On Waring numbers of Henselian rings and fields
General results

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By nth Waring number of R we mean

$$w_n(R) = \sup\{\ell_n(a) : \; a \in R, \; \ell_n(a) < \infty\}.$$

An overview of well known results

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$$w_2(\mathbb{Z}) = w_2(\mathbb{Q}) = 4$$

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Theorem (Choi, Dai, Lam, Reznick, 1982)

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Theorem (Choi, Lam, Reznick, 1995)

$$s + 2 \leq w_2(\mathbb{R}(x_1, x_2, \ldots, x_s)) \leq 2^s, \ \ s \in \mathbb{N}_{\geq 2}$$

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Theorem (Dai, Lam, Peng, 1980)

Let $s \in \mathbb{N}_+$ and $A_s = \mathbb{R}[x_1, x_2, \ldots, x_s]/(x_1^2 + x_2^2 + \cdots + x_s^2 + 1)$. Then $s_2(A_s) = s$.

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Theorem (Pfister, 1965)

If K is a nonreal field, then $s_2(K) = 2^d$ for some $d \in \mathbb{N}$. On the other hand, for each $d \in \mathbb{N}$ there exists a field K with $s_2(K) = 2^d$.

If $s_n(R) < \infty$ and $n! \in R^*$, then

$$w_n(R) \leq n w_n(\mathbb{Z})(s_n(R)+1)$$

as

$$n!x = \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{n-1}{r} [(x+r)^n - r^n].$$

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Theorem (Becker, 1982)

If K is a field and $d \in \mathbb{N}_+$, then

$$w_2(K) < \infty \iff w_{2d}(K) < \infty.$$

An overview of well known results

Theorem (Ryley, 1825)

$$w_3(\mathbb{Q}) = 3$$

as

$$(p^3 + qr)^3 + (-p^3 + pr)^3 + (-qr)^3 = a(6avp^2)^3,$$

$$\{p, q, r\} = \{a^2 + 3v^3, a^2 - 3v^3, 36a^2v^3\}.$$

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The identity of Israel

$$\left(\frac{27m^3 - n^9}{27m^2n^2 + 9mn^5 + 3n^8}\right)^3 + \left(\frac{-27m^3 + 9mn^6 + n^9}{27m^2n^2 + 9mn^5 + 3n^8}\right)^3 + \left(\frac{27m^2n^3 + 9mn^6}{27m^2n^2 + 9mn^5 + 3n^8}\right)^3 = m$$

shows that $w_3(K) \leq 3$ for **any** field K.

Henselian rings

We say that a local ring R is *Henselian*, if for every $f \in R[x]$ and $b \in R$ such that $f(b) \in \mathfrak{m}$ and $f'(b) \notin \mathfrak{m}$ there exists $a \in R$ such that $f(a) = 0$ and $a \equiv b \pmod{\mathfrak{m}}$.

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If R is a valuation ring with valuation ν, then R is Henselian if and only if for every $f \in R[x]$ and $b \in R$ such that $\nu(f(b)) > 2\nu(f'(b))$, then there exists an element $a \in R$ such that $f(a) = 0$ and $\nu(a - b) > \nu(f'(b))$.

Preliminary results

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Fact

Let $\varphi : R \to S$ be a homomorphism of rings. Then for any $x \in R$ and for any positive integer $n > 1$ the following inequality holds $\ell_n(\varphi(x)) \leq \ell_n(x)$. If φ is an epimorphism, then $w_n(S) \leq w_n(R)$.

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Fact

If S a localization of a ring R, then $w_n(S) \leq w_n(R)$.

Proof.

Follows from the fact that

$$\ell_{n,S}\left(\frac{a}{b}\right) \leq \ell_{n,R}(ab^{n-1}),$$

where $a, b \in R$ and $\frac{a}{b} \in S$. $\square$

Preliminary results

Let p be a prime integer and n be a positive integer written in the form $n = p^k m$, where p does not divide m and $k \geq 0$. We then say that m is the p-free part of n. We extend this definition to the case $p = 0$ and put $m = n$.

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Fact

Let R be a ring of prime characteristic p and $n > 1$ be a positive integer. If $n = p^k m$, where m is the p-free part of n, then

$$s_n(R) = s_m(R)$$

and

$$w_n(R) \leq w_m(R).$$

If we further assume that R is reduced, then

$$w_n(R) = w_m(R).$$

Theorem

Let R be a local ring with the maximal ideal $\mathfrak{m}$ and the residue field k. Let n be a positive integer and m be the $\mathrm{char}(k)$-free part of n. Assume that $\mathrm{char}(k) \nmid n$ or $\mathrm{char}(R) = \mathrm{char}(k)$ and R is reduced. Then, the following statements are true.

a) We have $s_n(R) \geq s_m(k)$ and $w_n(R) \geq w_m(k)$.

b) If R is Henselian and $s_m(k) < \infty$, then $s_n(R) = s_m(k)$ and $w_n(R) \leq \max\{w_m(k), s_m(k) + 1\}$.

c) If $f \in \mathfrak{m} \backslash \mathfrak{m}^2$ and $m > 1$, then $\ell_n(f) \geq s_m(k) + 1$. In particular, $w_n(R) \geq s_m(k) + 1$ on condition that $\mathfrak{m} \neq \mathfrak{m}^2$.

Corollary

Let R be a Henselian local ring with the maximal ideal $\mathfrak{m} \neq \mathfrak{m}^2$, residue field k and $s_m(k) < \infty$. Let n be a positive integer and m be the $\mathrm{char}(k)$-free part of n. Assume that $\mathrm{char}(k) \nmid n$ or $\mathrm{char}(R) = \mathrm{char}(k)$ and R is reduced. Then the following holds

$$w_n(R) = \begin{cases} \max\{w_m(k), s_m(k) + 1\} & \text{for } m > 1 \\ 1 & \text{for } m = 1 \end{cases}.$$

nth Waring numbers of Henselian discrete valuation rings with finite nth level

Theorem

Assume that R is a Henselian DVR and $s_m(R/\mathfrak{m}^{2\nu(m)+1}) < \infty$. Then $s_n(R) = s_m(R/\mathfrak{m}^{2\nu(m)+1})$ and

i) $w_n(R) = \max\{w_m(R/\mathfrak{m}^{2\nu(m)+1}), s_m(R/\mathfrak{m}^{2\nu(m)+1}) + 1\}$ if $m > 1$ and $n > 2\nu(m) + 1$;

ii) $w_n(R) = w_m(R/\mathfrak{m}^{2\nu(m)+1})$ if $m > 1$ and $n \leq 2\nu(m) + 1$;

iii) $w_n(R) = 1$ if $m = 1$.

Moreover, if $\mathrm{char}(R) \nmid n$ and every element of $R/\mathfrak{m}^{2\nu(m)+1}$ can be written as a sum of nth powers in $R/\mathfrak{m}^{2\nu(m)+1}$, then every element of R can be written as a sum of $w_n(R)$ nth powers in R.

nth Waring numbers of total rings of fractions of Henselian rings with finite nth level

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Theorem

Let R be a Henselian local ring with the total ring of fractions $Q(R) \neq R$ and the residue field k. Let n be a positive integer and m be the $\mathrm{char}(k)$-free part of n. Assume that $\mathrm{char}(k) \nmid n$ or $\mathrm{char}(R) = \mathrm{char}(k)$. Then,

$$w_n(Q(R)) \leq \begin{cases} s_m(k) + 1 & \text{for } m > 1 \\ 1 & \text{for } m = 1 \end{cases},$$

where the equality in the case of $m > 1$ holds under assumption that R is an integral domain, $s_m(Q(R)) = s_m(R)$ and there exists a nontrivial valuation $\nu : Q(R) \to \mathbb{Z} \cup \{\infty\}$ such that $R \subset R_\nu := \{f \in Q(R) \,|\, \nu(f) \geq 0\}$. Moreover, if $\mathrm{char}(k) \nmid n$, then every element of $Q(R)$ can be written as a sum of $w_n(Q(R))$ nth powers in $Q(R)$.

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Theorem

Let R be a Henselian valuation ring. If $s_m(R) < \infty$, then for each $g \in K$ we have

$$\begin{aligned}\ell_{n,K}(g) &= \inf \left\{ \ell^*_{m,R}(gh_1^n), \ell^*_{m,R}\left(\frac{g}{h_2^n}\right) \;\middle|\; h_1, h_2 \in R, \frac{g}{h_2^n} \in R \right\} \\ &= \inf \left\{ \ell^*_{m,R/I_{2\nu(n)}}(\overline{gh_1^n}), \ell^*_{m,R/I_{2\nu(n)}}\left(\overline{\frac{g}{h_2^n}}\right) \;\middle|\; h_1, h_2 \in R, \frac{g}{h_2^n} \in R \right\}.\end{aligned}$$

Moreover, if $\mathrm{char}(K) \nmid n$, then for every element $f \in K$ we have $\ell_n(f) < \infty$.

Corollary

Let R be a Henselian valuation ring with the field of fractions K. If $s_m(R) < \infty$, then we have the following inequality:

$$w_n(K) \leq s_n(R/I_{2\nu(n)}) + 1.$$

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Corollary

Assume additionally that R is a DVR. If $s_m(R) < \infty$ and $n > 2\nu(m) + 1$, then

$$w_n(K) = s_m(R/\mathfrak{m}^{2\nu(m)+1}) + 1.$$

nth Waring numbers of Henselian DVRs and their fields of fractions with infinite nth level

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Theorem

Let R be a DVR with the field of fractions K and the residue field k. Take a positive integer n such that $s_n(k) = \infty$. Then

$$w_n(K) = w_n(R) \geq w_n(k),$$

where the equality holds if R is Henselian.

To be continued... (:

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Thank you!