The Importance of Being Prime: A Tribute to Nick Baeth

July 14, 2023

Nick Baeth 1978 – 2021

Born: 1978 in Libby, Montana

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On the Importance of Being Prime

The importance of being prime*, a nontrivial generalization for nonunique factorizations

Nicholas R. Baeth and Scott T. Chapman

This paper is dedicated to the memory of Nick Baeth. It reflects not only the mathematics that he loved, but his passion for working with undergraduates.

Abstract. The notion of primeness is the key to the phenomenon of unique factorization. In particular, when unique factorization in a monoid fails, the arithmetic of that monoid is determined by the irreducible elements which are not prime. We illustrate this with examples of easy-to-understand monoids which are, for the most part, multiplicative submonoids of the natural numbers. Through these examples, we examine the ω-invariant, which offers a quantification of both primeness and nonunique factorization. We close by shifting gears and illustrating the same concepts in noncommutative semigroups, again by using relatively simple constructions involving positive integers.

1. PROLOGUE In many areas of mathematics, the notion of unique factorization plays a key role. In elementary number theory, for example, most of the fundamental results depend on the unique factorization of positive integers into products of prime integers. Almost all students of mathematics know what a prime number is, and likely recall that a prime is an integer larger than one which cannot be factored as a product of two integers, each larger than one. Once algebraic structures evolve beyond the set of integers, this well-understood

What IS so Important About Being Prime?

Theorem

An atomic monoid M is factorial if and only if every irreducible element of M is prime.

A Brief Digression: Expository/Survey Papers

- W. Gao and A. Geroldinger, Zero-sum problems in finite abelian groups: a survey, *Expo. Math.* (2006): **Citations: 322**.
- A. Geroldinger, Sets of lengths, *Amer. Math. Monthly* (2016): **Citations: 87**.
- F. Halter Koch, Halbgruppen mit divisorentheorie, *Expo. Math.* (1990): **Citations: 64**.
- N. Baeth and R. Wiegand, Factorization theory and decompositions of modules, *Amer. Math. Monthly* (2013): **Citations: 54**.
- P-J. Cahen and J-L. Chabert, What you should know about integer-valued polynomials?, **Amer. Math. Monthly** (2016), **Citations: 38**.

The Importance of Being Prime

What is in *The Importance of Being Prime?*

- The paper is presented like a play in three Acts with an Epilogue. The main definitions and ideas are introduced through basic (and some non-basic) examples most of which are submonoids of $(\mathbb{N}_0, +)$ or $(\mathbb{N}, x)$.
- Act 1: $\mathbb{Z}$ and its interesting children. An ending moral: An atomic monoid with a non-prime irreducible is not factorial.
- Act 2: A basic exploration of what non-unique factorization can entail (elasticity, semi-length functions, accepted elasticity, full elasticity, $\ldots$).
- Act 3: The Omega function which **measures** primeness.
- Epilogue: Primeness in the noncommutative setting.

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A Review of Nick's Research Career

The early years: 2005–2009. Papers focusing on Ring/Module Theory.

- A Krull-Schmidt theorem for one-dimensional CM local rings of FCMT, *J. Pure Appl. Algebra* 208 (2007).
- Bounds for indecomposable torsion-free modules, *J. Pure Appl. Algebra* 213 (2009), with M. R. Luckas.
- Direct sum decompositions over two-dimensional local domains, *Comm. Algebra* 37 (2009).

The Turn Toward Factorizaton Theory

The years 2009-2014. Nick's work veers toward Factorization Theory and Undergraduate Research

- Atoms of the relative block monoid, *Involve* 2 (2009), with J. C. Hoffmeier.
- Number theory in matrix semigroups, *Linear Algebra Appl.* 434 (2011), with V. Ponomarenko et. al.
- Irreducible divisor graphs and factorizations in atomic domains, *Comm. Algebra* 39 (2011), with M. Axtell and J. Stickles.
- Irreducible divisor graphs for numerical monoids, *Involve* 5 (2012), with D. Bachman and C. Edwards
- Factorization in k-furcus semigroups, *Involve* 5 (2012), with K. Cassity.
- Irreducible divisor simplicial complexes, *Involve* 6 (2013), with J. Hobson.

The Extension of Nick's Collaborators and his most significant work in Factorization Theory

Due to the influence of his time in Graz, the years 2014–2021 saw Nick's ring of collaborators grow which lead to the several major papers in Factorization Theory.

- Monoids of modules and arithmetic of direct-sum decompositions, *Pacific J. Math.*, 271 (2014), with A. Geroldinger.
- A semigroup-theoretical view of direct-sum decompositions, *J. Algebra Appl.*, 14 (2015), with A. Geroldinger, D. Grynkiewicz, and D. Smertnig.
- Factorization theory: From commutative to noncommutative settings, *J. Algebra*, 441 (2015), with D. Smertnig.
- Arithmetical invariants in local quaternion orders, *Acta Arith.*, 186 (2018), with D. Smertnig.
- Factorizations in upper triangular matrices over information semialgebras, *J. Algebra* 562 (2020), with F. Gotti.

Two of Nick's Later Papers

1. Multiplicative factorization in numerical semigroups, *International Journal of Algebra and Computation*, 30.02 (2020), with Matthew Enlow.
2. Complement-Finite ideals, *Algebraic, Number Theoretic, and Topological Aspects of Ring Theory*, to appear.

Multiplicative Factorization in Numerical Semigroups

Let $S = \langle n_1, n_2, \ldots, n_k \rangle$ be a numerical semigroup (obviously under $+$) with

$$n_1 < n_2 \cdots < n_k.$$

It is well known that

$$\rho(S) = \frac{n_k}{n_1}.$$

Set $S_1 = S \backslash \{0\} \cup \{1\}$. Clearly S_1 is a monoid under regular multiplication.

Question

Given S as above, what is $\rho(S_1)$?

More Multiplicative Factorization

Unlike the situation for S, the question does not have a simple or clean answer. In [1], Nick does the following.

- In Theorem 2.1 he characterizes the atoms of S_1 in terms of the set $\mathbb{P}\backslash S = \{q_1, \ldots, q_t\}$.
- In Theorem 3.3, he constructs the length sets of some particular elements in S_1.
- In Theorem 3.5, he argues that $\rho(S_1)$ is finite and accepted. He also finds some rough bounds on $\rho(S_1)$.
- He closes the paper with 3 questions (a couple are very involved) and most of them are still open.

We note that very few values of $\rho(S_1)$ are known. Even specializing to "managable" numerical semigroups (like $S = \langle n, n+1, \ldots, 2n-1\rangle$ or more generally $S = \langle n, n+k, n+2k, \ldots, n+tk\rangle$) does not simplify the problem.

Compliment-Finite Ideals

Let $F = \mathcal{F}(P)$ be a free abelian monoid with basis P, the set of primes in F. We take S to be a submonoid of F satisfying the following two properties.

(CF1) $\mid F \backslash S \mid < \infty$, and

(CF2) $fs \in S$ for all $f \in F$ and all $s \in S \backslash \{1\}$.

Then $S \backslash \{1\}$ is a complement-finite ideal of F. Since S is assumed to contain the identity 1, it is not an ideal of F. However, for convenience, we call any monoid S containing 1 and satisfying Properties (CF1) and (CF2) a complement-finite ideal (of F).

Examples

1. Numerical monoids.
2. Generalized Numerical monoids. These are affine submonoids of $\mathbb{N}_0^k$ with finite compliment.
3. Let G be an additive finite abelian group. We are all well acquainted with the submonoid of $\mathcal{F}(G)$ consisting of zero-sum sequences, the Block monoid $\mathcal{B}(G)$. Here is a similar less well-known construction. Let $\mathcal{F}_{\mathcal{B}}(G)$ be the submonoid of $\mathcal{F}(G)$ consisting of sequences which are NOT zero-sum free. It is clear that

$$\mathcal{B}(G) \subseteq \mathcal{F}_{\mathcal{B}}(G) = \mathcal{F}(G)\mathcal{B}(G) \subseteq \mathcal{F}(G)$$

and that $\mid \mathcal{F}(G)\backslash\mathcal{F}_{\mathcal{B}}(G) \mid < \infty$. Hence $\mathcal{F}_{\mathcal{B}}(G)$ is a compliment-finite ideal of $\mathcal{F}(G)$.

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Some Comments on Compliment-Finite Ideals

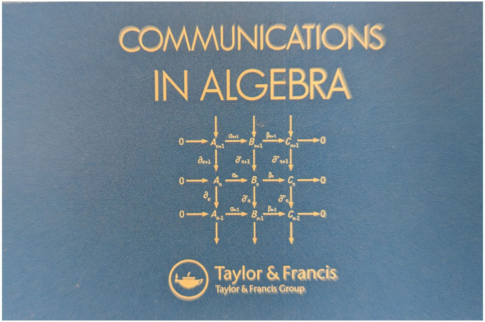
Paper [2] contains (among other things) the following.

- Compliment-finite ideals are never Krull but are C-monoids.
- Proposition 4.2 characterizes the irreducible elements of a compliment-finite ideal.
- Section 4 begins an examination of the arithmetic of a compliment-finite ideal.
- Section 5 examines the basic properties and arithmetic of $\mathcal{F}_{\mathcal{B}}(G)$.

Let's Always Remember Nick

- Nick is a perfect counterexample to the old adage that mathematicians do their best work very early in their careers. Even though he spent many years at liberal arts institutions with high teaching loads, the quality of his work, after his initial years at Central Missouri State, improved and appeared in stronger journals.
- While Nick is gone, his mathematical legacy is not.

Communications in Algebra



We have 5 Editors here who are happy to field questions about submitted papers:

Jim Coykendall

Pedro Garcia-Sanchez

Alfred Geroldinger

Felix Gotti

Alberto Facchini