularity.

Assumption 2.3. *The operator A has a bounded inverse.*

Remark 2.8. Under Assumption 2.1 this is equivalent to A injective with $\operatorname{Ran} A = \mathfrak{K}$. By Eq. (2.2), this implies that $A \restriction \mathfrak{D}_A$ has a bounded inverse as an operator from $\mathfrak{D}_A$ to $Q^{\perp}\mathfrak{K}$. In particular, $A\mathfrak{D}_A = Q^{\perp}\mathfrak{K}$. In this case the operator A_0 is also invertible, the inverse operator

$$A_0^{-1} = (AZ_A)^{-1} = P^{\perp}A^{-1} \restriction Q^{\perp}\mathfrak{K} \tag{2.18}$$

is bounded and defined on all of $Q^{\perp}\mathfrak{K}$.

Remark 2.9. Whenever $z(t)$ is a solution of Eq. (2.1), then $y(t) = \exp(-\lambda_0 t)z(t)$ is a solution of Eq. (2.1) with A replaced by $A - \lambda_0 M$. Hence if (A, M) is a regular operator pencil, then Assumption 2.3 means no further restriction.

Lemma 2.3. *Under the Assumptions 2.1—2.3 the operator A_0 defined in (2.13) is closed on $\mathfrak{D}(A_0) = P^{\perp}\mathfrak{D}_A$.*

Proof. Let (x_n) be a sequence in $\mathfrak{D}(A_0)$ which converges to x in $P^{\perp}\mathfrak{H}$, such that $A_0 x_n = AZ_A x_n$ is also convergent. By continuity of A^{-1} (Assumption 2.3) the sequence $z_n \equiv A^{-1}AZ_Ax_n = Z_Ax_n$ is convergent, and for each n, $z_n \in \mathfrak{D}_A$. Since Az_n converges and since A is closed on $\mathfrak{D}_A$ we find $z \in \mathfrak{D}_A$ and

$$Az = \lim Az_n = \lim AZ_Ax_n. \tag{2.19}$$

The projection $P^\perp$ is continuous, hence

$$P^\perp z = \lim P^\perp z_n = \lim P^\perp Z_Ax_n = \lim x_n = x.$$

From $z \in \mathfrak{D}_A$ we therefore conclude that $x = P^\perp z \in P^\perp\mathfrak{D}_A = \mathfrak{D}(A_0)$. Moreover, by (2.12), $z = Z_AP^\perp z$, which implies $Az = AZ_Ax$. Together with (2.19) this implies the result. □

By construction, for all $z \in \mathfrak{D}_A$, we have $Az = A_0x$, where $x = P^\perp z$. Moreover, for $z \in \mathfrak{D}(M)$, $Mz = M^\perp x$ with the invertible operator $M^\perp$ defined in Eq. (2.3). Hence we can replace the degenerate Cauchy problem Eq. (2.1) by

$$\frac{d}{dt}\,M^\perp x(t) = A_0\,x(t), \qquad x(0) = P^\perp z_0. \tag{2.20}$$

How to proceed now depends on the precise assumptions on M.

Assumption 2.4. *Let $\mathfrak{D}_A \subset \mathfrak{D}(M)$ and assume that at least one of the following statements is true:*

(Case a): The operator M has a closed range.

(Case b): The operator M has a closed domain.

Remark 2.10. If M is closed and densely defined (Assumption 2.1), we can give the equivalent formulation (using the closed graph theorem):

(Case a): $(M^\perp)^{-1}$ is bounded and defined on all of $Q^\perp\mathfrak{K}$.

(Case b): $M^\perp$ is bounded and defined on all of $P^\perp\mathfrak{H}$.

If Assumption 2.4 (Case a) is fulfilled, we may define the operator

$$A_1 = A_0(M^\perp)^{-1} \tag{2.21}$$

on the natural domain

$$\mathfrak{D}(A_1) = \{y \in Q^\perp\mathfrak{K} \mid (M^\perp)^{-1}y \in \mathfrak{D}(A_0)\} = M^\perp P^\perp\mathfrak{D}_A = M\mathfrak{D}_A, \tag{2.22}$$

where it is closed, because it is the product of a closed operator A_0 and a bounded operator $(M^\perp)^{-1}$. The operator A_1 is densely defined in the Hilbert space

$$\mathfrak{K}_0 = (M\mathfrak{D}_A)^c. \tag{2.23}$$

If Assumption 2.4 (Case b) is fulfilled, we define instead

$$A_2 = (M^\perp)^{-1}A_0. \tag{2.24}$$

This operator is closed on

$$\mathfrak{D}(A_2) = \{x \in P^\perp\mathfrak{D}_A \mid A_0x \in \mathrm{Ran}\,M\} = A_0^{-1}\mathrm{Ran}\,M, \tag{2.25}$$

because it is the product of a boundedly invertible operator $(M^{\perp})^{-1}$ with a closed operator A_0. The operator A_2 is densely defined in the Hilbert space

$$\mathfrak{H}_0 = (P^{\perp}\mathfrak{D}_A)^c. \tag{2.26}$$

Assumption 2.5. *A_1 (resp. A_2) generates a strongly continuous semigroup in $\mathfrak{K}_0$ (resp. $\mathfrak{H}_0$).*

Theorem 2.1. *The Assumptions 2.1—2.5 imply the following statements (Case a): For each initial value $z_0 \in \mathfrak{D}_A$ the degenerate Cauchy problem (2.1) has the unique strict solution*

$$z(t) = Z_A(M^{\perp})^{-1}e^{A_1 t}Mz_0. \tag{2.27}$$

(Case b): For each initial value $z_0 \in A^{-1}Ran\,M$ the unique strict solution of Eq. (2.1) is given by

$$z(t) = Z_A e^{A_2 t}P^{\perp}z_0. \tag{2.28}$$

Proof. (Case a): Let $z_0 \in \mathfrak{D}_A$. Then $Mz_0 \in \mathfrak{D}(A_1)$, cf. Eq. (2.22). By Assumption 2.5 the Cauchy problem

$$\frac{d}{dt}\,y(t) = A_1\,y(t), \qquad y(0) = Mz_0 \tag{2.29}$$

has a unique solution $y(t) = \exp(A_1 t)\,y(0)$ which is continuously differentiable and is in $\mathfrak{D}(A_1)$ for all $t \geq 0$. Since $(M^{\perp})^{-1}$ is bounded, $x = (M^{\perp})^{-1}y$ is again continuously differentiable with $x(t) \in P^{\perp}\mathfrak{D}_A$. Define $z(t) = Z_A x(t)$. Then $Mz = M^{\perp}x = y$ is continuously differentiable and

$$Az(t) = A_0 x(t) = A_1 y(t) = \frac{d}{dt}\,y(t) = \frac{d}{dt}\,Mz(t).$$

This shows that Az is continuous, and since A^{-1} is bounded, z is also continuous. Hence Eq. (2.27) is the unique strict solution of the degenerate Cauchy problem Eq. (2.1) in the sense of Definition 2.1.

(Case b): For $z_0 \in \mathfrak{D}_A$ with $Az_0 \in \operatorname{Ran} M$ we have $P^{\perp}z_0 \in \mathfrak{D}(A_2)$, cf. Eq. (2.25). Hence the Cauchy problem

$$\frac{d}{dt}\,x(t) = A_2\,x(t), \qquad x(0) = P^{\perp}z_0 \tag{2.30}$$

has the unique solution $x(t) = \exp(A_2 t)\,x(0)$, which is continuously differentiable and in $\mathfrak{D}(A_2)$ for all $t \geq 0$. Hence $z(t) = Z_A x(t) \in \mathfrak{D}_A$, and $Mz(t) = M^{\perp}x(t)$ is continuously differentiable, because $M^{\perp}$ is bounded. Hence

$$\frac{d}{dt}\,Mz(t) = M^{\perp}A_2 x(t) = A_0 x(t) = Az(t)$$

is continuous. This implies (by the boundedness of A^{-1}) that z is continuous. Therefore Eq. (2.28) is the unique strict solution of Eq. (2.1). $\square$

Remark 2.11. If $M^\perp$ is bounded and has a bounded inverse, then the cases a and b are equivalent and

$$\begin{aligned}&\mathfrak{D}(A_2) = (M^\perp)^{-1}\mathfrak{D}(A_1), \quad A_2 = (M^\perp)^{-1}A_1M^\perp,\\ &e^{A_2t} = (M^\perp)^{-1}e^{A_1t}M^\perp.\end{aligned}$$

Remark 2.12. If, in addition to the assumptions of the theorem, A is self-adjoint in $\mathfrak{H} = \mathfrak{K}$, and M is self-adjoint and positive, then it is useful to factorize the degenerate Cauchy problem in a more symmetric way. If $M^\perp$ is bounded and has a bounded inverse, then the same is true for the positive square root $(M^\perp)^{1/2}$. Write $u(t) = (M^\perp)^{1/2}x(t)$ to obtain the ordinary Cauchy problem $\dot{u}(t) = A_3u(t)$ with

$$A_3 = (M^\perp)^{-1/2}\,A_0\,(M^\perp)^{-1/2}.$$

Here the generator A_3 is self-adjoint, if it is densely defined.

The conditions necessary for Assumption 2.5 to hold are usually given in terms of the resolvent of the generator (Hille-Yoshida theorem). In the next lemma the resolvents of A_1 and A_2 are expressed in terms of the operators A and M.

Lemma 2.4. *Under the Assumptions 2.1—2.4 we find for λ in the resolvent set $\rho(A_1)$ of A_1 (Case a)*

$$(A_1 - \lambda)^{-1}Q^\perp = M(A - \lambda M)^{-1}Q^\perp,$$

and for $\lambda \in \rho(A_2)$ (Case b)

$$(A_2 - \lambda)^{-1}P^\perp = P^\perp(A - \lambda M)^{-1}M.$$

Proof. We note that $\mathfrak{D}(M) \supset \mathfrak{D}_A$ and hence $\mathfrak{D}(M^\perp) \supset \mathfrak{D}(A_0)$. Thus we calculate (in Case a):

$$\begin{aligned}(A_1 - \lambda)^{-1}Q^\perp &= \Big(A_0(M^\perp)^{-1} - \lambda M^\perp(M^\perp)^{-1}\Big)^{-1}Q^\perp\\ &= M^\perp(A_0 - \lambda M^\perp)^{-1}Q^\perp\\ &= M^\perp\Big((A - \lambda M^\perp P^\perp)Z_A\Big)^{-1}Q^\perp\\ &= M(A - \lambda M)^{-1}Q^\perp,\end{aligned}$$

and (in Case b):

$$\begin{aligned}(A_2 - \lambda)^{-1}P^\perp &= \Big((M^\perp)^{-1}A_0 - \lambda(M^\perp)^{-1}M^\perp\Big)^{-1}P^\perp\\ &= (A_0 - \lambda M^\perp)^{-1}M^\perp P^\perp\\ &= \Big((A - \lambda M^\perp P^\perp)Z_A\Big)^{-1}M\\ &= P^\perp(A - \lambda M)^{-1}M. \end{aligned} \tag{2.31}$$

In these calculations we used Eq. (2.12). □

Hence the assumptions on the resolvents of A_1 (or A_2) can be expressed in terms of the operators

$$S(\lambda) = M(A - \lambda M)^{-1}, \qquad \text{or} \quad R(\lambda) = (A - \lambda M)^{-1}M.$$

If, for some $\lambda \in \mathbb{C}$, the operator $S(\lambda)$ is bounded and everywhere defined, we say that λ belongs to the generalized resolvent set $\rho_S(A)$. If the same holds for $R(\lambda)$, we say $\lambda \in \rho_R(A)$. The operators $S(\lambda)$ and $R(\lambda)$ are pseudoresolvents, i.e., they satisfy the resolvent equation

$$S(\lambda) - S(\mu) = (\lambda - \mu)S(\lambda)S(\mu), \quad \text{all } \lambda \text{ and } \mu \text{ in } \rho_S(A),$$

and similarly for R. Thus the generalized resolvent sets are open subsets of $\mathbb{C}$. If Assumptions 2.1—2.4 hold, then

$$\rho_S(A) = \rho(A_1), \qquad \text{or} \quad \rho_R(A) = \rho(A_2)$$

(in Case a or b, respectively). From general semigroup theory we find, e.g., the following sufficient condition replacing Assumption 2.5:

Assumption 2.6. *Let $\{\lambda \in \mathbb{C} \mid Re\,\lambda > \omega\} \subset \rho_S(A)$ for some real constant ω, and assume that there exists a constant $1 \geq K > 0$ such that*

$$\|M(A - \lambda M)^{-1}Q^{\perp}\| \leq \frac{K}{Re\,\lambda - \omega}, \quad \text{for all } \lambda \text{ with } Re\,\lambda > \omega, \tag{2.32}$$

(or assume a similar condition on $P^{\perp}(A - \lambda M)^{-1}M$ in Case b).

Remark 2.13. In view of Remark 2.9 we can assume without loss of generality that the constant ω in Eq. (2.32) is negative.

Remark 2.14. The operator A_1 is invertible, its inverse is given by

$$(A_1)^{-1} = M^{\perp}(A_0)^{-1} = MA^{-1} \upharpoonright Q^{\perp}\mathfrak{K}.$$

Since $(A_1)^{-1}$ is closed and defined on all of $Q^{\perp}\mathfrak{K}$, it is bounded. The operator $T = MA^{-1}$ is the object of investigation in Favini's paper [5] and, e.g., in [7], [11]. Instead of (2.1), Favini considers the equation

$$\frac{d}{dt}T\,z(t) = z(t)$$

under suitable assumptions on T. These assumptions are formulated in terms of the operator $A(A - \lambda M)^{-1}$ and insure analyticity of the semigroup generated by the inverse of $T \upharpoonright Q^{\perp}\mathfrak{K}$ (which is just A_1).

In the general case we expect no continuous dependence on the initial conditions, because the operator Z_A in Eqs. (2.27) and (2.28) is not continuous. In special cases, however, the time evolution defined in Theorem 2.1 extends to a bounded operator.

Example 2.3. For simplicity, we assume that $M^\perp$ is bounded and has a bounded inverse. Hence $\rho_S(A) = \rho_R(A)$. Let A_2 be the generator of a bounded holomorphic semigroup (see [12], Section IX.1.6). This is the case, if $\rho_R(A)$ contains a sector $|\arg z| < \frac{\pi}{2} + \delta$, $\delta > 0$, and if

$$\|(A - \lambda M)^{-1} M\| \leq \frac{K}{|\lambda|} \quad \text{for } |\arg\lambda| < \tfrac{\pi}{2} + \delta - \epsilon \tag{2.33}$$

holds for any $\epsilon > 0$, with a suitable constant $K > 0$. Thus $\exp A_2 t$ is given by a Dunford-Taylor integral (where Γ is a suitable path within the sector mentioned above, see [12], p. 490)

$$e^{A_2 t} = -\frac{1}{2\pi \mathrm{i}} \int_\Gamma e^{\lambda t} \, (A_2 - \lambda)^{-1} \, d\lambda. \tag{2.34}$$

Inserting this into Eq. (2.28) and using

$$Z_A \, (A_2 - \lambda)^{-1} \, P^\perp = (A - \lambda M)^{-1} M = R(\lambda)$$

(cf. Eq. (2.31)) we obtain immediately

$$z(t) = -\frac{1}{2\pi \mathrm{i}} \Big(\int_\Gamma e^{\lambda t} \, R(\lambda) \, d\lambda \Big) z_0. \tag{2.35}$$

It is clear that this defines a bounded evolution operator on all of $\mathfrak{H}$. In this case $z(t)$ depends continuously on the initial value. As $t \to 0$, z might be discontinuous (in t), but $P^\perp z(t) \to P^\perp z_0$. For $z_0 \in \mathfrak{D}_A$, however, z is a strict solution and is therefore continuous in t.

3. Example: The Dirac equation in the nonrelativistic limit

In this section we show that the Dirac equation of relativistic quantum mechanics has a singularity in the nonrelativistic limit which can be understood in the framework of degenerate Cauchy problems. The Dirac equation is a system of linear partial differential equations which has the following abstract structure (after subtraction of the "rest-energy" mc^2, see [18], Chapters 5 and 6 for details)

$$\frac{d}{dt} \, \psi(t) = -\mathrm{i} \Big(cD + mc^2(\tau - \mathbf{1}) + V \Big) \, \psi(t). \tag{3.1}$$

Here $m > 0$ denotes a mass, c the speed of light. τ is a unitary involution (which means $\tau^2 = \tau^*\tau = \tau\tau^* = \mathbf{1}$) in the given Hilbert space $\mathfrak{H}$, the operator D is self-adjoint and anticommutes with τ,

$$\tau \mathfrak{D}(D) \subset \mathfrak{D}(D), \qquad D\tau + \tau D = 0 \quad \text{on } \mathfrak{D}(D).$$

The operator V is symmetric, commutes with τ,

$$\tau \mathfrak{D}(V) \subset \mathfrak{D}(V), \qquad V\tau - \tau V = 0 \quad \text{on } \mathfrak{D}(V),$$

and is relatively bounded with respect to D (but not necessarily with relative bound less than 1).

Remark 3.1. Here we work in a setting which is much more general than relativistic quantum mechanics. In [18] it is shown how the concrete special cases fit into the abstract approach. Here we only mention that the Dirac equation in its usual form is obtained in the special case $\mathfrak{H} = \mathbb{C}^4 \otimes L^2(\mathbb{R}^3)$, $D = -\mathrm{i}\alpha \cdot \nabla$, $\tau = \beta$, where α_1, α_2, α_3, and β are the famous Dirac matrices. In this case V is multiplication by a diagonal matrix-valued function (electric or scalar potential).

The Hilbert space decomposes into a direct sum of the eigenspaces belonging to the eigenvalues ± 1 of τ, $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$. We write $\mathfrak{H}_\pm = P_\pm \mathfrak{H}$, where $P_\pm$ are the orthogonal projections

$$P_\pm = \tfrac{1}{2}\Big(\mathbf{1} \pm \tau\Big). \tag{3.2}$$

In order to investigate the nonrelativistic limit ($c = \infty$) we make for $0 < c < \infty$ the transformation $\psi(t) \to z(t) = (P_+ + cP_-)\psi(t)$ and multiply (3.1) with $P_+ + P_-/c$ from the left. This gives the following equation for z:

$$\frac{d}{dt}\Big(P_+ + P_-/c^2\Big)\, z(t) = -\mathrm{i}\Big(D + VP_+ - 2mP_- + VP_-/c^2\Big)\, z(t).$$

In the nonrelativistic limit $c = \infty$ this becomes formally

$$\frac{d}{dt}\, P_+\, z(t) = -\mathrm{i}\Big(D + VP_+ - 2mP_-\Big)\, z(t). \tag{3.3}$$

This is a degenerate Cauchy problem of the form (2.1), where $M = P_+$ is a self-adjoint projection operator (hence it is bounded and has closed range and both cases (a) and (b) in Assumption 2.4 apply here). The theory described in Section 2 gives the tools for the factorization of the degenerate equation (3.3).

Theorem 3.1. *Under the above assumptions $A = -i(D+VP_+-2mP_-)$ is closed on the domain*

$$\mathfrak{D}_A = \Big\{ z \in \mathfrak{H} \;\Big|\; P_+ z \in \mathfrak{D}(D^2),\, P_- z = \frac{1}{2m}\, D\, P_+ z \Big\}. \tag{3.4}$$

For all initial values $z_0 \in \mathfrak{D}_A$ the degenerate abstract Dirac equation (3.3) has a unique strict solution $z(t)$ which is in $\mathfrak{D}_A$ for all t. Moreover, $z(t)$ is given by

$$z(t) = \Big(1 + \frac{1}{2m}\, D\Big)\, \exp(A_0 t)\, P_+\, z_0, \tag{3.5}$$

where

$$A_0 = -i\Big(\frac{D^2}{2m} + V\Big) P_+. \tag{3.6}$$

The operator iA_0 is self-adjoint on $\mathfrak{D}(A_0) = P_+ \mathfrak{D}_A = \mathfrak{D}(D^2) P_+ \subset \mathfrak{H}_+$.

Proof. First we note that V is bounded relative to D^2 with infinitesimal relative bound, hence $D^2/2m+V$ is self-adjoint on $\mathfrak{D}(D^2)$. The self-adjointness of $\mathrm{i}A_0$ follows immediately. In the notation of Section 2 we have $Q=P=P_-$, because $M=P_+$ is a self-adjoint projection. Note that D maps $\operatorname{Ker} M\cap\mathfrak{D}(D)$ into $\operatorname{Ran} M$ and $\operatorname{Ran} M\cap\mathfrak{D}(D)$ into $\operatorname{Ker} M$. We find

$$\begin{aligned}
QAP^{\perp} &= P_-AP_+ = -\mathrm{i}DP_+ = -\mathrm{i}P_-D,\\
P^{\perp}AQ &= P_+AP_- = -\mathrm{i}DP_- = -\mathrm{i}P_+D,\\
QAP &= P_-AP_- = 2m\mathrm{i}P_-,\\
Q^{\perp}AP^{\perp} &= P_+AP_+ = -\mathrm{i}VP_+.
\end{aligned}$$

The domain $\mathfrak{D}_A=\{z\in\mathfrak{D}(A)\mid QAz=0\}$ is according to Lemma 2.2 characterized as the set of all z with P_+z and P_-z in $\mathfrak{D}(A)=\mathfrak{D}(D)$ and

$$P_-z=\frac{1}{2m}\,D\,P_+z.$$

From this we may conclude that $P_+z\in\mathfrak{D}(D^2)$. Hence $\mathfrak{D}_A$ can be described as in Eq. (3.4), $P_+\mathfrak{D}_A=P_+\mathfrak{D}(D^2)$, and we can define the operator A_0 as in Eq. (2.15) to end up with Eq. (3.6). For all $z\in\mathfrak{D}_A$ we have $Az=A_0P_+z$ and the closedness of A on $\mathfrak{D}_A$ follows easily from the self-adjointness (and hence closedness) of A_0 on $P_+\mathfrak{D}_A$. We have $A_0=A_1=A_2$ because $M^{\perp}=(M^{\perp})^{-1}=1\restriction\mathfrak{H}_+$. Let $x(t)=\exp(A_0t)P_+z_0$ be a solution of the factorized problem with an initial value P_+z_0 in $\mathfrak{D}(A_0)$. Let Z_A be defined as in Eq. (2.11) as the operator which maps $P^{\perp}\mathfrak{D}(D^2)$ onto $\mathfrak{D}_A$,

$$Z_A=\left(1+\frac{1}{2m}\,D\right)P_+.$$

We note that Z_A is well defined (but generally unbounded) on $\mathfrak{D}(D)\supset\mathfrak{D}(A_0)$. In order to show the continuity of the solution $z(t)=Z_Ax(t)$ of the degenerate problem, we had to assume that A^{-1} is bounded (see Assumption 2.3). This is not true here, but a somewhat tedious calculation shows that

$$\begin{aligned}
(\mathrm{i}A-\lambda M)^{-1} &= \Big(D+(V-\lambda)P_+-2mP_-\Big)^{-1}\\
&= \left(Z_A+\frac{D+\lambda}{2m}P_-\right)R(\lambda)+Z_A\Big(R(\lambda)-R_0(\lambda)\Big)\,\frac{D}{2m}\,P_-,
\end{aligned}$$

where

$$R(\lambda)=\left(\frac{D^2}{2m}+VP_+-\lambda\right)^{-1},\qquad R_0(\lambda)=\left(\frac{D^2}{2m}-\lambda\right)^{-1}.$$

Since $D^2/2m$ is nonnegative and V is infinitesimally relatively bounded, $R(\lambda)$ and $R_0(\lambda)$ are bounded for λ sufficiently negative or for λ with $\operatorname{Im}\lambda\neq 0$. This implies also the boundedness of $(\mathrm{i}A-\lambda M)^{-1}$, because $DR(\lambda)D$ and $DR_0(\lambda)D$ are bounded. Hence there exists a $\lambda_0\in\mathbb{C}$ such that $(A-\lambda_0M)^{-1}$ is bounded and we can apply Theorem 2.1 with A replaced by $A-\lambda_0M$ (cf. Remark 2.9).

Note that $Z_{A-\lambda_0 M} = Z_A$ and A_0 has to be replaced by $A_0 - \lambda_0$. Theorem 2.1 now implies that $y(t) = Z_A \exp(A_0 t - \lambda_0 t) P_+ z_0$ is a strict solution of the degenerate problem with $A - \lambda_0 M$. The solution $z(t)$ of the original problem is, according to Remark 2.9, finally given by $\exp(\lambda_0 t) y(t)$. $\square$

Remark 3.2. For the applications in quantum mechanics the reduced Cauchy problem in $\mathfrak{H}_+ = (\operatorname{Ker} M)^\perp$ is the Schrödinger equation: If D is defined as in Remark 3.1 above, then the properties of Dirac matrices imply that $D^2 = -\Delta$ is the Laplace operator.

Remark 3.3. The time evolution $z(t) = Z_A \exp(A_0 t)\, P_+ z_0$ can be defined for all initial values $z_0 \in \mathfrak{D}(D^2)$ and $z(t)$ is even a strict solution if $z_0 \in \mathfrak{D}_A$. Nevertheless the time evolution operator is unbounded and cannot be extended to all of $\mathfrak{H}$.

For example, if $V = 0$, Z_A commutes with $\exp(A_0 t)$ and

$$z(t) = \exp\left(-\mathrm{i}\frac{D^2}{2m} t\right) \left(1 + \frac{1}{2m}\, D\right) P_+ z_0$$

implies that

$$\|z(t)\|^2 = \|P_+ z_0\|^2 + \frac{1}{2m} \|D P_+ z_0\|^2. \tag{3.7}$$

Hence the domain of the evolution operator cannot be extended to initial values $z_0 \notin \mathfrak{D}(D)$. Eq. (3.7) implies that (in the case $V = 0$) $Z_A \exp(A_0 t)\, P_+$ is continuous from $\mathfrak{D}(D)$, equipped with the graph-norm, into the Hilbert space $\mathfrak{H}$. A more detailed investigation the time evolution of the Dirac equation in the non-relativistic limit, scattering theory and relativistic corrections in the framework of degenerate Cauchy problems will be published elsewhere.

4. The inhomogeneous equation

In this section we briefly discuss the modifications necessary in order to treat the inhomogeneous equation

$$\frac{d}{dt} M\, z(t) = A\, z(t) + f(t). \tag{4.1}$$

We make the following simplifying assumption.

Assumption 4.1. *In addition to Assumptions 2.1–2.5 let $M^\perp = M \restriction P^\perp \mathfrak{H}$ be bounded and have a bounded inverse. Furthermore, let A_0 be densely defined in $P^\perp \mathfrak{H}$.*

Some consequences of this assumption are described in the Remark 2.11. We also have $\mathfrak{H}_0 = P^\perp \mathfrak{H}$, and $\mathfrak{K}_0 = Q^\perp \mathfrak{K}$, cf. Eqs. (2.23) and (2.26).

A necessary condition for z to be a solution of Eq. (4.1) is now given by

$$z(t) = Z_A P^\perp z(t) - (QAP)^{-1} Q f(t), \quad \text{for all } t \geq 0 \tag{4.2}$$

where Z_A is defined as in Eq. (2.11). This is an immediate consequence of the requirement that $Az(t) + f(t)$ be in $\operatorname{Ran} M$. We can easily restrict (4.1) to

$$\begin{aligned} \frac{d}{dt} M^\perp x(t) &= A_0\, x(t) + \Big(Q^\perp - Q^\perp AP(QAP)^{-1}Q\Big) f(t) \\ &= A_0\, x(t) + Y_A\, f(t), \end{aligned} \tag{4.3}$$

where $A_0 = AZ_A = Y_A A$ is given as in Eqs. (2.13) and (2.17). Under the assumptions made above there are two equivalent methods of dealing with Eq. (4.3) (cf. cases (a) and (b) described in Section 2), either by transforming to

$$\frac{d}{dt}\, y(t) = A_1\, y(t) + Y_A\, f(t), \tag{4.4}$$

(where $y(t) = M^\perp x(t)$) or to

$$\frac{d}{dt}\, x(t) = A_2\, x(t) + (M^\perp)^{-1} Y_A\, f(t), \tag{4.5}$$

where (as in Section 2)

$$A_1 = A_0 (M^\perp)^{-1}, \quad A_2 = (M^\perp)^{-1} A_0.$$

Since we assume that A^{-1} is bounded, we can define $g(t) = P^\perp A^{-1} f(t)$ and write Eq. (4.5) as

$$\frac{d}{dt}\, x(t) = A_2 \Big(x(t) + g(t)\Big). \tag{4.6}$$

Whenever $g(t)$ is in $\mathfrak{D}(A_2) = P^\perp \mathfrak{D}_A$, a solution of this equation is given by:

$$\begin{aligned} x(t) &= e^{A_2 t} P^\perp z_0 + \int_0^t e^{A_2(t-s)} A_2 g(s)\, ds \\ &= e^{A_2 t} P^\perp z_0 + A_2 \int_0^t e^{A_2(t-s)} g(s)\, ds. \end{aligned}$$

It is well known (see [12], Section IX.1.5) that the expression

$$h(t) = A_2 \int_0^t e^{A_2(t-s)} g(s)\, ds \tag{4.7}$$

is well defined whenever $g(t)$ is a continuously differentiable function of t (which is the case if f is continuously differentiable). In order to obtain a solution of the original degenerate problem we have to build the expression

$$z(t) = Z_A x(t) - (QAP)^{-1} Q f(t) \tag{4.8}$$

which is only possible if $x(t) \in \mathfrak{D}(Z_A)$.

Remark 4.1. In the important special case where the factorization operator Z_A is bounded, the solution $z(t)$ is well defined for all initial values z_0 and for all continuously differentiable functions f. It is a strict solution in the sense of Definition 2.1 only if $z(t) \in \mathfrak{D}_A$ and $z_0 \in \mathfrak{D}_A$.

Remark 4.2. Another special case, where $z(t)$, given by Eq. (4.8), is well defined is obtained if A and M satisfy Eq. (2.33) (see, e.g., [11]). In this case A_2 generates a bounded analytic semigroup and $h(t)$ is in $\mathfrak{D}(A_2)$, because $A_2^2 \exp(A_2 t)$ is bounded.

Finally we discuss the inhomogeneous degenerate equation for functions f with values in $\operatorname{Ran} M$. The following theorem is a direct analogue of Theorem 2.1. Under the simplifying assumptions of this section (cf. also Remark 2.11) it can be formulated as follows.

Theorem 4.1. *Let f be a continuously differentiable function with values in $\operatorname{Ran} M$. Under Assumption 4.1, Eq. (4.1) has a unique strict solution z for each initial value $z_0 \in \mathfrak{D}_A$. It is given by*

$$\begin{aligned} z(t) &= Z_A\, x(t) - (QAP)^{-1} Q\, f(t) = Z_A\, x(t), \\ x(t) &= e^{A_2 t} P^{\perp} z_0 + \int_0^t e^{A_2(t-s)}\, (M^{\perp})^{-1} f(s)\, ds. \end{aligned}$$

Proof. For every initial value $P^{\perp} z_0 \in \mathfrak{D}(A_2) = P^{\perp}\mathfrak{D}_A$ the function $t \to x(t)$ is continuously differentiable, and $x(t) \in \mathfrak{D}(A_2) = P^{\perp}\mathfrak{D}_A$ for all $t \geq 0$ (see [12], Section IX.1.5). Therefore $z(t) = Z_A x(t)$ is well defined. Using the continuity of $M^{\perp}$ we find that $M z(t) = M^{\perp} x(t)$ is continuously differentiable with

$$\begin{aligned} \frac{d}{dt}\, M z(t) &= M^{\perp}\, \frac{d}{dt}\, x(t) \\ &= M^{\perp}\Big(A_2\, x(t) + (M^{\perp})^{-1} f(t)\Big) \\ &= A z(t) + f(t). \end{aligned}$$

This shows that $Az(t)$ and hence $z(t) = A^{-1} A z(t)$ is continuous. Hence z is a strict solution in the sense of Definition 2.1. This finishes the proof. Let us finally note that $x(t)$ can be defined for all z_0, and for each $t \geq 0$

$$f(t) = \int_0^t e^{A_2(t-s)}\, (M^{\perp})^{-1} f(s)\, ds$$

is in $\mathfrak{D}(A_2)$ and hence $Z_A f(t)$ is well defined. But $Z_A \exp(A_2 t) P^{\perp}$ need not be defined for initial values not in $\mathfrak{D}_A$, because Z_A may be unbounded. $\square$

5. Some remarks about control theory

In this section we make some remarks concerning the basic concepts of control theory under the simplifying assumptions of the previous section. Leaving a more exhaustive treatment of degenerate control problems to further investigation, our purpose here is to illustrate the possible usefulness of our approach by discussing a special case to be described in the following.

In order to define a degenerate control problem, we need an operator B mapping a control space U into $\mathfrak{K}$, and an operator C mapping $\mathfrak{H}$ into the output space V, U and V being suitable Banach spaces. The system of equations

$$\frac{d}{dt} M\, z(t) = A\, z(t) + B\, u(t), \tag{5.1}$$

$$v(t) = Cz(t). \tag{5.2}$$

will be called the control problem defined by the ordered quadrupel (M, A, B, C). Besides the Assumption 4.1 we assume that B and C are bounded and satisfy

$$\operatorname{Ran} B \subset (\operatorname{Ran} M)^c, \qquad \operatorname{Ker} C \supset \operatorname{Ker} M. \tag{5.3}$$

Hence Theorem 4.1 applies to this situation and Eq. (5.1) has a unique strict solution $z(t, u, z_0)$ whenever Bu is continuously differentiable and $z_0 \in \mathfrak{D}_A$.

Remark 5.1. The output $v(t) = Cx(t)$ is well defined and continuous in t for all initial values $z_0 \in \mathfrak{H}$. This is because by Eq. (5.3) $Cz = Cx$ holds for any z and $x = P^\perp z$. Hence $CZ_A = CP^\perp Z_A = C$ is bounded and $CZ_A \exp(A_2 t)P^\perp$ extends to all of $\mathfrak{H}$.

We will also consider the "dual problem" defined by $(-M^*, A^*, C^*, B^*)$. Because of Eq. (5.3) we find that

$$\operatorname{Ran} C^* \subset (\operatorname{Ran} M^*)^c, \qquad \operatorname{Ker} B^* \supset \operatorname{Ker} M^*. \tag{5.4}$$

Under the Assumptions 2.1–2.5 of Section 2 we can easily restrict (5.1), (5.2) to

$$\frac{d}{dt} M^\perp\, x(t) = A_0\, x(t) + B\, u(t), \tag{5.5}$$

$$v(t) = Cx(t).$$

where A_0 is given as in Eqs. (2.14) and (2.15).

Because of Eq. (5.4), the dual problem can be restricted in a similar way. When defining the factorization operators Z_{A^*} and Y_{A^*} for the dual system one has to take into account that the roles of Q and P are interchanged. Hence

$$Z_{A^*} = Q^\perp - Q(PA^*Q)^{-1}PA^*Q^\perp = (Y_A)^*, \qquad Y_{A^*} = (Z_A)^*$$

and we can define $(A^*)_0 = A^* Z_{A^*} = Y_{A^*} A^*$. We see immediately that

$$(A^*)_0 = P^\perp A^* Q^\perp - P^\perp A^* Q(PA^*Q)^{-1}PA^*Q^\perp = (A_0)^*. \tag{5.6}$$

With Assumption 4.1 the two formulations Case a and b of Section 2 are equivalent, and we have two equivalent methods of dealing with (5.5): Method (a) shows that the control problem (M, A, B, C) in $\mathfrak{H}$ can be reduced to $(\mathbf{1}, A_1, B_1, C_1)$ in $Q^{\perp}\mathfrak{K}$, where

$$A_1 = A_0(M^{\perp})^{-1}, \quad B_1 = B, \quad C_1 = C(M^{\perp})^{-1}.$$

On the other hand, by method (b) we obtain an equivalent problem in $P^{\perp}\mathfrak{H}$ which is defined by the operators $(\mathbf{1}, A_2, B_2, C_2)$, where

$$A_2 = (M^{\perp})^{-1}A_0, \quad B_2 = (M^{\perp})^{-1}B, \quad C_2 = C.$$

The dual problem $(-M^*, A^*, C^*, B^*)$ can be factorized in a similar way. This leads to an ordinary control problem $(-\mathbf{1}, (A^*)_1, (C^*)_1, (B^*)_1)$ in $P^{\perp}\mathfrak{H}$ (=factorization by method a) and to $(-\mathbf{1}, (A^*)_2, (C^*)_2, (B^*)_2)$ in $Q^{\perp}\mathfrak{K}$ (method b). Using Eq. (5.6) and

$$(M^*)^{\perp} = (M^{\perp})^*,$$

we obtain the following relations

$$\begin{aligned}
&(A^*)_1 = (A^*)_0(M^{*\perp})^{-1} = ((M^{\perp})^{-1}A_0)^* = (A_2)^*,\\
&(C^*)_1 = C^* = (C_2)^*,\\
&(B^*)_1 = B^*(M^{*\perp})^{-1} = ((M^{\perp})^{-1}B)^* = (B_2)^*,
\end{aligned}$$

and by similar calculations

$$(A^*)_2 = (A_1)^*, \quad (C^*)_2 = (C_1)^*, \quad (B^*)_2 = (B_1)^*.$$

In the control theory dealing with nondegenerate systems it is customary to define a dual system for each control problem. According to this definition, the dual of the ordinary control problem $(\mathbf{1}, A_1, B_1, C_1)$ obtained by method (a) is given by $(-\mathbf{1}, (A_1)^*, (C_1)^*, (B_1)^*)$. From the result above this dual system is the same as $(-\mathbf{1}, (A^*)_2, (C^*)_2, (B^*)_2)$, which is just the control problem obtained from $(-M^*, A^*, C^*, B^*)$ by factorization via method (b).

Hence we find the following diagram which exhibits the compatibility of duality and factorization:

$$\begin{tikzcd}
(M, A, B, C) \arrow[r, leftrightarrow, "\text{duality}"] \arrow[d, leftrightarrow, "\text{(a)}"'] & (-M^*, A^*, C^*, B^*) \arrow[d, leftrightarrow, "\text{(b)}"] \\
(1, A_1, B_1, C_1) \arrow[r, leftrightarrow, "\text{duality}"] & (-1, (A^*)_2, (C^*)_2, (B^*)_2)
\end{tikzcd}$$

There is, of course, a corresponding diagram with (a) and (b) exchanged:

$$\begin{array}{ccc} (M,A,B,C) & \overset{\text{duality}}{\longleftrightarrow} & (-M^*,A^*,C^*,B^*) \\ \text{(b)}\updownarrow & & \updownarrow\text{(a)} \\ (1,A_2,B_2,C_2) & \overset{\text{duality}}{\longleftrightarrow} & (-1,(A^*)_1,(C^*)_1,(B^*)_1) \end{array}$$

This shows that methods (a) and (b) are dual methods of factorization.

In control theory one is also interested in the possibility of feedback control. This is achieved by an operator K defined on a domain $\mathfrak{D}(K) \subset \mathfrak{H}$ and mapping into the control space U. The feedback-control system is now defined by

$$\frac{d}{dt} M\, z(t) = (A + BK)\, z(t). \tag{5.7}$$

We assume that $\mathfrak{D}(K) \supset \mathfrak{D}(A)$, hence the operator BK is relatively bounded with respect to A. We can factorize (5.7) as in Section 2. The range condition (5.3) implies that $QB = 0$ and hence $\mathfrak{D}_{A+BK} = \mathfrak{D}_A$. We define $(A + BK)_0 = (A + BK)Z_{A+BK}$ on $P^\perp\mathfrak{D}_A$, where

$$\begin{aligned} Z_{A+BK} &= \mathbf{1} - \Big(Q(A+BK)P\Big)^{-1} Q(A+BK) \\ &= \mathbf{1} - (QAP)^{-1}QA = Z_A \quad \text{on } P^\perp\mathfrak{D}_A. \end{aligned}$$

For the factorized problem we therefore obtain (by method a) the generator $A_1 + B_1K_1$, where $K_1 = KZ_A(M^\perp)^{-1}$ and (by method b) $A_2 + B_2K_2$ where $K_2 = K$ (here A_j and B_j, $j = 1, 2$, are defined as above). The solution of the degenerate feedback-system (5.7) is therefore given by

$$z(t) = S_{A+BK}(t)\, z_0$$

with the (possibly unbounded) evolution operator

$$S_{A+BK}(t) = Z_A\, e^{(A_2+B_2K_2)t}\, P^\perp = Z_A\, (M^\perp)^{-1}\, e^{(A_1+B_1K_1)t}\, M^\perp\, P^\perp.$$

The standard notions of control theory can now be generalized to degenerate problems as follows.

Definition 5.1. The degenerate control system (M, A, B, C) is called *approximately controllable*, if for all $\epsilon > 0$, all $T > 0$ and all $z_0, z_1 \in (\mathfrak{D}_A)^c$ there exists a control function u such that $\|z(T, u, z_0) - z_1\| < \epsilon$.

It is called *observable*, if the dual system $(-M^*, A^*, C^*, B^*)$ is approximately controllable.

The degenerate control system is called *stabilizable*, if there exists a bounded operator K such that $S_{A+BK}(t)$ is bounded for $t \geq 0$ with $\|S_{A+BK}(t)\| \leq \mu \exp(-\omega t)$, $\mu \geq 1$, $\omega > 0$.

The degenerate control system is called *detectable*, if the dual system is stabilizable.

Theorem 5.1. *Let (M, A, B, C) satisfy the assumptions formulated in this section and assume that the factorization operator Z_A is bounded. Then the degenerate system (M, A, B, C) is controllable (observable, stabilizable, detectable) if and only if the ordinary control system $(1, A_j, B_j, C_j)$ ($j = 1$ or 2) is controllable (observable, stabilizable, detectable).*

Proof. The proof is immediate after the observations made in this section. If Z_A is bounded, then it extends to a bounded operator defined everywhere on $(P^\perp \mathfrak{D}_A)^c = P^\perp \mathfrak{H}$ and maps this subspace onto $(\mathfrak{D}_A)^c$. A solution of the degenerate problem can thus be defined according to Theorem 4.1 for all initial values z_0 and is given by

$$\begin{aligned} z(t, u, z_0) &= Z_A\, x(t, u, P^\perp z_0), \\ x(t, u, x_0) &= e^{A_2 t}\, x_0 + \int_0^t e^{A_2(t-s)}\, B_2\, u(s)\, ds. \end{aligned}$$

Here $x(t, u, x_0)$ is a solution of the system factorized by method (b). For controllability, we just note that $\|z(T, u, z_0) - z_1\| \leq \|Z_A\|\, \|x(T, u, x_0) - x_1\|$, where $x_{0,1} = P^\perp z_{0,1}$. Hence if x_1 can be approximated by a solution $x(T, u, x_0)$ of the factorized system, then $Z_A x(T, u, x_0)$ is a solution of the degenerate problem which approximates the given vector z_1.

If the factorized system is stabilizable this means that there exists a bounded operator K_2 such that $(A_2 + B_2 K_2)$ is generator of an exponentially stable semigroup. Hence $\|S_{A+BK}(t)\| \leq \|Z_A\|\, \mu\, \exp(-\omega t)$, where $\mu \geq 1$ and $\omega > 0$. The result follows since $\|Z_A\| \geq 1$.

In the same way one shows that controllability and stabilizability of the system factorized according to method (a) implies the corresponding properties of the degenerate system. In a similar way controllability (stabilizability) of the degenerate system implies these properties for the factorized systems.

Concerning the properties observability and detectability, which are defined with the help of the dual system, the commuting diagrams above together with the preceeding part of this proof show the equivalence of the degenerate and the factorized systems. □

Remark 5.2. Under the assumptions of the theorem and according to our observations in Section 2, the degenerate control system is stabilizable, if there exists an operator K and constants $\mu \geq 1$ and $\omega > 0$ such that

$$\|(A + BK - \lambda M)^{-1} M\| \leq \frac{\mu}{\operatorname{Re}\lambda - \omega} \quad \text{for all } \lambda \text{ with } \operatorname{Re}\lambda > \omega.$$

Acknowledgment . We would like to thank W. Desch and F. Kappel for stimulating discussions and helpful suggestions.

References

[1] Carroll, R. W. and Showalter, R. E.: *Singular and degenerate Cauchy problems*, Mathematics in Science and Engineering **127**, Academic Press, New York, San Francisco, London 1976.

[2] Dai, L.: *Singular Control Systems*, Lecture Notes in Control and Information Sciences **118**, Springer-Verlag, Berlin, Heidelberg, New York 1989.

[3] Favini, A., *Laplace transform method for a class of degenerate evolution problems*, Rend. Mat. **12**, 511–536 (1979).

[4] Favini, A., *Controllability conditions of linear degenerate evolution systems*, Appl. Math. Optim. **6**, 153–167 (1980).

[5] Favini, A., *Abstract potential operator and spectral methods for a class of degenerate evolution problems*, J. Diff. Equat. **39**, 212–225 (1981).

[6] Favini, A., *Degenerate and singular evolution equations in Banach space*, Math. Ann. **273**, 17–44 (1985).

[7] Favini, A. and Plazzi, P., *On some abstract degenerate problems of parabolic type–1. The linear case*, Nonlinear Analysis **12**, 1017–1027 (1988).

[8] Favini, A. and Plazzi, P., *On some abstract degenerate problems of parabolic type–2. The nonlinear case*, Nonlinear Analysis **13**, 23–31 (1989).

[9] Favini, A. and Plazzi, P., *On some abstract degenerate problems of parabolic type–3: Applications to linear and nonlinear problems*, Osaka J. Math. **27**, 323–359 (1990).

[10] Favini, A. and Yagi, A., *Multivalued linear operators and degenerate evolution problems*, Annali di Mat. Pura Appl. (1992).

[11] Favini, A. and Yagi, A., *Space and time regularity for degenerate evolution equations*, J. Math. Soc. Japan **44**, 331–350 (1992).

[12] Kato, T.: *Perturbation Theory for Linear Operators (2nd ed)*, Grundlehren der mathematischen Wissenschaften **132**, Springer-Verlag, Berlin, Heidelberg, New York 1980.

[13] Lagnese, J., *Singular differential equations in Hilbert space*, SIAM J. Math. Anal. **4**, 623–637 (1973).

[14] Rutkas, A. G., *The Cauchy problem for the equation $Ax'(t) + Bx(t) = f(t)$.*, Differencialnye-Uravnenija **11**, 1996–2010 (1975).

[15] Showalter, R. E., *Nonlinear degenerate evolution equations and partial differential equations of mixed type*, SIAM J. Math. Anal. **6**, 25–42 (1975).

[16] Showalter, R. E., *Degenerate parabolic initial-boundary value problems*, J. Diff. Eq. **31**, 296–312 (1979).

[17] Showalter, R. E., *Initial and final-value problems for degenerate parabolic evolution systems*, Indiana University Mathematics Journal **28**, 883–893 (1979).

[18] Thaller, B.: *The Dirac Equation*, Text and Monographs in Physics, Springer Verlag, Berlin, Heidelberg, New York 1992.

Inst. f. Mathematik, Universität Graz, Heinrichstr. 36, A-8010 Graz, Austria

E-mail address: `bernd.thaller@kfunigraz.ac.at, sigrid.thaller@kfunigraz.ac.at`