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Non-integrally closed Kronecker function rings and integral domains with a unique minimal overring

Joint work with K. Alan Loper.

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1. Kronecker function rings.

Let D be an integral domain. For a polynomial $f \in D[t]$, the content is the ideal $c(f) \subseteq D$ generated by the coefficients of f.

The Nagata ring of D is $D(t) = S^{-1}(D[t])$ where

$$S = \{f \in D[t] \mid c(f) = D\}.$$

This ring was known long before Nagata.

Let $D = \mathcal{O}_K$ be the ring of integers of a number field. Then $\mathcal{O}_K$ is a Dedekind domain, but often is not a PID (equivalently not UFD, not Bezout).

Kronecker: $\mathcal{O}_K(t)$ is always a PID.

More in general, if D is a Prüfer domain, $D(t)$ is a Bezout domain and its localizations at prime ideals are valuation rings of the form $V(t)$ where V is a valuation overring of D.

However if D is not Prüfer, the ring $D(t)$ is also not Prüfer, and the most of valuation overrings do not appear as localizations.

Krull generalized Kronecker's construction to arbitrary integrally closed domains in a different way.

If $D = \overline{D}$, then $D = \bigcap_{V \in \text{Zar}(D)} V$. The Kronecker function ring of D is

$$Kr(D) = \bigcap_{V \in \text{Zar}(D)} V(t).$$

This ring is always a Bezout domain and for every $V \supseteq D$, the ring $V(t)$ appears as localization of $Kr(D)$ at some prime ideal.

Furthermore, $Kr(D) = D(t)$ if and only if D is a Prüfer domain.

2. What if D is not integrally closed?

We still would like to write $D = \bigcap_{A \in \mathcal{F}} A$ and define

$$Kr^{\mathcal{F}}(D) = \bigcap_{A \in \mathcal{F}} A(t).$$

Which kind of rings do we want in $\mathcal{F}$?

Observation.
The following conditions are equivalent for an integral domain A such that $D \subseteq A \subseteq \mathcal{Q}(D)$:

(1) A is maximal with respect to the property of excluding some element $x \in \mathcal{Q}(D) \setminus D$.

(2) A admits a unique minimal overring.

(3) A cannot be expressed as intersection of proper overrings.

We call a domain satisfying these equivalent conditions a maximal excluding domain. Clearly D is the intersection of all its maximal excluding overrings.

3. Maximal excluding domains

Some properties of maximal excluding domains are already known since past work of Gilmer, Heinzer and few more authors. Let A be maximal with respect to excluding an element x of its quotient field. Then:

- A is local and its unique minimal overring $A[x]$ has at most two maximal ideals.

- A is integrally closed if and only if it is a valuation domain with branched maximal ideal.

- If A is not integrally closed, then $A[x]$ is an integral extension. Ex. $K[[x^2, x^3]] \subseteq K[[x]]$.

- If $A[x]$ has two maximal ideals, then $\overline{A}$ is a Prüfer domain with two maximal ideals. Ex. $\mathbb{Z}_{(5)}[5i] \subseteq \mathbb{Z}_{(5)}[i]$.

- If A and $A[x]$ share the same maximal ideal, then $\overline{A}$ is a valuation domain sharing the same maximal ideal. Ex. $\mathbb{Q} + X\mathbb{Q}(i)[[X]] \subseteq \mathbb{Q}(i)[[X]]$.

In general the integral closure of a maximal excluding domain may not be a Prüfer domain (Gilmer and Hoffman first gave such an example).

Examples can be constructed using generalized power series rings.

Example.
$A = K[[\frac{y}{x^k}, k \in \mathbb{Z} \setminus \{0\}]]$ is maximal excluding with unique minimal overring equal to $A[y] = \overline{A}$. The integral closure is a one-dimensional PVD, not Prüfer.

Question. If a maximal excluding domain is one-dimensional, is the integral closure either Prüfer or a PVD?

We gave a positive answer in the case of generalized power series ring with exponents in the positive part of a totally ordered abelian group.

Example. Let $G = \mathbb{Z} \oplus \mathbb{Z}[\sqrt{2}]$ ordered lexicographically. Let A be the generalized power series ring $K[[x^s,\, s \in S]]$ where $S = S_1 \cup S_2 \cup S_3 \subseteq G_{\geq 0}$ with

$$S_1 = \{(0, a + b\sqrt{2}),\, a, b \geq 0\} \quad S_2 = \{g \in G,\, g > (1, 0)\}$$

$$S_3 = \{(1, c + d\sqrt{2}),\, c + d\sqrt{2} > 0, cd < 0\}.$$

This A is maximal excluding. Its integral closure is isomorphic to

$$K[[x, y]] + zK((x, y))[[z]],$$

which is very far from being Prüfer or PVD (it has infinitely many primes of the same height).

Question. Is the complete integral closure of a maximal excluding domain always a Prüfer domain?

4. Nagata rings vs Kronecker function rings.

We study rings of the form $Kr^{\mathcal{F}}(D) = \bigcap_{A \in \mathcal{F}} A(t)$ where $\mathcal{F}$ is a family of (maximal excluding) overrings of D such that $D = \bigcap_{A \in \mathcal{F}} A$.

Theorem.
Suppose that the integral closure of D is a Prüfer domain. Then for every family $\mathcal{F}$,

$$Kr^{\mathcal{F}}(D) = D(t).$$

Corollary.
Given an integral domain D, the following conditions are equivalent:

(1) The integral closure of D is a Prüfer domain.

(2) The operation of Nagata ring extension commutes with intersection for arbitrary collections of overrings of D.

However if A is maximal excluding and the integral closure is not Prüfer, we still have that $Kr^{\mathcal{F}}(D) = D(t)$ since the only possible defining family is $\mathcal{F} = \{D\}$.

5. Constructions of non-integrally closed Kronecker function rings.

Construction 1.

Let $F' \subseteq F$ be a finite Galois extension. Let D be a local domain with maximal ideal $\mathfrak{m}_D$ and quotient field F such that:

- $D' = D \cap F'$ is integrally closed.
- $\overline{D'}^F = \overline{D} = D[\theta_1, \ldots, \theta_n]$ where $F = \theta_1 F' + \ldots + \theta_n F'$.
- A few more technical assumptions.

For V a valuation overring of D', set $A_V = V[\mathfrak{m}_D]_{(\mathfrak{m}_D, \mathfrak{m}_V)}$.

Let $\mathcal{F}$ be the collection of all the maximal excluding overrings of D containing A_V for some V. Set $R = Kr^{\mathcal{F}}(D)$.

Theorem.
The integral closure of R is $Kr(\overline{D})$ and for every maximal ideal $\mathfrak{p}$ of R, the localization is $R_{\mathfrak{p}} = A_V(t)$.

Examples.

- $D = K[[x^2, x^3, y]],\ D' = K[[x^2, y]]$.

- $D = \mathbb{Q} + (x, y)\mathbb{Q}(i)[[x, y]],\ D' = \mathbb{Q}[[x, y]]$.

Question Is a ring of the form $Kr^{\mathcal{F}}(D)$ locally at a maximal ideal equal to the Nagata ring of some overring of D?

Construction 2.

Let A be a semilocal domain with s maximal ideals. Let D be an integral domain with the same quotient field of A such that:

- $D = \overline{D} \cap A$.
- All the residue fields of D have cardinality at least s.

Set $R = Kr(\overline{D}) \cap A(t)$.

Theorem.
The integral closure of R is $Kr(\overline{D}) \cap \overline{A}(t)$. Every maximal ideal $\mathfrak{p}$ of R is the center of $V(t)$ for some valuation overring V of D. The localization at $\mathfrak{p}$ is equal to $R_{\mathfrak{p}} = (A \cap V)_{\mathfrak{m}_V \cap A}(t)$.

In particular, if $\overline{A}$ is a Prüfer domain, we get $\overline{R} = Kr(\overline{D})$.

If A and V have no common overrings (except from the quotient field) then $(A \cap V)_{\mathfrak{m}_V \cap A}$ is equal to V.

Hence, if $\overline{A}$ is Prüfer, R is locally equal to $Kr(\overline{D})$ at all but finitely many maximal ideals.

Examples.

- $D = K[[x^2, x^3, y]] = K[[x, y]] \cap K((y))[[x^2, x^3]]$.

  In this case $\overline{A}$ is Prüfer and only one localization of R at a maximal ideal is not a valuation domain.

  This localization is equal to the Nagata ring of the ring $T = \pi^{-1}(K[[y]])$ where $\pi : K((y))[[x^2, x^3]] \to K((y))$.

- Let $B = K[[\frac{y}{x^k}, k \in \mathbb{Z} \setminus \{0\}]]$ be maximal excluding with integral closure not Prüfer.

  $D = B[[z]] = B[[z]][y] \cap K((z))[[\frac{y}{x^k}, k \in \mathbb{Z} \setminus \{0\}]]$.

  In this case the integral closure of R is not a Prüfer domain.

Bibliography:

LG, K. Alan Loper, *Non-integrally closed Kronecker function rings and integral domains with a unique minimal overring.* Preprint, ArXiv:2304.03723 (2023)

Thanks for your attention!