Graded Semigroups

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July 11, 2023

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Graded Rings

- Given a group Γ, a (unital associative) ring R is Γ-*graded* if

$$R = \bigoplus_{\alpha \in \Gamma} R_\alpha,$$

where each R_α is an additive subgroup of R (called the *degree* α *component*), and $R_\alpha R_\beta \subseteq R_{\alpha\beta}$ for all $\alpha, \beta \in \Gamma$ (with $R_\alpha R_\beta$ consisting of all sums of elements of the form rp, for $r \in R_\alpha$ and $p \in R_\beta$).

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3. For any ring R and set X (of commuting or non-commuting variables), the polynomial ring $R[X]$ is $\mathbb{Z}$-graded, via letting $R[X]_n$ be the set of homogeneous polynomials of degree n.

Example 4 - Leavitt Path Algebras

Let K be a field and $E = (E^0, E^1, \mathbf{s}, \mathbf{r})$ a directed graph (with E^0 the vertex set, E^1 the edge set, and $\mathbf{s}, \mathbf{r} : E^1 \to E^0$ the source and functions).

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The *Leavitt path K-algebra $L_K(E)$ of E* is the K-algebra generated by $\{v \mid v \in E^0\} \cup \{e, e^{-1} \mid e \in E^1\}$, subject to the relations:
(V) $vw = \delta_{v,w} v$ for all $v, w \in E^0$,
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$L_K(E)$ is $\mathbb{Z}$-graded, via setting

$$L_K(E)_n = \Big\{ \sum_i a_i p_i q_i^{-1} \in L_K(E) \,\Big|\, |p_i| - |q_i| = n \Big\},$$

where $|e_1 \cdots e_n| = n$ is the *length* of the path $e_1 \cdots e_n$ ($e_1, \dots, e_n \in E^0$).

Graded Semigroups

- Each of the (nontrivial) examples above ($R\Gamma$, $R[X]$, $L_K(E)$) happens to be either a semigroup ring or a quotient of a semigroup ring, and acquires its grading from the semigroup structure (base group, monomials, products of paths).

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- Similar constructions have appeared elsewhere, e.g., J. M. Howie's "semigroups with length", papers of E. Ilić-Georgijević.
- There is also literature on graded groupoids, which are related to semigroups.
- But no systematic treatment of graded semigroups had been performed before.

Definition

Let S be semigroup (with zero) and Γ a group. Then S is Γ-*graded* if

$$S = \bigcup_{\alpha \in \Gamma} S_\alpha,$$

where $S_\alpha \subseteq S$ and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in \Gamma$, and $S_\alpha \cap S_\beta = \{0\}$ for all distinct $\alpha, \beta \in \Gamma$.

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Equivalently, S is Γ-*graded* if there is a map $\phi : S \setminus \{0\} \to \Gamma$ such that $\phi(st) = \phi(s)\phi(t)$, whenever $st \neq 0$. Here $S_\alpha = \phi^{-1}(\alpha) \cup \{0\}$ for each $\alpha \in \Gamma$.

Examples

1. Let Γ be a group and $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ a Γ-graded ring. Then $\bigcup_{\alpha \in \Gamma} R_\alpha$ is a Γ-graded (multiplicative) semigroup, which is strongly graded iff R is.

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3. Let Γ be a group, $S = \langle x_i \mid r_k = s_k \rangle$ a semigroup defined by generators and relations, and $\phi : \{x_i\} \to \Gamma$ any function such that $\phi(r_k) = \phi(s_k)$ (extending ϕ to words in the x_i by concatenation). Then S is Γ-graded.

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4. Let Γ be a group, X a set with a distinguished element $0_X \in X$, $\phi : X\backslash\{0_X\} \to \Gamma$ a map, and $\mathcal{T}(X)$ the semigroup of all functions $\psi : X \to X$ such that $\psi(0_X) = 0_X$. For all $\alpha \in \Gamma$ let $X_\alpha = \phi^{-1}(\alpha) \cup \{0_X\}$ and $\mathcal{T}(X)_\alpha = \left\{\psi \in \mathcal{T}(X) \mid \psi(X_\beta) \subseteq X_{\alpha\beta}\ \forall \beta \in \Gamma\right\}$. Then

$$\mathcal{T}^{\mathrm{gr}}(X) = \bigcup_{\alpha\in\Gamma} \mathcal{T}(X)_\alpha$$

is a Γ-graded subsemigroup of $\mathcal{T}(X)$ (which is strongly Γ-graded if and only if $|X_\alpha| = |X_\beta|$ for all $\alpha, \beta \in \Gamma$), and every Γ-graded semigroup embeds in such a semigroup.

Example 5 - Graph Inverse Semigroup

Given a directed graph $E = (E^0, E^1, \mathbf{s}, \mathbf{r})$, the *graph inverse semigroup* $\mathcal{S}(E)$ *of* E is the semigroup (with zero) generated by the vertex set E^0 and the edge set E^1, together with $\{e^{-1} \mid e \in E^1\}$, satisfying the relations:
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Each nonzero element of $\mathcal{S}(E)$ is of the form pq^{-1}, for some paths p, q in E, where $(e_1 \cdots e_n)^{-1} = e_n^{-1} \cdots e_1^{-1}$ for $e_1, \dots, e_n \in E^1$ and $v^{-1} = v$ for $v \in E^0$.

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$\mathcal{S}(E)$ is an inverse semigroup, with $(pq^{-1})^{-1} = qp^{-1}$ for all paths p, q. (A semigroup S is an *inverse semigroup* if for each $s \in S$ there is a unique $t \in S$ satisfying $sts = s$ and $tst = t$.)

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Connections with Leavitt Path Algebras

- Given a field K and a directed graph E, the (contracted) semigroup ring $K[\mathcal{S}(E)]$ is called the *Cohn path K-algebra $C_K(E)$ of E*, and the ring

$$K[\mathcal{S}(E)] \Big/ \Big\langle v - \sum_{e \in \mathbf{s}^{-1}(v)} ee^{-1} \;\Big|\; v \in E^0 \text{ is regular} \Big\rangle,$$

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Theorem (Finite Graph Case)

Let E be a finite nonempty graph. Then the following are equivalent.

1. E has no sinks (i.e., vertices that emit no edges).
2. $L_K(E)$ is strongly graded in the natural $\mathbb{Z}$-grading, for any field K.
3. $\mathcal{S}(E)$ is locally strongly graded in the natural $\mathbb{Z}$-grading. (I.e., for all $n, m \in \mathbb{Z}$ and $s \in \mathcal{S}(E)_{n+m} \setminus \{0\}$, there exists $t \in \mathcal{S}(E)_n \mathcal{S}(E)_m \setminus \{0\}$ such that $t = su$ for some idempotent $u \in \mathcal{S}(E)$).

Semigroup Rings

- Given a ring R and a semigroup S, we denote by $R[S]$ the contracted semigroup ring (where the zero of S is identified with the zero of RS). An arbitrary element of $R[S]$ is of the form $\sum_{s \in S} r^{(s)} s$, where $r^{(s)} \in R$, and all but finitely many of the $r^{(s)}$ are zero.

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- If Γ is a group and S is a Γ-graded semigroup, then $R[S]$ is a Γ-graded ring, via setting

$$R[S]_\alpha = \Big\{ \sum_{s\in S} r^{(s)}s \mid s \in S_\alpha \text{ whenever } r^{(s)} \neq 0 \Big\},$$

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Proposition

Let Γ be a group, S a Γ-graded semigroup, and R a ring. Then S is a strongly Γ-graded semigroup if and only if $R[S]$ is a strongly Γ-graded ring (in the induced grading).

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Theorem (M. Cohen & S. Montgomery, 1984)

Let Γ be a group and R a Γ-graded ring. Then $R\text{-}\mathrm{Gr}$ is isomorphic to $R\#\Gamma\text{-}\mathrm{Mod}$, where $R\#\Gamma$ is the *smash product* of R and Γ.

S-Sets

- Let S a semigroup. A set X is a *left S-set* or *S-act*, if there is an action of S on X, such that $s(tx) = (st)x$ for all $s, t \in S$ and $x \in X$.

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Theorem

Let Γ be a group and S a Γ-graded inverse semigroup. Then S is strongly graded if and only if $S\text{-}\mathrm{Gr}$ is naturally equivalent to $S_\varepsilon\text{-}\mathrm{Mod}$.

Smash Product

Given a group Γ and a Γ-graded semigroup S, define the *smash product of S with* Γ as

$$S\#\Gamma = \{sP_\alpha \mid s \in S \setminus \{0\}, \alpha \in \Gamma\} \cup \{0\}.$$

Also, define a binary operation on $S\#\Gamma$ by

$$(sP_\alpha)(tP_\beta) = \begin{cases} stP_\beta & \text{if } st \neq 0 \text{ and } t \in S_{\alpha\beta^{-1}} \\ 0 & \text{otherwise} \end{cases}$$

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Theorem

Let Γ be a group and S a Γ-graded semigroup with local units (i.e., for every $s \in S$ there exist idempotents $u, v \in S$ such that $us = s = sv$). Then $S\#\Gamma$ is a semigroup, and S-Gr is isomorphic to $S\#\Gamma$-Mod.

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Proposition

Let Γ be a group, S a Γ-graded semigroup, and R a ring. Then $R[S\#\Gamma] \cong R[S]\#\Gamma$.

Morita Theory

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- If S is a monoid (or has common local units), then S-$\mathrm{FAct} = S$-Mod.
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Theorem

Let Γ be a group, and S and T be Γ-graded semigroups with local units. If S and T are graded Morita equivalent, then they are Morita equivalent.

Thank you!