APPROXIMATION OF DEGENERATE CAUCHY PROBLEMS

S. THALLER * AND B. THALLER

Institut für Mathematik, Universität Graz, Heinrichstrasse 36,
A-8010 Graz, Austria

Abstract. We investigate the approximation of a linear homogenuous degenerate Cauchy problem $\frac{d}{dt}Mz(t) = Az(t), z(0) = z_0$, where M and A are closed, densely defined operators in a Hilbert space $\mathfrak{H}$ and M has a nontrivial kernel. Analoguously to the Trotter-Kato Theory for nondegenerate equations we obtain conditions when the convergence of the pseudo resolvents of M and A is equivalent to the convergence of the solutions of the given Cauchy problem. We study the connection between these approximations and approximations of the factorized nondegenerate Cauchy problem connected with the given problem and we give an example where factorization and approximation can be interchanged.

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1. Introduction

In this paper we investigate the approximation of a homogeneous degenerate abstract Cauchy problem given in a Hilbert space $\mathfrak{H}$,

$$\frac{d}{dt} M\, z(t) = A\, z(t), \qquad z(0) = z_0, \tag{1.1}$$

by degenerate Cauchy problems

$$\frac{d}{dt} M^n\, z_n(t) = A^n\, z_n(t), \qquad z_n(0) = z_{n0} \tag{1.2}$$

in some subspaces $\mathfrak{H}^n$. The operators M, M^n, A and A^n are closed and densely defined operators in $\mathfrak{H}$ resp. $\mathfrak{H}^n$. We give conditions on these operators such that the strict solutions of equations (1.2) converge to the strict solution of (1.1). In Section 2 we recapitulate results of [8] and [9] where we have shown under which conditions a Cauchy problem (1.1) can be factorized into a nondegenerate Cauchy problem

$$\frac{d}{dt} x(t) = A_1 x(t) \tag{1.3}$$

on the factor space $\mathfrak{H}/\operatorname{Ker} M$, when strict solutions of (1.1) exist, and when the solutions are described by a semigroup. In Section 3 a result for approximating (1.1) by (1.2) similarily to the Trotter-Kato Theorem for nondegenerate Cauchy problems ([4], [11]) is obtained. We show that if we impose certain conditions on M, M^n, A and A^n the convergence of the pseudo resolvents of A^n and M^n to the pseudoresolvent of A and M in the Hilbert space norm is equivalent to the convergence of the solutions of (1.2) to the solutions of (1.1) in the Hilbert space norm uniformly on compact t-intervals. As the connection between solutions of (1.1) and (1.3) is described by an unbounded linear operator Z_A, it is in general not possible to obtain an approximation of the degenerate equation (1.1) by approximating the factorized nondegenerate problem (1.3). Even if this operator Z_A is bounded it is in general not possible to interchange factorization and approximation, i.e., factorizing the approximating equations (1.2) does not lead to the same result as approximating the factorized Cauchy problem (1.3). In Section 4 we describe a special situation where this is possible. The subspaces $\mathfrak{H}^n$ are finite dimensional and the operators A^n and M^n are defined by $A^n = \tilde{\Pi} A \Pi$ and $M^n = \tilde{\Pi} M \Pi$, where Π is a projection on the subspaces $\mathfrak{H}^n$ and $\tilde{\Pi}$ is a projection on the range of $M\Pi$. This construction of A^n and M^n leads to Galerkin approximation of the factorized equation (1.3) and there is a continuous dependence between the approximating solutions of (1.3) and (1.1). Approximations of degenerate Cauchy problems have been investigated by Lamm-Rosen [5], Rosen-Raghu [7], and Mao-Reich-Rosen [6]. They all used the setting of Carroll-Showalter [1], where the norm of the Hilbert space is defined by the operator M

which has to be positive semidefinite. Therefore, the approximating equations are situated in Hilbert spaces with different norms.

2. Preliminaries

In this section we give a short summary of results in [8] and [9] which we will need for this paper. We consider the homogeneous degenerate abstract Cauchy problem

$$\frac{d}{dt}Mz(t) = Az(t), \qquad z(0) = z_0 \tag{2.1}$$

where M and A are closed, densely defined, linear operators mapping a Hilbert space $\mathfrak{H}$ into itself. For convenience, we assume that the operator M is bounded and has a closed range.

Notation . Let T be a linear operator. By $\mathfrak{D}(T)$, $\operatorname{Ker} T$, and $\operatorname{Ran} T$ we denote its domain, kernel, and range, respectively. The restriction of T to a set $\mathfrak{D}$ smaller than the domain of T will be denoted by $T \restriction \mathfrak{D}$, the closure of T will be denoted by T^c and the adjoint operator will be written as T^*. The degenerate Cauchy problem (2.1) will be denoted by (M, A).

Definition 2.1. A strict solution of the degenerate Cauchy problem Eq. (2.1) is a continuous function $z : [0, \infty) \longrightarrow \mathfrak{H}$ such that $z(t) \in \mathfrak{D}(A) \cap \mathfrak{D}(M)$ for all $t \geq 0$, Mz is continuously differentiable, and Eq. (2.1) holds.

Let P denote the orthogonal projection onto $\operatorname{Ker} M$, Q the projection onto $\operatorname{Ker} M^*$ and let $P^\perp = \mathbf{1} - P$, $Q^\perp = \mathbf{1} - Q$. Our assumptions on M imply that $M^\perp = M \restriction P^\perp\mathfrak{H}$ is bounded and defined on all of $P^\perp\mathfrak{H}$. Moreover, $M^\perp$ is invertible and $(M^\perp)^{-1}$ is bounded on $Q^\perp\mathfrak{H}$.

Remark 2.1. Our restrictive assumptions on the operator M can be weakened, but they are convenient, because they simplify certain domain conditions and enable us to apply all results of [8] and [9]. In a forthcoming paper [10] the theory of degenerate Cauchy problems will be treated under more relaxed conditions on M.

Any strict solution of (2.1) must be in the set

$$\begin{aligned}\mathfrak{D}_A &= \{z \in \mathfrak{D}(A) \mid Az \in (\operatorname{Ran} M)^c\} &(2.2)\\ &= \{z \in \mathfrak{D}(A) \mid QAz = 0\} &(2.3)\end{aligned}$$

The following assumption is necessary in order to obtain uniqueness of strict solutions.

Assumption 2.1. *For every $x \in P^\perp\mathfrak{D}_A$ the set $(P^\perp)^{-1}\{x\} \cap \mathfrak{D}_A$ consists of precisely one element $z \in \mathfrak{D}_A$.*

An equivalent way of stating this assumption would be to require

$$\operatorname{Ker} M \cap \mathfrak{D}_A = \{0\}.$$

Assumption 2.2. *There is a real constant* ω *such that for all* λ *with* $\lambda > \omega$ *the operator* $(A - \lambda M) \restriction \mathfrak{D}_A$ *has a bounded inverse which is defined on all of* $\operatorname{Ran} M$*. Moreover, there exists a constant* $0 < K \leq 1$ *such that*

$$\|P^{\perp}(A - \lambda M)^{-1} M\| \leq \frac{K}{\lambda - \omega} \tag{2.4}$$

for all $\lambda > \omega$*.*

Remark 2.2. By Assumption 2.1 we can define a linear operator Z_A on $P^{\perp}\mathfrak{D}_A$ by $z = Z_A x$, where z is the unique element in the set $(P^{\perp})^{-1}\{x\} \cap \mathfrak{D}_A$. This operator Z_A is the inverse of the projection $P^{\perp}$ on $\mathfrak{D}_A$, i.e.,

$$Z_A P^{\perp} = \mathbf{1} \quad \text{on } \mathfrak{D}_A, \qquad P^{\perp} Z_A = \mathbf{1} \quad \text{on } P^{\perp}\mathfrak{D}_A \tag{2.5}$$

We define the operator A_0, $\mathfrak{D}(A_0) = P^{\perp}(\mathfrak{D}_A)$, by

$$A_0 = A Z_A \tag{2.6}$$

and we assume that $\mathfrak{D}(A_0) = P^{\perp}(\mathfrak{D}_A)$ is dense in $P^{\perp}\mathfrak{H}$.

Remark 2.3. In [8] and [9] it was shown that $(A - \lambda M)Z_A = A_0 - \lambda M^{\perp}$ is closed, A_0 is closed on $P^{\perp}\mathfrak{D}_A$, and the operator $A_1 = (M^{\perp})^{-1} A_0$ is closed on $\mathfrak{D}(A_1) = P^{\perp}\mathfrak{D}_A$. Furthermore,

$$(A_1 - \lambda)^{-1} = P^{\perp}(A - \lambda M)^{-1} M \qquad \text{on } P^{\perp}\mathfrak{H} \tag{2.7}$$

and therefore Eq. (2.4) implies that A_1 generates a c_0-semigroup on $P^{\perp}\mathfrak{H}$.

The following theorem was proved in [8]:

Theorem 2.1. *Let Assumptions 2.1 and 2.2 be fulfilled. Then the degenerate Cauchy problem (2.1) has a unique strict solution* $z(t)$ *for each initial value* $z_0 \in \mathfrak{D}_A$ *and* $z(t) = Z_A x(t)$*, where* $x(t) = e^{A_1 t} P^{\perp} z_0$ *is the solution of the nondegenerate Cauchy problem*

$$\frac{d}{dt} x(t) = A_1 x(t), \qquad x(0) = x_0 = P^{\perp} z_0. \tag{2.8}$$

Remark 2.4. In general, the operator Z_A is not closable. A necessary and sufficient condition for the closability of Z_A is that $\operatorname{Ker} M \cap \mathfrak{D}_A^c = \{0\}$.

Remark 2.5. If Z_A is closable we denote by $\mathfrak{H}_0$ the domain of the closure ${Z_A}^c$. By Lemma 2.2 in [9] $\mathfrak{H}_0$ is a dense subspace of $(P^{\perp}\mathfrak{D}_A)^c$ and, equipped with the graph norm

$$\|x\|_G^2 = \|Z_A^c x\|^2, \tag{2.9}$$

$\mathfrak{H}_0$ is a Hilbert space which is isometrically isomorphic to the closure of $\mathfrak{D}_A$ with respect to the norm in $\mathfrak{H}$. The isomorphism from $\mathfrak{H}_0$ onto $\mathfrak{D}_A$ is given by Z_A, its inverse by $P^{\perp}$.

If $\mathfrak{H}_0$ is an invariant set for the semigroup $e^{A_1 t}$, i.e., $e^{A_1 t}\mathfrak{H}_0 \subset \mathfrak{H}_0$, then the operators $T(t) = Z_A^c e^{A_1 t} P^\perp$ are a semigroup on the Hilbert space $\mathfrak{D}_A^c$. In this case the homogeneous degenerate Cauchy problem (2.1) is completely equivalent to a nondegenerate Cauchy problem on the Hilbert space $(\mathfrak{H}_0, \|.\|_G)$.

It can be shown [10] that Assumption 2.2 together with Z_A closable and $\mathfrak{H}_0$ invariant under $e^{A_1 t}$ are implied by the requirement, that for some $K < 1$

$$\|(A - \lambda M)^{-1} M\| \le \frac{K}{\lambda - \omega} \tag{2.10}$$

for all λ with $\lambda > \omega$.

3. Approximation

In this section we will approximate a degenerate Cauchy problem (2.1) in the sense of Trotter-Kato. We will use the well-known Trotter-Kato Theorem for nondegenerate Cauchy problems in the version given in [3]:

Theorem 3.1. *Let Z and X_n be Banach spaces with norm $\|.\|$, $\|.\|_n$, $n = 1, 2, ...$, respectively, and X be a closed linear subspace of Z. For every $n = 1, 2, ...$ there exist bounded linear operators $P_n : X \longrightarrow X_n$ and $E_n : X_n \longrightarrow Z$ satisfying*

(A1) $\|P_n\| \le M_1, \|E_n\| \le M_2$ where M_1, M_2 are independent of n,

(A2) $P_n E_n = \mathbf{1}_n$, where $\mathbf{1}_n$ is the identity operator on X_n.

Let $A, A_n \in G(M, \omega)$ and let $T(t)$ and $T_n(t)$ be the semigroups generated by A and A_n on X and X_n respectively. Then the following statements are equivalent:

(a) There exists a $\lambda_0 \in \rho(A) \cap \bigcap_{n=1}^{\infty} \rho(A)$ such that, for all $x \in X$,
$\|E_n(A_n - \lambda_0)^{-1} P_n x - (A - \lambda_0)^{-1} x\| \longrightarrow 0$ as $n \longrightarrow \infty$.

(b) For every $x \in X$ and $t \ge 0$, $\|E_n T_n(t) P_n x - T(t)x\| \longrightarrow 0$ as $n \longrightarrow \infty$ uniformly on bounded $t-$intervals.

If (a) or (b) is true, then (a) holds for all λ with $\lambda > \omega$.

We now consider the degenerate Cauchy problem (2.1) in the case where the solution is described by a c_0-semigroup: Let A and M fulfill the assumptions made in Section 2 for factorizing the degenerate Cauchy problem (2.1) into the nondegenerate problem (2.8)

$$\frac{d}{dt} x(t) = A_1 x(t)$$

with $A_1 = (M^\perp)^{-1} A_0 = (M^\perp)^{-1} A Z_A$. Furthermore, assume that Z_A is closable and $e^{A_1 t}(P^\perp \mathfrak{D}_A)^c \subset (P^\perp \mathfrak{D}_A)^c$, i.e., $e^{A_1 t}$ is a c_0-semigroup $T(t) = Z_A^c e^{A_1 t} P^\perp$ in $\mathfrak{D}_A^c$ where $z(t) = T(t) z_0$, $z_0 \in \mathfrak{D}_A$,is a strict solution of (2.1), and its generator

$\tilde{A} = Z_A^c A_1 P^\perp$ is densely defined in $\mathfrak{D}_A$. For this situation we now apply the Trotter-Kato Theorem:

Proposition 3.1. *Let $(\mathfrak{D}_A^c)^N$ be subspaces of $\mathfrak{D}_A^c$ and let $\mathcal{P}^N : \mathfrak{D}_A^c \longrightarrow (\mathfrak{D}_A^c)^N$ be orthogonal projections with $\mathcal{P}^N z \longrightarrow z$ for all $z \in \mathfrak{D}_A^c$. Furthermore, let $\tilde{A}^N, \tilde{A}$ be infinitesimal generators of c_0-semigroups $T^N(t), T(t)$ on $(\mathfrak{D}_A^c)^N$ and $(\mathfrak{D}_A^c)$, respectively, satisfying $\tilde{A}^N \in G(K, \omega)$, $\tilde{A} \in G(K, \omega)$, and assume that there exists a $\lambda \in \rho(\tilde{A}) \cap \bigcap_{N=1}^\infty \rho(\tilde{A}^N)$ such that for all $z \in \mathfrak{D}_A^c$*

$$\|(\tilde{A}^N - \lambda)^{-1} \mathcal{P}^N z - (\tilde{A} - \lambda)^{-1} z\| \longrightarrow 0 \qquad \text{as } N \longrightarrow \infty.$$

Then for all $z \in \mathfrak{D}_A^c$

$$\|T^N(t)\mathcal{P}^N z - T(t)z\| \longrightarrow 0 \qquad \text{as } N \longrightarrow \infty$$

uniformly in t on any compact interval.

Proof. Apply Theorem 3.1 with $Z = \mathfrak{D}_A^c$, $X_n = (\mathfrak{D}_A^c)^N$, $P_n = \mathcal{P}^N$ and $E_n = \iota^N$, where ι^N is the canonical injection, $\iota^N : (\mathfrak{D}_A^c)^N \longrightarrow \mathfrak{D}_A^c$. □

Remark 3.2. Instead of $\rho(\tilde{A})$, the resolvent set of the generator $\tilde{A}$, we could also take the resolvent set of the pseudoresolvent $(A - \lambda M)^{-1} M$.

Remark 3.3. Proposition 3.1 does not say whether the operators $\tilde{A}^N$ belong to an approximating degenerate Cauchy problem or not.

In the next Theorem we want to give conditions for degenerate equations approximating a given equation (2.1).

Theorem 3.2. *Consider the degenerate Cauchy problems*

$$\frac{d}{dt} M z(t) = A z(t)$$

and

$$\frac{d}{dt} M^n z_n(t) = A^n z_n(t) \qquad \text{for all } n \geq 1. \tag{3.1}$$

Let Assumption 2.1 be fulfilled, and assume that there exist $K < 1$ and ω such that

$$\|(A - \lambda M)^{-1} M\| \leq \frac{K}{\lambda - \omega} \tag{3.2}$$

and

$$\|(A^n - \lambda M^n)^{-1} M^n\| \leq \frac{K}{\lambda - \omega} \tag{3.3}$$

for all $n \geq 0$ and $\lambda > \omega$. Let $\mathcal{P}^n$ denote the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{D}_{A^n}^c$. Then the following statements are equivalent:

a) There is a $\lambda \in \rho(M,A) \cap \bigcap_{n=1}^{\infty} \rho(M^n, A^n)$ *such that for all* $z \in \mathfrak{D}_A^c$

$$\|(A^n - \lambda M^n)^{-1} M^n \mathcal{P}^n z - (A - \lambda M)^{-1} M z\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

b) For all $z \in \mathfrak{D}_A^c$ *and for all* $t_0 \geq 0$

$$\|T_n(t)\mathcal{P}^n z - T(t)z\| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

uniformly on bounded t-intervals, where $T(t)$ *and* $T_n(t)$ *are* c_0*-semigroups describing the solutions of (2.1) and (3.1), respectively.*

Proof. By condition (3.2) and Remark 2.5 we find that

$$\tilde{A} = Z_A^c A_1 P^{\perp}$$

is the generator of a c_0-semigroup $T(t)$ and the solution of (2.1) is given by $z(t) = T(t)z_0$ (cf. [9],Theorem 2.2). In the same way we get by (3.3) for all $n \geq 0$ generators

$$\tilde{A}^n = Z_{A^n}(A^n)_1 P_{M^n}^{\perp}$$

of c_0-semigroups $T_n(t)$ which describe the solutions of (3.1) by $z_n(t) = T_n(t)z_{n0}$. Now we can apply Theorem (3.1), setting $\mathfrak{H} = Z$, $\mathfrak{D}_A^c = (X, \|.\|_{\mathfrak{H}})$, $\mathfrak{D}_{A^n}^c = (X_n, \|.\|_{\mathfrak{H}})$ and $\mathcal{P}^n = P_n$, $\mathbf{1} = E_n$. Hence conditions $(A1)$ and $(A2)$ in Theorem (3.1) are fulfilled. Equations (3.2) and (3.3) imply that $\tilde{A}$ and $\tilde{A}^n \in G(K,\omega)$ for all $n \geq 0$. For the resolvent sets $\rho(\tilde{A})$ and $\rho(\tilde{A}^n)$ we get

$$\rho(\tilde{A}) = \rho(A_1) = \rho(M, A),$$

where $\rho(M,A)$ isthe resolvent set of the pseudoresolvent $(A - \lambda M)^{-1}M$, and

$$\rho(\tilde{A}^n) = \rho(M^n, A^n) \qquad \text{for all } n \geq 0.$$

Clearly, the resolvents of $\tilde{A}$ and $\tilde{A}^n$ are given by

$$(\tilde{A} - \lambda)^{-1} = (A - \lambda M)^{-1} M$$

and

$$(\tilde{A}^n - \lambda)^{-1} = (A^n - \lambda M^n)^{-1} M^n \text{for all } n \geq 0$$

(c.f.[8]). □

Example 3.1. Let $\mathfrak{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and consider the one-dimensional Dirac equation

$$\mathrm{i}\frac{d}{dt}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{c^2} \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} V & -\mathrm{i}\frac{d}{dx} \\ -\mathrm{i}\frac{d}{dx} & -1 + \frac{V}{c^2} \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}, \tag{3.4}$$

where c denotes the speed of light, V is a symmetric operator on $\mathfrak{D}(V) \supset W^{2,2}(\mathbb{R})$ describing the potential. If we take the nonrelativistic limit $c \longrightarrow \infty$ (c.f. [8], [9], [2]), we obtain a degenerate Cauchy problem on $\mathfrak{H}$,

$$\frac{d}{dt}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix} = \begin{pmatrix}-\mathrm{i}V & -\frac{d}{dx}\\ -\frac{d}{dx} & \mathrm{i}\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix} \tag{3.5}$$

with matrix-operators

$$M = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \qquad \text{and } A = \begin{pmatrix}-\mathrm{i}V & -\frac{d}{dx}\\ -\frac{d}{dx} & \mathrm{i}\end{pmatrix}. \tag{3.6}$$

We have $M = P^{\perp} = Q^{\perp}$ and $\mathrm{i}A$ is symmetric on $\mathfrak{D}(A) = (\mathfrak{D}(V) \cap W^{1,2}(\mathbb{R})) \oplus W^{1,2}(\mathbb{R})$ (c.f. [9]). Furthermore, it was proved in [9] that $P^{\perp}\mathfrak{D}_A = W^{2,2}(\mathbb{R})$, $Z_A f = \begin{pmatrix}f\\ \mathrm{i}f'\end{pmatrix}$ on $P^{\perp}\mathfrak{D}_A$, $\mathrm{i}A_0 = \mathrm{i}AZ_A = -d^2/dx^2 + V$ is self-adjoint on $\mathfrak{D}(A_0) = W^{2,2}(\mathbb{R})$, $T(t) = Z_A^c e^{A_0 t} P^{\perp}$, $t \in \mathbb{R}$, is a strongly continuous group on $\mathfrak{D}_A^c \subset \mathfrak{H}$ and the operators $T(t)$ are bounded uniformly in t. Hence, (3.5) is factorizable into a nondegenerate problem and (3.2) is fulfilled, because

$$(A - \lambda M)^{-1} M = \begin{pmatrix}\left(-\mathrm{i}(-\frac{d^2}{dx^2} + V) - \lambda\right)^{-1} & 0\\ \mathrm{i}\frac{d}{dx}\left(-\mathrm{i}(-\frac{d^2}{dx^2} + V) - \lambda\right)^{-1} & 0\end{pmatrix}.$$

Setting $c^2 = n$, $n = 1, 2, ...$, the equations (3.4) can be written in the form of Eq. (3.1) with

$$M^n = \begin{pmatrix}1 & 0\\ 0 & \frac{1}{n}\end{pmatrix}, \qquad A^n = \begin{pmatrix}-\mathrm{i}V & -\frac{d}{dx}\\ -\frac{d}{dx} & \mathrm{i} - \mathrm{i}\frac{V}{n}\end{pmatrix}.$$

These operators define a sequence of non-degenerate Cauchy problems, where the operator $\mathrm{i}(M^n)^{-1}A_n$ is the self-adjoint Dirac operator.

The calculations in [2] imply that the condition *a*) of Theorem 3.2 is fulfilled with $\mathcal{P}^n = \mathbf{1}$. To this purpose we note that the operator $B(c)(H(c) - mc^2 - \lambda)^{-1}B(c)^{-1}$ which occurs in Lemma 2.1 in [2] is just another way of writing $(A^n - \lambda M^n)^{-1}M^n$. Lemma 2.1 in [2] now implies that $(A^n - \lambda M^n)^{-1}M^n$ converges in operator norm to $(A - \lambda M)^{-1}M$.Therefore, the solutions of (3.4) converge to the solution of (3.5) uniformly on bounded t-intervals as $n \longrightarrow \infty$.

Remark 3.4. Assume that the Hilbert space can be written as an orthogonal direct sum

$$\mathfrak{H} = \operatorname{Ran} M \oplus \operatorname{Ker} M. \tag{3.7}$$

The operator M has the matrix representation $M = \begin{pmatrix}M^{\perp} & 0\\ 0 & 0\end{pmatrix}$ with respect to this decomposition of the Hilbert space. We can regularize the degenerate Cauchy problem (2.1) by replacing M by

$$M^n = \begin{pmatrix} M^\perp & 0 \\ 0 & \frac{1}{n} \end{pmatrix}.$$

Lemma 3.1. *Let Assumption 2.1 be fulfilled and let M be as described above. Assume for some $K < 1$ that $\|(A - \lambda M)^{-1} M\| \leq K/\lambda$, for all $\lambda > 0$. Then for $\lambda > 0$ and for n large enough, there exist constants $K' < 1$ and $C > 0$ such that $\|(A - \lambda M^n)^{-1} M^n\| \leq K'/\lambda$ and*

$$\|(A - \lambda M^n)^{-1} M^n - (A - \lambda M)^{-1} M\| \leq \frac{C}{\lambda n} \to 0, \qquad \text{as } n \to \infty. \tag{3.8}$$

Hence Theorem 3.2 implies that the strict solutions of the regularized system (with M replaced by M^n) approximate the strict solutions of (2.1).

Proof. Let $\lambda > 0$. Since $M^\perp$ is bounded and bounded invertible, we obtain

$$\|(A - \lambda M)^{-1}\| \leq \frac{K\|(M^\perp)^{-1}\|}{\lambda} = \frac{\hat{K}}{\lambda}.$$

From $\|M - M^n\| = 1/n$ we find

$$\begin{aligned}
\|(A - &\lambda M^n)^{-1} M^n\| = \\
&= \|(1 + \lambda(A - \lambda M)^{-1}(M - M^n))^{-1}(A - \lambda M)^{-1}(M^n - M + M)\| \\
&\leq \|(1 + \lambda(A - \lambda M)^{-1}(M - M^n))^{-1}\| \, \|(A - \lambda M)^{-1}\| \, \|M^n - M\| + \\
&+ \|(1 + \lambda(A - \lambda M)^{-1}(M - M^n))^{-1}\| \, \|(A - \lambda M)^{-1} M\| \\
&\leq \frac{1}{1 - \hat{K}/n} \left(\frac{\hat{K}}{n} + K \right) \frac{1}{\lambda} \leq \frac{K'}{\lambda},
\end{aligned}$$

where $K' < 1$ provided $K < 1$ and n is large enough. Similarly, we obtain

$$\begin{aligned}
\|(A - &\lambda M^n)^{-1}\| \leq \frac{1}{1 - \hat{K}/n} \frac{\hat{K}}{\lambda} \leq \tilde{K}/\lambda, \\
\|(A - &\lambda M^n)^{-1} - (A - \lambda M)^{-1}\| = \\
&= \|(A - \lambda M^n)^{-1} \lambda (M^n - M)(A - \lambda M)^{-1}\| \\
&\leq \frac{\tilde{K}}{\lambda} \frac{\lambda}{n} \frac{\hat{K}}{\lambda} \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

and finally

$$\begin{aligned}
\|(A - &\lambda M^n)^{-1} M^n - (A - \lambda M)^{-1} M\| \\
&\leq \left\| \left((A - \lambda M^n)^{-1} - (A - \lambda M)^{-1} \right) M^n \right\| + \|(A - \lambda M)^{-1}\| \, \|M^n - M\| \\
&\leq \left(2\|M\| \frac{\tilde{K}\hat{K}}{\lambda} + \frac{\hat{K}}{\lambda} \right) \frac{1}{n} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

□

4. Galerkin approximation

Let us now consider the case where the given Hilbert space $\mathfrak{H}$ is approximated by finite dimensional subspaces and the approximating operators are the restrictions to these subspaces. We will see that if the operators in the factorized system are Galerkin approximations with some projection $\mathcal{P}^n$, the operators in the degenerate system are restricted by two different projections and in this case we can interchange factorization and approximation.

Denote by (M, A) a degenerate Cauchy problem in the Hilbert space $\mathfrak{H}$, assume that it is factorizable and that Z_A is bounded. Let U be a finite dimensional subspace of $\mathfrak{D}_A$ and let

$$\Pi : \mathfrak{D}_A^c \longrightarrow U$$

be a projection onto this subspace U. Define a projection $\mathcal{P}$ from $P^\perp \mathfrak{D}_A^c$ in $P^\perp U = P^\perp \Pi(\mathfrak{D}_A^c)$ by

$$\mathcal{P} : x \mapsto P^\perp \Pi z \tag{4.1}$$

where $x = P^\perp z$ and $z = Z_A x$. Π can always be chosen such that $\mathcal{P}$ is an orthogonal Projection. (For the right choice of Π start with an orthogonal projection $\mathcal{P} : P^\perp \mathfrak{D}_A^c \longrightarrow P^\perp U$ and define Π by $\Pi z = Z_A \mathcal{P} P^\perp z$).

Hence we have

$$\mathcal{P} P^\perp z = P^\perp \Pi z \quad \text{for all } z \in \mathfrak{D}_A^c \tag{4.2}$$

and

$$\mathcal{P} x = P^\perp \Pi Z_A x \quad \text{for all } x \in P^\perp \mathfrak{D}_A^c. \tag{4.3}$$

Define the operator $\mathcal{M} : \mathfrak{D}_A^c \longrightarrow \mathfrak{H}$ by $\mathcal{M} = M\Pi$. As the subspace U is assumed to be finite dimensional the range of $\mathcal{M}$ is also finite dimensional and we can define $\tilde{\Pi}$ as the projection of $Q^\perp \mathfrak{H}$ on $\operatorname{Ran} \mathcal{M}$. Hence,

$$\mathcal{M} = \tilde{\Pi} \mathcal{M} = \tilde{\Pi} M \Pi \quad \text{on } \mathfrak{D}_A^c. \tag{4.4}$$

In a similar way we define a restriction $\mathcal{A}$ of the operator A as an operator mapping $U \subset \mathfrak{D}_A^c$ into $\mathfrak{H}$ by

$$\mathcal{A} = \tilde{\Pi} A \Pi \tag{4.5}$$

Lemma 4.1. *Let (M, A) be a degenerate Cauchyproblem and assume that it is factorizable to a nondegenerate Cauchy problem $\frac{d}{dt} x(t) = A_1 x(t)$, where A_1 is defined as in section (2). Consider furthermore the degenerate Cauchy problem $(\mathcal{M}, \mathcal{A})$ with operators $\mathcal{M}$ and $\mathcal{A}$ defined by (4.5) and (4.4) and denote its factorized equation by $\frac{d}{dt} x(t) = \mathcal{A}_1 x(t)$ with $\mathcal{A}_1 = (\mathcal{M}^\perp)^{-1} \mathcal{A} Z_{\mathcal{A}}$. Then we have:*

1. $\Pi \mathfrak{D}_A = \mathfrak{D}_{\mathcal{A}}$
2. $P^\perp = P^\perp_{\mathcal{M}}$ *on $\mathfrak{D}_{\mathcal{A}}$, where $P^\perp_{\mathcal{M}}$ denotes the orthogonal projection from $\mathfrak{H}$ to $(Ker\, \mathcal{M})^\perp$*
3. $Z_A = Z_{\mathcal{A}}$ *on $\mathcal{P} P^\perp \mathfrak{D}_{\mathcal{A}}$*

Proof. 1. We have $\mathfrak{D}_A = \{z \in \mathfrak{D}(A) \mid Az \in (\operatorname{Ran} M)^c\}$ and similarily for the equation $(\mathcal{M}, \mathcal{A})$ we get $\mathfrak{D}_{\mathcal{A}} = \{z \in \mathfrak{D}(\mathcal{A}) \mid \mathcal{A}z \in (\operatorname{Ran} \mathcal{M})^c\}$. By definition of $\mathcal{M}$ and $\tilde{\Pi}$ we have

$$\begin{aligned}\operatorname{Ran} \mathcal{M} &= \operatorname{Ran} \tilde{\Pi} M \Pi \\ &= \tilde{\Pi} \operatorname{Ran} M\Pi \\ &= \operatorname{Ran} M\Pi \\ &= \tilde{\Pi} \operatorname{Ran} M.\end{aligned}$$

For all $z \in \mathfrak{D}_A$ we have $\Pi z \in U \subset \mathfrak{D}_A$ and hence $A\Pi z \in (\operatorname{Ran} M)^c$. This implies that $\tilde{\Pi} A \Pi z \in \tilde{\Pi} \operatorname{Ran} M = \operatorname{Ran} \mathcal{M}$, $z \in \mathfrak{D}(\mathcal{A})$ and $\mathcal{A}z \in \operatorname{Ran} \mathcal{M}$, hence $z \in \mathfrak{D}_{\mathcal{A}}$ and therefore $\Pi \mathfrak{D}_A \subset \mathfrak{D}_{\mathcal{A}}$. Conversely, by definition of $\mathcal{A}$, $\mathfrak{D}(\mathcal{A}) \subset U$, hence, $\mathfrak{D}_{\mathcal{A}} \subset \Pi \mathfrak{D}_A$.

2. By definition of $\mathcal{M}$, $M \restriction \Pi \mathfrak{D}_A = \mathcal{M}$. This implies $P_{\mathcal{M}}^{\perp} \restriction \Pi \mathfrak{D}_A = P^{\perp} \restriction \Pi \mathfrak{D}_A$.

3. $\mathcal{P} P^{\perp} \mathfrak{D}_A =$ (by (4.2)) $P^{\perp} \Pi \mathfrak{D}_A =$ (by 1.) $P^{\perp} \mathfrak{D}_{\mathcal{A}} =$ (by 2.) $P_{\mathcal{M}}^{\perp} \mathfrak{D}_{\mathcal{A}}$. Therefore, we get $Z_A P^{\perp} \Pi \mathfrak{D}_A = \Pi \mathfrak{D}_A = \mathfrak{D}_{\mathcal{A}} = Z_{\mathcal{A}} P_{\mathcal{M}}^{\perp} \mathfrak{D}_{\mathcal{A}}$.

□

Lemma 4.2. *Consider the degenerate Cauchy problems* (M, A) *and* $(\mathcal{M}, \mathcal{A})$ *with the same assumptions as in Lemma 4.1. Then we have*

$$\mathcal{P} A_1 \mathcal{P} = \mathcal{A}_1 \qquad \text{on } P^{\perp} \mathfrak{D}_{\mathcal{A}} = \mathcal{P} P^{\perp} \mathfrak{D}_A \tag{4.6}$$

Proof. By definition, $\mathcal{A}_1$ is defined on $P^{\perp} \mathfrak{D}_{\mathcal{A}}$. The operator $\mathcal{P} A_1 \mathcal{P}$ is defined on $\mathcal{P} P^{\perp} \mathfrak{D}_A$ which is by (4.2) and Lemma 4.1 equal to $P^{\perp} \mathfrak{D}_{\mathcal{A}}$. By definition of A_1 we get for all $x \in \mathcal{P} P^{\perp} \mathfrak{D}_A$:

$$\mathcal{P} A_1 \mathcal{P} x = \mathcal{P} (M^{\perp})^{-1} A Z_A \mathcal{P} x \tag{4.7}$$

$\mathcal{P} x = P^{\perp} \Pi Z_A x$ for all $x \in P^{\perp} \mathfrak{D}_A$. Therefore we have

$$Z_A \mathcal{P} x = \Pi Z_A x \qquad \text{for all } x \in \mathcal{P} P^{\perp} \mathfrak{D}_A. \tag{4.8}$$

By construction of $\mathcal{M}$ we get $\mathcal{M}^{\perp} \mathcal{P} = \tilde{\Pi} \mathcal{M}^{\perp}$ and this implies

$$\mathcal{P} (M^{\perp})^{-1} = (\mathcal{M}^{\perp})^{-1} \tilde{\Pi} \tag{4.9}$$

Inserting (4.8) and (4.9) into (4.7) yields

$$\begin{aligned}\mathcal{P} A_1 \mathcal{P} x &= (\mathcal{M}^{\perp})^{-1} \tilde{\Pi} A \Pi Z_A x \\ &= (\mathcal{M}^{\perp})^{-1} \mathcal{A} Z_{\mathcal{A}} x \\ &= \mathcal{A}_1 x \qquad \text{for all } x \in \mathcal{P} P^{\perp} \mathfrak{D}_A = P^{\perp} \mathfrak{D}_{\mathcal{A}}.\end{aligned}$$

□

Theorem 4.1. *Let* (M, A) *be a degenerate Cauchy problem, assume that it is factorizable to a nondegenerate Cauchy problem and that the operator* Z_A *is bounded. For every* $n = 1, 2, ...$ *let* U^n *be a finite dimensional subspace of* $\mathfrak{D}_A^c$ *and define* *for each* U^n *the projection* Π^n, $\tilde{\Pi}^n$ *and* $\mathcal{P}^n$ *as before. Define the Operators* M^n *and* A^n *by*

$$M^n = \tilde{\Pi}^n M \Pi^n, \qquad A^n = \tilde{\Pi}^n A \Pi^n \qquad \text{on } \Pi^n \mathfrak{D}_A \tag{4.10}$$

Furthermore, let

1. $P^\perp \Pi^n Z_A x \longrightarrow x$ *for all* $x \in P^\perp \mathfrak{D}_A^c$
2. *there exist constsants* K *and* ω *such that*

$$\|P^\perp (A - \lambda M)^{-1} M\| \leq \frac{K}{\lambda - \omega} \tag{4.11}$$

   and for all $n = 1, 2, ...$

$$\|P^\perp_{M^n} A^n - \lambda M^n)^{-1} M^n\| \ \leq \frac{K}{\lambda - \omega} \tag{4.12}$$

3. *there is a* $\lambda \in \rho(M, A) \cap \bigcap_{n=1}^\infty \rho(M^n, A^n)$ *such that for all* $x \in P^\perp \mathfrak{D}_A^c$

$$\|P^\perp (A^n - \lambda M^n)^{-1} M^n \mathcal{P}^n x - P^\perp (A - \lambda M)^{-1} M x\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.13}$$

For all $n \geq 1$ *let* $z_n(t)$ *be the strict solution of the degenerate Cauchy problem* (M^n, A^n)*. Then*

$$z_n(t) \longrightarrow z(t) \text{ uniformly on compact } t\text{-intervals.}$$

Proof. Condition 1. implies $\mathcal{P}^n x \longrightarrow x$ for all $x \in P^\perp \mathfrak{D}_A^c$. By condition (4.11) the operators A_1 and $(A^n)_1$ generate semigroups of the class (M, ω). Condition (4.13) implies that the resolvents of $(A^n)_1$ converge. Using Lemma 4.2 we get $(A^n)_1 = (A_1)^n$. So all the conditions for the Trotter-Kato Theorem for the nondegenerate Cauchy problems

$$\frac{d}{dt} x(t) = A_1 x(t) \tag{4.14}$$

and

$$\frac{d}{dt} x_n(t) = (A_1)^n x_n(t) \tag{4.15}$$

are fulfilled and therefore

$$x_n(t) \longrightarrow x(t)$$

uniformly on bounded t-intervals. By Lemma 4.1 we have $z_n(t) = Z_{A^n} x_n(t) = Z_A x_n(t)$ on $\mathcal{P}^n \mathfrak{D}_A^c$. As Z_A is assumed to be bounded we get

$$z_n(t) \longrightarrow z(t) \qquad \text{uniformly on bounded } t\text{-intervals}$$

□

Remark 4.1. The situation in Theorem 4.1 is an example when we can interchange factorization and approximation. In this case it doesn't matter wether we factorize the degenerate equation first and approximate the nondegenerate system by finite dimensional equations or if we approximate the degenerate equation first. In general, we cannot expect the approximating solutions of the factorized equation to approximate the degenerate problem, because Z_A need not be bounded and, in addition, we get for each equation a different Z_{A^n}. Furthermore, approximating subspaces of $P^\perp \mathfrak{D}_A$ need not lead to subspaces of $\mathfrak{D}_A$.

Remark 4.2. If we want the projections $\mathcal{P}^n$ to be orthogonal, we can first define the projections $\mathcal{P}^n$ and then the projections Π^n in order to guarantee orthogonality. In the case where we consider the factorized problem in the graph norm of Z_A both $\mathcal{P}^n$ and Π^n are orthogonal at the same time because then Z_A is an isometric isomorphism between $\mathfrak{D}_A^c$ and $P^\perp \mathfrak{D}_A^c$.

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Inst. f. Mathematik, Universität Graz, Heinrichstrasse 36,, A-8010 Graz, Austria

E-mail address: `bernd.thaller@kfunigraz.ac.at`

E-mail address: `sigrid.thaller@kfunigraz.ac.at`