On non-associative algebras generated by gyrogroups

Teerapong Suksumran

Department of Mathematics, Faculty of Science
Chiang Mai University, Thailand

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Outline

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Concrete example of a gyrogroup: Möbius gyrogroup

Set $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$. Möbius addition [1], $\oplus_M$, is given by

$$a \oplus_M b = \frac{a+b}{1+\bar{a}b} \tag{1}$$

for all $a, b \in \mathbb{D}$.

- $\oplus_M$ is a binary operation on $\mathbb{D}$.
- 0 is an identity of $\mathbb{D}$.
- Given $a \in \mathbb{D}$, $-a$ is an inverse of a.
- $\oplus_M$ is non-associative and non-commutative.

[1] A. Ungar, *The holomorphic automorphism group of the complex disk*, Aequationes Mathmematicae **47** (1994)

Concrete example of a gyrogroup: Möbius gyrogroup

- $(\mathbb{D}, \oplus_M)$ satisfies a law similar to the associative law:

$$\begin{aligned} a \oplus_M (b \oplus_M c) &= (a \oplus_M b) \oplus_M \mathrm{gyr}[a, b]c \\ (a \oplus_M b) \oplus_M c &= a \oplus_M (b \oplus_M \mathrm{gyr}[b, a]c), \end{aligned}$$

where $\mathrm{gyr}[a, b]$ is an automorphism of $\mathbb{D}$ given by

$$\mathrm{gyr}[a, b]z = \omega z, \quad z \in \mathbb{D}, \tag{2}$$

and $\omega = \dfrac{1 + a\bar{b}}{1 + \bar{a}b}$ is a unit complex number for all $a, b \in \mathbb{D}$.

Gyrogroups: An axiom approach

Let G be a non-empty set with a binary operation $\oplus$. The pair $(G, \oplus)$ is called a gyrogroup if the following conditions hold:

1. $\exists e \in G\ \forall a \in G,\ a \oplus e = a = e \oplus a$ (identity element)
2. $\forall a \in G \exists b \in G,\ b \oplus a = e = a \oplus b$ (inverse element)
3. $\forall a, b \in G\ \exists \mathrm{gyr}[a, b], \mathrm{gyr}[b, a] \in \mathrm{Aut}\,(G, \oplus)$ such that
    - $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c$ (left gyroassociative law)
    - $(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b, a]c)$ (right gyroassociative law)
4. $\forall a, b \in G,$
    - $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a, b]$ (left loop property)
    - $\mathrm{gyr}[a, b \oplus a] = \mathrm{gyr}[a, b]$ (right loop property)

Groups and gyrogroups

Recall the gyroassociative law

$$\begin{aligned} a \oplus (b \oplus c) &= (a \oplus b) \oplus \mathrm{gyr}[a, b]c \\ (a \oplus b) \oplus c &= a \oplus (b \oplus \mathrm{gyr}[b, a]c) \end{aligned}$$

- Every group is a gyrogroup by defining $\mathrm{gyr}[a, b]$ to be the identity automorphism.
- Any gyrogroup with trivial gyroautomorphisms is a group.

Groups and gyrogroups

GROUP	GYROGROUP
group identity 1	gyrogroup identity e
inverse element a^{-1}	inverse element $\ominus a$
the associative law	the gyroassociative law
subgroup	subgyrogroup
normal subgroup	normal subgyrogroup
quotient group	quotient gyrogroup
group homomorphism	gyrogroup homomorphism
group isomorphism	gyrogroup isomorphism
abelian group	gyrocommutative gyrogroup
$\vdots$	$\vdots$

Construction of a gyrogroup algebra

Throughout the remaining of this talk, let $G = \{a_1, a_2, \ldots, a_n\}$ be a *finite* gyrogroup of order n with a_1 being the identity of G, and let $\mathbb{F}$ be a field.

Define $\mathbb{F}[G]$ to be the set of all finite formal sums of elements of G with coefficients from $\mathbb{F}$, that is,

$$\mathbb{F}[G] = \left\{ \sum_{i=1}^{n} \lambda_i a_i \colon \lambda_i \in \mathbb{F}, i = 1, 2, \ldots, n \right\}. \tag{3}$$

Construction of a gyrogroup algebra

Define the following operations on $\mathbb{F}[G]$:

$$\begin{aligned} \left(\sum_{i=1}^{n} \alpha_i a_i\right) + \left(\sum_{i=1}^{n} \beta_i a_i\right) &= \sum_{i=1}^{n} (\alpha_i + \beta_i) a_i, \\ \lambda \left(\sum_{i=1}^{n} \alpha_i a_i\right) &= \sum_{i=1}^{n} (\lambda \alpha_i) a_i, \\ \left(\sum_{i=1}^{n} \alpha_i a_i\right)\left(\sum_{i=1}^{n} \beta_i a_i\right) &= \sum_{i=1}^{n} \left(\sum_{\substack{j,k \\ a_j \oplus a_k = a_i}} \alpha_j \beta_k \right) a_i. \end{aligned} \tag{4}$$

Since the linear equations $x \oplus a = b$ and $a \oplus y = b$ in the variables x and y have unique solutions in G for all $a, b \in G$, specification of any two of a, b, c in the equation $a \oplus b = c$ uniquely determines the third. Hence, the index condition used in (4) makes sense.

Gyrogroup algebras

Theorem 1 (Gyrogroup algebras)

The set $\mathbb{F}[G]$, equipped with the operations defined by (4), is a unital non-associative algebra over $\mathbb{F}$. If G is a group, then $\mathbb{F}[G]$ becomes the usual group ring. If G is a gyrogroup with non-trivial gyroautomorphisms, then $\mathbb{F}[G]$ is not associative.

The algebra $\mathbb{F}[G]$ constructed above is called the *gyrogroup algebra* of G over $\mathbb{F}$.

Some properties of gyrogroup algebras

By convention the terms with zero coefficients of a formal sum in $\mathbb{F}[G]$ are omitted. We remark that the base field $\mathbb{F}$ appears in $\mathbb{F}[G]$ under the identification

$$\lambda \leftrightarrow \lambda a_1.$$

Furthermore, the original gyrogroup G appears in $\mathbb{F}[G]$ under the identification

$$a_i \leftrightarrow 1a_i.$$

Theorem 2

Every finite gyrogroup can be embedded into a nonassociative algebra.

Some properties of gyrogroup algebras

Theorem 3

The gyrogroup G is a basis for $\mathbb{F}[G]$ as a vector space. In particular, the dimension of $\mathbb{F}[G]$ equals $|G|$.

Proof. Let $A = \sum_{i=1}^{n} \alpha_i a_i$.

Clearly, $A = \alpha_1(1a_1) + \alpha_2(1a_2) + \cdots + \alpha_n(1a_n)$. This proves that G spans $\mathbb{F}[G]$.

If $\beta_1(1a_1) + \beta_2(1a_2) + \cdots + \beta_n(1a_n) = 0$, where $\beta_i \in \mathbb{F}$, then $\sum_{i=1}^{n} \beta_i a_i = 0$. By definition, $\beta_i = 0$ for all i. This proves that G is linearly independent. $\square$

Some properties of gyrogroup algebras

Theorem 4

The base field $\mathbb{F}$ is contained in the center of $\mathbb{F}[G]$.

Proof. Let $\lambda \in \mathbb{F}$. For all $\sum_{i=1}^{n} \lambda_i a_i \in \mathbb{F}[G]$,

$$\begin{aligned}(\lambda a_1)\left(\sum_{i=1}^{n} \lambda_i a_i\right) &= \sum_{i=1}^{n} (\lambda\lambda_i)(a_1 \oplus a_i) \\ &= \sum_{i=1}^{n} (\lambda\lambda_i) a_i \\ &= \sum_{i=1}^{n} (\lambda_i\lambda)(a_i \oplus a_1) \\ &= \left(\sum_{i=1}^{n} \lambda_i a_i\right)(\lambda a_1).\end{aligned}$$

Hence, λa_1 commutes with all the elements of $\mathbb{F}[G]$ $\square$.

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