Affine Semigroups of Maximal Projective Dimension

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Contents

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and non-negative integers respectively.

Numerical Semigroup

A submonoid S of $\mathbb{N}$ is called a numerical semigroup if $\mathbb{N} \setminus S$ is finite. Equivalently, there exist $m_0, m_1, \ldots, m_p \in \mathbb{N}$ with $\gcd(m_0, m_1, \ldots, m_p) = 1$ such that

$$S := \langle m_0, m_1, \ldots, m_p \rangle = \left\{ \sum_{i=0}^{p} \lambda_i m_i \mid \lambda_i \in \mathbb{N} \right\}.$$

Here S is called the numerical semigroup generated by $m_0, m_1, \ldots, m_p$.

- Let f be the largest integer such that $f \notin S$, then f is called the Frobenius number of S, and denoted by $\mathrm{F}(S)$.

- An element $f \in \mathbb{Z} \setminus S$ is called a pseudo-Frobenius number if $f + s \in S$ for all $s \in S \setminus \{0\}$. We will denote the set of pseudo-Frobenius numbers of S by $\mathrm{PF}(S)$.

- A numerical semigroup S is symmetric if $\mathrm{PF}(S) = \{\mathrm{F}(S)\}$.

- A numerical semigroup S is pseudo symmetric if $\mathrm{PF}(S) = \{\mathrm{F}(S), \mathrm{F}(S)/2\}$.

Affine Semigroup (pointed)

An affine semigroup is a finitely generated submonoid S of $\mathbb{N}^r$ minimally generated by $a_1, \ldots, a_n$, and denoted by $S = \langle a_1, \ldots, a_n \rangle$. The cardinality of the minimal generating set of S is called the embedding dimension of S, denoted by $e(S)$.

Affine Semigroup Ring

Let S be an affine semigroup in $\mathbb{N}^r$ minimally generated by $a_1, \ldots, a_n$. The semigroup ring $\mathbb{K}[S] = \mathbb{K}[\mathbf{t}^{a_1}, \ldots, \mathbf{t}^{a_n}]$ of S is a $\mathbb{K}$-subalgebra of the polynomial ring $\mathbb{K}[t_1, \ldots, t_r]$ over the field $\mathbb{K}$, where $\mathbf{t}^{a_i} = t_1^{a_{i1}} \cdots t_d^{a_{ir}}$ for $a_i = (a_{i1}, \ldots, a_{ir})$.

- Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and define a map

$$\pi : R \to \mathbb{K}[S]$$

$$x_i \mapsto \mathbf{t}^{a_i},\ i = 1, \ldots, n.$$

Note that π is a surjective $\mathbb{K}$-algebra homomorphism, and thus

$$\mathbb{K}[S] \cong \frac{R}{\mathrm{Ker}(\pi)}.$$

- Set $\deg x_i = a_i$ for all $i = 1, \ldots, n$. With this grading R is a multi-graded ring. For a monomial $\mathbf{x}^u := x_1^{u_1} \cdots x_n^{u_n}$, the S-degree of $\mathbf{x}^u$ is defined as $\deg_S \mathbf{x}^u = \sum_{i=1}^{n} u_i a_i$.

- Let I_S denote the kernel of π. Then

$$I_S = \left(\mathbf{x}^u - \mathbf{x}^v \mid \deg_S \mathbf{x}^u = \deg_S \mathbf{x}^v\right).$$

Therefore, I_S is a graded homogeneous ideal of R. Thus, $\mathbb{K}[S]$ has a graded structure inherited from R.

pseudo-Frobenius elements in Affine Semigroups

- Let S be the affine semigroup minimally generated by $\{a_1, \ldots, a_n\} \subseteq \mathbb{N}^r$. Consider the cone of S in $\mathbb{Q}^r_{\geq 0}$,

$$\mathfrak{C}(S) := \left\{ \sum_{i=1}^{n} \lambda_i a_i \mid \lambda_i \in \mathbb{Q}_{\geq 0}, i = 1, \ldots, n \right\}$$

and set $\mathcal{H}(S) := (\mathfrak{C}(S) \setminus S) \cap \mathbb{N}^r$.

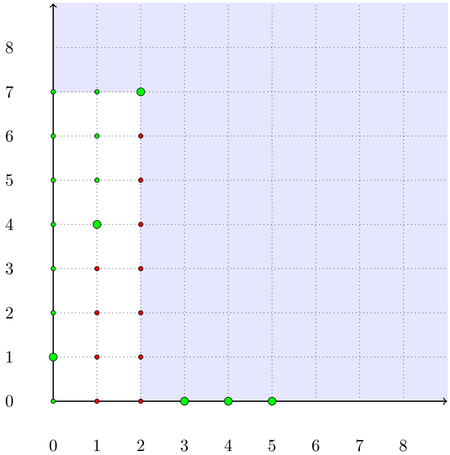
Definition

An element $f \in \mathcal{H}(S)$ is called a **pseudo-Frobenius element** of S if $f + s \in S$ for all $s \in S \setminus \{0\}$. The set of pseudo-Frobenius elements of S is denoted by $\mathrm{PF}(S)$. In particular,

$$\mathrm{PF}(S) = \{f \in \mathcal{H}(S) \mid f + a_j \in S, \ \forall j \in [1, n]\}.$$

Example:

Let $S = \langle (0,1), (3,0), (4,0), (5,0), (1,4), (2,7) \rangle$.



- $\mathcal{H}(S)$= set of all red points.
- $\mathrm{PF}(S) = \{(1,3), (2,6)\}$.

Pseudo-Frobenius elements in Affine Semigroups

Remark

Pseudo-Frobenius elements may not exist. Let

$$S = \langle (2,0), (1,1), (0,2) \rangle.$$

Then S is the subset of points in $\mathbb{N}^2$ whose sum of coordinates is even. Thus, we have that $\mathcal{H}(S) + S = \mathcal{H}(S)$. Therefore $\mathrm{PF}(S) = \emptyset$.

- If $\mathcal{H}(S)$ is finite then the set of pseudo-Frobenius elements is always non-empty.

MPD-semigroup

Let $R = \mathbb{K}[x_1, \ldots, x_n]$, we say that $S = \langle a_1, \ldots, a_n \rangle$ satisfies the **maximal projective dimension** (MPD) property if

$$\mathrm{pdim}_R \mathbb{K}[S] = n - 1.$$

Equivalently, $\mathrm{depth}_R \mathbb{K}[S] = 1$.

- (Garcia-Garcia et al., 2019), proved that S is an MPD-semigroup if and only if $\mathrm{PF}(S) \neq \emptyset$.
- In particular, if S is a MPD-semigroup then $b \in S$ is the S-degree of the $(n-2)$th minimal syzygy of $\mathbb{K}[S]$ if and only if
$$b \in \left\{ a + \sum_{i=1}^{n} a_i \mid a \in \mathrm{PF}(S) \right\}.$$

- The cardinality of $\mathrm{PF}(S)$ is equal to the last Betti number of $\mathbb{K}[S]$. We call it the Betti-type of S.

Example

Let $S = \langle a_1 = (2, 11), a_2 = (3, 0), a_3 = (5, 9), a_4 = (7, 4)\rangle$. Then, by Macaulay2, we have graded minimal free resolution of $\mathbb{K}[S]$,

$$0 \to R(-(81, 93)) \oplus R(-(94, 82)) \to R^6 \to R^5 \to R \to \mathbb{K}[S] \to 0.$$

Therefore, $\mathrm{pdim}_R \mathbb{K}[S] = 3$. Hence, S is MPD. Also, we have

$$\mathrm{PF}(S) = \left\{ (81, 93) - \sum_{i=1}^{4} a_i, (94, 82) - \sum_{i=1}^{4} a_i \right\}.$$

Therefore, $\mathrm{PF}(S) = \{(64, 89), (77, 58)\}$.

Gluing of MPD-semigroups

Definition

Let $G(S)$ be the group generated by S. Let A be the minimal generating system of S and $A = A_1 \cup A_2$ be a nontrivial partition of A. Let S_i be the submonoid of $\mathbb{N}^d$ generated by $A_i, i \in 1, 2$. Then $S = S_1 + S_2$. We say that S is the **gluing** of S_1 and S_2 along s if
(1) $s \in S_1 \cap S_2$ and,
(2) $G(S_1) \cap G(S_2) = s\mathbb{Z}$.

Theorem (– , Goel, Sengupta)

Let S be a gluing of S_1 and S_2. Then S is MPD if and only if S_1 and S_2 are MPD. Moreover,

$$\mathrm{PF}(S) = \{f + g + s \mid f \in \mathrm{PF}(S_1), g \in \mathrm{PF}(S_2)\}.$$

Sketch of proof:

- If S_1 and S_2 are MPD-semigroups then by [Garcia-Garcia et. el, 2020], S is an MPD-semigroup.
- Let the embedding dimensions of S_1 and S_2 are n_1 and n_2 respectively. Suppose without loss of generality that S_1 is not an MPD-semigroup. Therefore, we have

  $$\mathrm{pdim}_{R_1} \mathbb{K}[S_1] < n_1 - 1,$$

  where $R_1 = k[x_1, \ldots, x_{n_1}]$.

- Also, by Auslander-Buchsbaum formula,

$$\text{pdim}_{R_2}\mathbb{K}[S_2] \leq n_2 - 1,$$

where $R_2 = k[x_1, \ldots, x_{n_2}]$.

- For $R = k[x_1, \ldots, x_{n_1+n_2}]$, we have

$$\text{pdim}_R\mathbb{K}[S] = \text{pdim}_{R_1}\mathbb{K}[S_1] + \text{pdim}_{R_2}\mathbb{K}[S_2] + 1 < n_1 + n_2 - 1.$$

Since, S is MPD, this is a contradiction.

- Now set, $T = \{f + g + s \mid f \in \text{PF}(S_1), g \in \text{PF}(S_2)\}$. Then $T \subset \text{PF}(S)$.
- Now, by the minimal graded free resolution of semigroup ring associated to gluing of affine semigroups (see Gimenez and Srinivasan, 2019), we can deduce that

$$|\,\text{PF}(S)| = |\,\text{PF}(S1)| \cdot |\,\text{PF}(S2)|.$$

- Therefore, to complete the proof, it is sufficient to show that if $f + g + d, f' + g' + d \in T$ such that $f + g + d = f' + g' + d$ then $f = f'$ and $g = g'$.

Unboundedness of Betti-type

Motivated by an example of Jafari and Yaghmaei (2022), we construct the following class of examples.

- Let $a \geq 3$ be an odd natural number and $p \in \mathbb{Z}^+$. Define

$$S_{a,p} = \langle (a,0), (0,a^p), (a+2,2), (2,2+a^p) \rangle.$$

- Define the set

$$\Delta = \{(a^p(a+2) - (\ell+2)a - 2, a^p(\ell+2) - 2) \mid 0 \leq \ell < a^p - 1\}.$$

Proposition (– , Sengupta)

$S_{a,p}$ is an MPD-semigroup and $\Delta \subseteq \mathrm{PF}(S_{a,p})$.

Unboundedness of Betti-type

Theorem (– , Sengupta)

For each $e \geq 4$, there exists a class of MPD-semigroups of embedding dimension e in $\mathbb{N}^2$ such that the Betti-type is not a bounded function in terms of the embedding dimension e.

Definition

Let $\prec$ be a term order on $\mathbb{N}^d$. Then $F(S)_{\prec} = \max_{\prec} \mathcal{H}(S)$, if it exists, is called a **Frobenius element** of S. Note that Frobenius elements of S may not exist. However, if $|\mathcal{H}(S)| < \infty$, then S has Frobenius elements.

The Hilbert series of an affine semigroup algebra $\mathbb{K}[S]$ is defined as

$$\mathrm{H}(\mathbb{K}[S], \mathbf{t}) = \sum_{s \in S} \mathbf{t}^s,$$

the formal sum of all monomials $\mathbf{t}^s = t_1^{s_1} \cdots t_r^{s_r}$, where $s \in S$. It can be written as a rational function of the form

$$\mathrm{H}(\mathbb{K}[S], \mathbf{t}) = \frac{\mathcal{K}(t_1, \ldots, t_r)}{\prod_{i=1}^{n}(1 - \mathbf{t}^{a_i})},$$

where $\mathcal{K}(t_1, \ldots, t_r)$ is a polynomial in $\mathbb{Z}[t_1, \ldots, t_r]$.

Let $\exp(LT_{\prec}\ \mathcal{K}(\mathbb{K}[S];\mathbf{t}))$ be the exponent of the leading term of $\mathcal{K}(\mathbb{K}[S];\mathbf{t}))$ with respect to $\prec$.

Theorem (– , Goel, Sengupta)

Let $S = \langle a_1, \ldots, a_n \rangle \subseteq \mathbb{N}^r$ be a $\mathcal{C}$-semigroup such that $\mathfrak{C}(S) = \mathbb{Q}^r_{\geq 0}$. Then $\mathrm{F}(S)_{\prec} = \exp(LT_{\prec}\ \mathcal{K}(\mathbb{K}[S];\mathbf{t})) - \sum_{i=1}^{n} a_i$ for any term order $\prec$.

Example

Let $S = \langle a_1 = (0,1), a_2 = (2,0), a_3 = (3,0), a_4 = (1,3) \rangle$.

- $\mathrm{cone}(S) = \mathbb{Q}^2_{\geq 0}$ and $\mathcal{H}(S) = \{(1,0),(1,1),(1,2)\}$ is finite.
- Therefore, $\mathrm{F}(S)_{\prec} = (1,2)$ for any term order $\prec$.

We have,

$$\mathrm{H}(\mathbb{K}[S];\mathbf{t}) = \frac{1 - t_1^6 - t_1^3t_2^3 - t_1^4t_2^3 - t_1^2t_2^6 + t_1^6t_2^3 + t_1^7t_2^3 + t_1^4t_2^6 + t_1^5t_2^6 - t_1^7t_2^6}{(1-t_2)(1-t_1^2)(1-t_1^3)(1-t_1t_2^3)}.$$

Hence,

$$\mathrm{F}(S)_{\prec} = \exp(LT_{\prec}\ \mathcal{K}(\mathbb{K}[S];\mathbf{t})) - \sum_{i=1}^{4} a_4 = (7,6) - (6,4) = (1,2).$$

$\prec$-symmetric semigroups

Definition

Fix a term order $\prec$ such that $F(S)_\prec = \max_\prec \mathcal{H}(S)$ exists.

- If $\mathrm{PF}(S) = \{F(S)_\prec\}$, then S is called a **$\prec$-symmetric semigroup**.
- If $\mathrm{PF}(S) = \{F(S)_\prec, F(S)_\prec/2\}$, then S is called **$\prec$-pseudo-symmetric**.

$\prec$-symmetric semigroups

- If $\mathcal{H}(S)$ is a non-empty finite set, then S is said to be a $\mathcal{C}$-semigroup, where $\mathcal{C}$ denotes the cone of the semigroup. When S is a $\mathcal{C}$-semigroup, we give a characterization of $\prec$-symmetric and $\prec$-pseudo-symmetric semigroups.

Theorem (– , Goel, Sengupta)

Let S be a $\mathcal{C}$-semigroup and let $F(S)_{\prec}$ denote the Frobenius element of S with respect to an order $\prec$. Then S is a $\prec$-symmetric semigroup if and only if for each $g \in \text{cone}(S) \cap \mathbb{N}^d$ we have:

$$g \in S \iff F(S)_{\prec} - g \notin S.$$

$\prec$-symmetric semigroups

Theorem (– , Goel, Sengupta)

Let S be a $\mathcal{C}$-semigroup and let $F(S)_\prec$ denote the Frobenius element of S with respect to an order $\prec$. Then S is a $\prec$-pseudo-symmetric semigroup if and only if $F(S)_\prec$ is even, and for each $g \in \text{cone}(S) \cap \mathbb{N}^d$ we have:

$$g \in S \iff F(S)_\prec - g \notin S \text{ and } g \neq F(S)_\prec/2.$$

Wilf's Conjecture

⋆ **Conjecture**(Wilf, 1978) Let S be a numerical semigroup. Then the following inequality is true for every numerical semigroup.

$$\mathrm{F}(S) + 1 \leq e(S) \cdot |\{s \in S \mid s < \mathrm{F}(S)\}|.$$

Example

Let $S = \langle 5, 7, 9 \rangle$. Then,

- $e(S) = 3$.
- $S = \{0, 5, 7, 9, 10, 12, 14, 15 \longrightarrow\}$.
- $\mathrm{F}(S) = 13$.
- $\{s \in S \mid s < \mathrm{F}(S)\} = \{0, 5, 7, 9, 10, 12\}$.
- $\mathrm{F}(S) + 1 = 14 < 3 \cdot 6 = 18$.

- Let S be a $\mathcal{C}$-semigroup and $\prec$ be a monomial order satisfying that every monomial is preceded only by a finite number of monomials. Define the Frobenius number of S as

$$\mathcal{N}(F(S)_{\prec}) = |\mathcal{H}(S)| + |\{g \in S \mid g \prec F(S)_{\prec}\}|$$

Extended Wilf's conjecture. (Garcia-Garcia et. al., 2018) Let S be a $\mathcal{C}$-semigroup and $\prec$ be a monomial order satisfying that every monomial is preceded only by a finite number of monomials. Then

$$\mathcal{N}(F(S)_{\prec}) + 1 \leq e(S) \cdot |\{g \in S \mid g \prec F(S)_{\prec}\}|$$

Extended Wilf's conjecture

Theorem (– , Goel, Sengupta)

Let S be a $\mathcal{C}$-semigroup with full cone. If S is $\prec$-symmetric or $\prec$-pseudo-symmetric semigroup, then extended Wilf's conjecture holds.

- $\mathcal{C}$-semigroups with full cone have been studied in the literature as generalized numerical semigroups. A generalized version of Wilf's conjecture has also been studied with this terminology, and the generalized Wilf's conjecture for generalized numerical semigroups implies the extended Wilf's conjecture for $\mathcal{C}$-semigroups with full cone (see Cisto et el., 2020).

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Thank you for your attention!