On the zero-sum invariants over $C_n \rtimes_s C_2$

Conference on Rings and Factorizations

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Partially supported by
FAPEMIG (RED-00133-21 and APQ-02546-21) and NAWI Graz

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Graz, July 10–14, 2023

Summary

1. Zero-sum problems
2. The invariants
3. On the group $C_n \rtimes_s C_2$
4. An open problem

Zero-sum problems

Let G be a finite group multiplicatively written.

The zero-sum problems study conditions to ensure that a given sequence over G has a non-empty subsequence (with some prescribed property which include lengths, weights, repetitions, etc) whose product of the terms (in some order) equals 1, the identity of G.

The terminology "zero-sum problems" relies on the abelian groups, where an additive notation is used.

Sequences over groups

A sequence S over G is a finite and unordered element of the free abelian monoid $\mathcal{F}(G)$ equipped with the concatenation product denoted by $\cdot$.

$$S = g_1 \cdot \ldots \cdot g_k = \prod^{\bullet}_{g \in G} g^{[\mathsf{v}_g(S)]} \in \mathcal{F}(G).$$

Remark: g^2 is the square of g and $g^{[2]} = g \cdot g$ is a two-terms sequence.

$T \in \mathcal{F}(G)$ is a subsequence of S if $T \mid S$ as elements of $\mathcal{F}(G)$, that is, if $\mathsf{v}_g(T) \le \mathsf{v}_g(S)$ for every $g \in G$. In this case,

$$S \cdot T^{[-1]} = \prod^{\bullet}_{g \in G} g^{[\mathsf{v}_g(S) - \mathsf{v}_g(T)]}.$$

If $K \subset G$, then let $S_K = \prod^{\bullet}_{g \in K} g^{[\mathsf{v}_g(S)]}$.

$\pi(S) = \{g_{\tau(1)} \ldots g_{\tau(k)};\ \tau$ is a permutation of $[1, k]\}$ is the set of products of S.

$\Pi(S) = \bigcup\limits_{\substack{T \mid S \\ |T| \geq 1}} \pi(T) \subset G$ is the set of subproducts of S.

We say that S is:

- product-one sequence if $1 \in \pi(S)$;
- n-product-one sequence if $1 \in \pi(S)$ and $|S| = n$;
- product-one free if $1 \notin \Pi(S)$;
- n-product-one free if $1 \notin \pi(T)$ for any $T \mid S$ with $|T| = n$.

The invariants

The small Davenport constant of G is defined by

$$\mathsf{d}(G) := \sup\{|S|;\ S \in \mathcal{F}(G) \text{ is product-one free}\}.$$

The Gao constant of G, $\mathsf{E}(G)$, is the smallest positive integer such that every sequence $S \in \mathcal{F}(G)$ with $|S| \geq \mathsf{E}(G)$ has a $|G|$-product-one subsequence.

- $\mathsf{E}(G) \geq \mathsf{d}(G) + |G|$.

  Gao conjecture: equality holds.

  It has been proven for abelian groups.

The direct and inverse problems

Fixed a finite group G, the direct problems consist on obtaining the precise values of the constants, while the inverse problems consist on obtaining the structure of ($|G|$-)product-one free sequences of large (or maximal) length.

Goal: introduce the inductive method to obtain the direct and inverse problems over non-abelian groups.

On the cyclic groups

Let $C_n = \langle y \mid y^n = 1 \rangle$ the cyclic group of order n.

We have $\mathsf{d}(C_n) = n - 1$ and $\mathsf{E}(C_n) = 2n - 1$.

Proposition (inverse problem for $\mathsf{E}(C_n)$, Gao 1997)

Let $n \geq 2$ and $S \in \mathcal{F}(C_n)$ with $|S| = 2n - k$, where $2 \leq k \leq \lfloor n/2 \rfloor + 2$. If S is n-product-one free, then there exists $a \cdot b \mid S$ such that $C_n = \langle ab^{-1} \rangle$, $\min\{\mathsf{v}_a(S), \mathsf{v}_b(S)\} \geq n - 2k + 3$.

In particular, $|S| = 2n - 2 \Rightarrow S = (a \cdot b)^{[n-1]}$.

On the dihedral groups

Let $D_{2n} = C_n \rtimes_{-1} C_2 = \langle x, y \mid x^2 = y^n = 1, yx = xy^{-1} \rangle$ be the dihedral group of order $2n$.

We have $\mathsf{d}(D_{2n}) = n$ and $\mathsf{E}(D_{2n}) = 3n$.

Proposition (inverse problem for $\mathsf{E}(D_{2n})$, Oh–Zhong 2020)

Let $n \geq 4$ and $S \in \mathcal{F}(D_{2n})$ of length $|S| = \mathsf{E}(D_{2n}) - 1 = 3n - 1$. Then S is $2n$-product-one free $\iff \exists \alpha, \beta \in D_{2n}$, $t_1, t_2, t_3 \in \mathbb{Z}$ such that $D_{2n} \cong \langle \alpha, \beta \mid \alpha^2 = \beta^n = 1, \beta\alpha = \alpha\beta^{-1} \rangle$, $\gcd(t_1 - t_2, n) = 1$ and

$$S = (\beta^{t_1})^{[2n-1]} \cdot (\beta^{t_2})^{[n-1]} \cdot \alpha\beta^{t_3}.$$

On the group $C_n \rtimes_s C_2$

Let

$$G_{n,s} := C_n \rtimes_s C_2 = \langle x, y \mid x^2 = y^n = 1, yx = xy^s \rangle,$$

where $s^2 \equiv 1 \pmod{n}$ but $s \not\equiv \pm 1 \pmod{n}$.

We have $\mathrm{d}(G_{n,s}) = n$ (trivial). The inverse problem consists of the sequences $\beta^{[n-1]} \cdot \alpha\beta^t$, where $G_{n,s} = \langle \alpha, \beta \mid \alpha^2 = \beta^n = 1, yx = xy^s \rangle$.

It is possible to factorize

$$n = n_1 n_2,$$

where $s \equiv -1 \pmod{n_1}$ and $s \equiv 1 \pmod{n_2}$.

The main result

Theorem (Avelar–Brochero Martínez–R. 2023)

Let n and s be as before and suppose additionally that $n_1 \geq 5$ when n is odd. We have

$$\mathsf{E}(G_{n,s}) = 3n.$$

Moreover, the $2n$-product-one free sequences of length $3n-1$ are

$$S = (\beta^{t_1})^{[2n-1]} \cdot (\beta^{t_2})^{[n-1]} \cdot \alpha\beta^{t_3}.$$

We already have $\mathsf{E}(G_{n,s}) \geq \mathsf{d}(G_{n,s}) + |G| = 3n$.

The case n odd

Let $S \in \mathcal{F}(G_{n,s})$ with $|S| = 3n$, where n is odd and $n_1 \geq 5$.

Let $H = \langle x, y^{n_2} \rangle \cong D_{2n_1}$, $H \lhd G_{n,s}$, so that $G_{n,s}/H \cong C_{n_2}$.

Since $|S| > \mathsf{E}(C_{n_2}) = 2n_2 - 1$, we may decompose

$$S = T_1 \cdot \ldots \cdot T_{3n_1-1} \cdot R,$$

where the T_i's are n_2-product-H subsequences and $|R| = n_2$.

Since $3n_1 - 1 = \mathsf{E}(H) - 1$, we use the inverse problem to ensure that if $h_i \in \pi(T_i)$, then w.l.o.g.

$$h_i = \begin{cases} y^{t_1 n_2} & \text{for } i \in [1, 2n_1 - 1], \\ y^{t_2 n_2} & \text{for } i \in [2n_1, 3n_1 - 2], \\ xy^{t_3 n_2} & \text{for } i = 3n_1 - 1. \end{cases}$$

Any decomposition of S as before must satisfy the equality above.

Notice that

- $\pi(T_i) = \{h_i\}$ and $|(T_i)_{x\langle y\rangle}|$ is even for $i \in [1, 3n_1 - 2]$.
- $xy^\alpha \cdot xy^\beta = xy^\beta \cdot xy^\alpha \iff \alpha \equiv \beta \pmod{n_1}$,
- $xy^\alpha \cdot y^\gamma = y^\gamma \cdot xy^\alpha \iff \gamma \equiv 0 \pmod{n_1}$.

It follows that if $|(T_i)_{x\langle y\rangle}| > 0$, then $h_i = 1$.

General idea: split into subcases and in each of them we guarantee we can avoid the sequence $h_1 \cdot \ldots \cdot h_{3n_1-1}$, by (for instance) changing the order of products or obtaining anoter term in $x\langle y^{n_2}\rangle$.

The inverse problem runs similarly. $\square$

The case n even

Suppose n is even.

Let $H = \langle y^{n_1} \rangle \cong C_{n_2}$, $H \lhd G_{n,s}$, so that $G_{n,s}/H \cong D_{2n_1}$.

It suffices to show that if $|S| = 2n$, then S contains an n-product-one subsequence.

We may decompose $S = T_1 \cdot \ldots \cdot T_{2n_2-1} \cdot R$, where the T_i's are n_1-product-H subsequences and $|R| = n_1$.

Since $\mathsf{E}(H) = 2n_2 - 1$, S contains a $2n$-product-one subsequence.

Therefore $\mathsf{E}(G_{n,s}) = 3n$.

The inverse problems run similarly. $\square$

An open problem

Let $G_{m,n,s} \cong C_n \rtimes_s C_m$, where $ord_n(s) = m$. Consider the following assertions.

(a) $\mathsf{E}(G_{m_0,n_0,s^2}) = m_0 n_0 + m_0 + n_0 - 2$.

(b) If $S \in \mathcal{F}(G_{m_0,n_0,s^2})$ has length $|S| = \mathsf{E}(G_{m_0,n_0,s^2}) - 1$ and has no $m_0 n_0$-product-one subsequence, then

$$S = (y^{\alpha})^{[\ell n_0 - 1]} \boldsymbol{\cdot} (y^{\beta})^{[m_0 n_0 + n_0 - \ell n_0 - 1]} \boldsymbol{\cdot} \prod^{\bullet}_{1 \le i \le m_0 - 1} x^{\omega} y^{\gamma_i},$$

where $\gcd(\alpha - \beta, n_0) = 1$, $\gcd(\omega, m_0) = 1$ and $\ell \in [1, m_0]$.

It is proven that if (a) and (b) hold, then $\mathsf{E}(G_{m,n,s}) = mn + m + n - 2$, where $m = 2m_0$ and $n = 2n_0$.

Assuming (a) and (b), can we solve the inverse problem?

That's all, folks. Thank you!