On Waring numbers of henselian rings
Applications

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Some definitions - recall

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We define *the n-length of $a \in R$*:

$$\ell_n(a) = \ell_{n,R}(a) = \inf \left\{ g \in \mathbb{N}_+ : \ a = \sum_{j=1}^{g} a_j^n \text{ for some } a_1, \ldots, a_g \in R \right\}$$

and *the nth level of R* as

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and *the nth level of R* as

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We call R real if $s_2(R) = \infty$.
By nth Waring number of R we mean

$$w_n(R) = \sup\{\ell_n(a) : \ a \in R, \ \ell_n(a) < \infty\}.$$

Waring numbers of the rings of power series

Theorem

Let k be a field n, s be positive integers and m is the $\mathrm{char}(k)$-free part of n.

a) We have

$$s_n(k[[x_1, \ldots x_s]]) = s_n(k((x_1, \ldots x_s))) = s_n(k((x_1)) \ldots ((x_s))) = s_m(k).$$

b) If $s_n(k) < \infty$, then

$$w_n(k[[x_1, \ldots, x_s]]) = \begin{cases} \max\{w_m(k), s_m(k)+1\} & \text{for } m > 1 \\ 1 & \text{for } m = 1 \end{cases},$$

$$w_n(k((x_1, \ldots x_s))) = w_n(k((x_1)) \ldots ((x_s))) = \begin{cases} s_m(k)+1 & \text{for } m > 1 \\ 1 & \text{for } m = 1 \end{cases}, \tag{1}$$

$$w_n(k((x_1)) \ldots ((x_s))[[x_{s+1}]]) = \begin{cases} s_m(k)+1 & \text{for } m > 1 \\ 1 & \text{for } m = 1 \end{cases}. \tag{2}$$

Waring numbers of the rings of power series

Theorem

Let k be a field such that $s_n(k) = \infty$. Then the following holds

$$w_n(k((x_1))\dots((x_{s-1}))[[x_s]]) = w_n(k((x_1))\dots((x_s))) = w_n(k),$$

$$w_n(k[[x_1,\dots,x_s]]) \geq w_n(k((x_1,\dots,x_s))) \geq w_n(k).$$

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We put $k(V)$ to be the field of fractions of $k[V]$, provided that V is an irreducible algebraic set.

Waring numbers of coordinate rings

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We put $k(V)$ to be the field of fractions of $k[V]$, provided that V is an irreducible algebraic set.
We say that the point $x \in V$ is a regular point, if the ring $k[V]_{\mathfrak{m}_x}$ is a regular local ring, where $\mathfrak{m}_x$ is the maximal ideal of polynomial functions vanishing in x.

Waring numbers of coordinate rings

Theorem

Let V be an irreducible algebraic subset of k^s, different from the point, which admits a regular point.

a) If $s_n(k) < \infty$, then

$$w_n(k[V]) \geq \max\{w_m(k), s_m(k) + 1\},$$

$$w_n(k(V)) \geq s_m(k) + 1,$$

where m is the $\mathrm{char}(k)$-free part of n.

b) If $s_n(k) = \infty$, then

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Remark

If $\dim V \geq 3$, then $w_2(\mathbb{R}[V]) = \infty$ (Choi, Dai, Lam, Reznick, 1982) meanwhile $w_2(\mathbb{R}) = 1$.

Waring numbers of coordinate rings

Corollary

Let V be an irreducible algebraic subset of k^s, different from the point, which admits a regular point. Assume that $\mathrm{char}(k) \neq 2$ and $s_2(k) < \infty$. Then $w_2(k[V]) = w_2(k(V)) = s_2(k) + 1$.

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Proof.

Write

$$a = \left(\frac{a+1}{2}\right)^2 - \left(\frac{a-1}{2}\right)^2.$$

□

Sums of two n-th powers

Corollary

Let R be a Henselian local ring with the total ring of fractions $Q(R) \neq R$ and a residue field k. Take an odd positive integer $n > 1$. Assume that $\mathrm{char}(k) \nmid n$ or R is rank-1 valuation ring with $\mathrm{char}(R) \nmid n$. Then, for every element $f \in Q(R)$ there exists a presentation

$$f = f_1^n + f_2^n$$

for some $f_1, f_2 \in Q(R)$.

Theorem (recall)

Let R be a Henselian DVR with quotient field K and $n > 1$ be an odd positive integer. Denote by m the $\mathrm{char}(K)$-free part of n. Then

$$w_n(K) = \begin{cases} 1 & \textit{if} \quad m = 1, \\ 2 & \textit{if} \quad m > 1. \end{cases}$$

Waring numbers of rings and fields of p-adic numbers

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Corollary

For any prime number p and any odd integer n we have $w_n(\mathbb{Q}_p) = 2$.

Theorem

Let p be an odd prime number, k be a positive integer and d be a positive integer not divisible by p. Then

$$w_{dp^{k-1}(p-1)}(\mathbb{Z}_p) = w_{dp^{k-1}(p-1)}(\mathbb{Q}_p) = p^k.$$

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Theorem

Let p be an odd prime number, k be a positive integer and d be an odd positive integer not divisible by p. Assume additionally that $\frac{dp^{k-1}(p-1)}{2} > 1$. Then the following equalities hold:

$$\begin{aligned} w_{\frac{dp^{k-1}(p-1)}{2}}(\mathbb{Z}_p) &= \frac{p^k - 1}{2}, \\ w_{\frac{dp^{k-1}(p-1)}{2}}(\mathbb{Q}_p) &= 2. \end{aligned}$$

Waring numbers of rings and fields of p-adic numbers

Theorem

Let $k, d > 0$ be positive integers, with d odd. Then the following holds:

$$w_{2^k d}(\mathbb{Z}_2) = w_{2^k d}(\mathbb{Q}_2) = \begin{cases} 4 & \text{if } k = 1, d = 1 \\ 15 & \text{if } k = 2, d = 1 \\ 2^{k+2} & \text{if } k > 2 \text{ or } d \geq 3 \end{cases}.$$

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Theorem

Let n be a positive integer. Then for any prime p satisfying $p > (n-1)^4$ we have the following formulas:

$$w_n(\mathbb{Z}_p) = w_n(\mathbb{Q}_p) = \begin{cases} 2 & \text{if } (n, p-1) \mid \frac{p-1}{2}, \\ 3 & \text{otherwise.} \end{cases}$$

p	$w_3(\mathbb{Z}_p)$	$w_3(\mathbb{F}_p)$	$s_3(\mathbb{F}_p)$
3	4	1	1
7	3	3	1
$p \equiv 1 \pmod 3$, $p \neq 7$	2	2	1
$p \equiv 2 \pmod 3$	2	1	1

Table: $w_3(\mathbb{Z}_p)$, of course $w_3(\mathbb{Q}_p)) = 2$ for any prime p.

Waring numbers of rings and fields of p-adic numbers

p	$w_4(\mathbb{Z}_p)$	$w_4(\mathbb{Q}_p)$	$w_4(\mathbb{F}_p)$	$s_4(\mathbb{F}_p)$
2	15	15	1	1
5	5	5	4	4
13	3	3	3	2
29	4	4	3	3
17,41	3	2	3	1
37,53,61	3	3	2	2
73	2	2	2	1
$p \equiv 3 \pmod 4$, $p < 81$	3	3	2	2
$p \equiv 1 \pmod 8$, $p > 81$	2	2	2	1
$p \not\equiv 1 \pmod 8$, $p > 81$	3	3	2	2

Table: $w_4(\mathbb{Z}_p)$ and $w_4(\mathbb{Q}_p)$.

p	$w_5(\mathbb{Z}_p)$	$w_5(\mathbb{F}_p)$	$s_5(\mathbb{F}_p)$
5	3	1	1
11	5	5	1
$p \not\equiv 1 \pmod 5$	2	1	1
$p \equiv 1 \pmod 5$, $p \geq 131$	2	2	1
$p \equiv 1 \pmod 5$, $p < 131$, $p \neq 11$	3	3	1

Table: $w_5(\mathbb{Z}_p)$, of course $w_5(\mathbb{Q}_p) = 2$ for any prime p.

Waring numbers of local rings and their henselizations and completions

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Theorem

Let $(R, \mathfrak{m})$ be a local ring with residue field k, maximal ideal $\mathfrak{m} \neq \mathfrak{m}^2$ and $s_n(k) < \infty$. Assume that $\mathrm{char}(k) \nmid n$ or $\mathrm{char}(R) = \mathrm{char}(k) = p$ and the rings $R^h, \widehat{R}$ are reduced. Then

a) $s_n(R) \geq s_n(R^h) = s_n(\widehat{R})$

b) $w_n(R) \geq w_n(R^h) = w_n(\widehat{R})$.

Theorem

Let $(R, \mathfrak{m})$ be a DVR. Then the following inequality holds:

$$w_n(R) \geq w_n(R^h) = w_n(\widehat{R}).$$

If we denote by K, K^h and $\widehat{K}$ their fields of fractions, respectively, then

$$w_n(K) \geq w_n(K^h) = w_n(\widehat{K}).$$

Waring numbers of local rings and their henselizations and completions

Definition

Let R be a ring and $n > 1$ be a positive integer. We say that a prime ideal $\mathfrak{p} \subset R$ is an n-good ideal if $\mathfrak{p}R_{\mathfrak{p}} \neq (\mathfrak{p}R_{\mathfrak{p}})^2$, $s_n(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) < \infty$, and one of the following conditions hold:

1. $\mathrm{char}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \nmid n$
2. $\mathrm{char}(R_{\mathfrak{p}}) = \mathrm{char}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) = p \mid n$ and the $\mathfrak{p}R_{\mathfrak{p}}$-adic completion of $R_{\mathfrak{p}}$ is reduced.
3. $R_{\mathfrak{p}}$ is a DVR.

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3. $R_{\mathfrak{p}}$ is a DVR.

Theorem

Let R be a ring and $n > 1$ be a positive integer. Then

$$w_n(R) \geq \sup w_n(\widehat{R_{\mathfrak{p}}}),$$

where supremum runs over all n-good ideals of R.

An integral domain R is called an almost Dedekind domain, if for every maximal ideal $\mathfrak{m} \subset R$, the localization $R_{\mathfrak{m}}$ is a DVR. In particular, a Noetherian almost Dedekind domain is a Dedekind domain.

Waring numbers of local rings and their henselizations and completions

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Theorem

Let R be an almost Dedekind domain with a fraction field K and $n > 1$ be a positive integer. Then, the following inequalities hold:

$$w_n(R) \geq \sup_{\mathfrak{m}} w_n(\widehat{R_{\mathfrak{m}}})$$

and

$$w_n(K) \geq \sup_{\mathfrak{m}} w_n(\mathrm{Frac}(\widehat{R_{\mathfrak{m}}})),$$

where supremum is taken over all maximal ideals.

Waring numbers of local rings and their henselizations and completions

Corollary

Let K be a number field with its ring of integers $\mathcal{O}_K = \mathbb{Z}[\alpha_1, ..., \alpha_s]$. Then the following inequalities hold

$$w_n(\mathcal{O}_K) \geq \sup_{p-\text{prime}} w_n(\mathbb{Z}_p[\alpha_1, ..., \alpha_s]),$$

$$w_n(K) \geq \sup_{p-\text{prime}} w_n(\mathbb{Q}_p[\alpha_1, ..., \alpha_s]).$$

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Corollary

1. $w_4(\mathbb{Q}) \geq 15$[a],
2. $w_6(\mathbb{Q}) \geq 9$,
3. $w_8(\mathbb{Q}) \geq 32$.

[a] $w_4(\mathbb{Q}) \in \{15, 16\}$

Waring numbers of local rings and their henselizations and completions

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Example

$$w_4(\mathbb{Z}[\sqrt{2}]) \geq \sup_{p-\text{prime}} w_4(\mathbb{Z}_p[\sqrt{2}]) = w_4(\mathbb{Z}_2[\sqrt{2}]) = 7,$$

$$w_4(\mathbb{Q}(\sqrt{2})) \geq \sup_{p-\text{prime}} w_4(\mathbb{Q}_p(\sqrt{2})) = w_4(\mathbb{Q}_2(\sqrt{2})) = 7.$$

Waring numbers of local rings and their henselizations and completions

Problems:

1. Assume that $\mathfrak{m} \neq \mathfrak{m}^2$ and $\mathrm{char}(R) = \mathrm{char}(k) \mid n$. In this case, nilpotents may occur in the completion, and we do not know how to compute $w_n(R)$. This is because the Frobenius map is not injective.

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2. Assume that $\mathfrak{m} \neq \mathfrak{m}^2$ and $\mathrm{char}(R) \neq \mathrm{char}(k) \mid n$. This is equivalent to $n \in \mathfrak{m}$. Here, we are not able to compute $w_n(R)$ if R is NOT a DVR. In particular, our theory cannot be applied to the case $n = p$ and $R = \mathbb{Z}_p[[x]]$. It can be shown by different methods that for $n = p$, $w_p(\mathbb{Z}_p[[x]])$ is finite. In n is a multiple of p nothing is known in this case.

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3. The last case deals with arbitrary n and $\mathfrak{m} = \mathfrak{m}^2$. In most cases we have an upper bound for $w_n(R)$, however these bounds may be sharp. Completion of such a ring degenerates into the residue field. Hence, it is possible $w_n(R^h) > w_n(\widehat{R})$.

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This is known for $n = 2$. If $\frac{1}{2} \in R$ then $w_2(R) = w_2(K)$ (Kneser and Colliot-Thélène).

Thank you!

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