On the class semigroup of a class of C-monoids

Jun Seok Oh

Jeju National University

Conference on Rings and Factorizations 2023, July 10, 2023

Outline

1. C-monoids

2. Results

Monoids

- A monoid means a commutative cancellative semigroup with identity element, so that any monoid H has its quotient group $\mathsf{q}(H)$.

For a monoid H, we call

- $H' = \{x \in \mathsf{q}(H) \mid \exists\, N \in \mathbb{N} \text{ such that } x^n \in H \text{ for all } n \geq N\}$ the seminormalization of H,
- $\widetilde{H} = \{x \in \mathsf{q}(H) \mid x^N \in H \text{ for some } N \in \mathbb{N}\}$ the root-closure of H,
- $\widehat{H} = \{x \in \mathsf{q}(H) \mid \exists\, c \in H \text{ such that } cx^n \in H \text{ for all } n \in \mathbb{N}\}$ the complete integral closure of H.

Then, $H \subseteq H' \subseteq \widetilde{H} \subseteq \widehat{H} \subseteq \mathsf{q}(H)$, and H is called

- seminormal (resp., root-closed, or completely integrally closed) if $H = H'$ (resp., $H = \widetilde{H}$, or $H = \widehat{H}$).

The class semigroup

Let $H \subseteq F$ be monoids.

- For $y, y' \in F$, $y \sim_H y'$ on $F \quad \Leftrightarrow \quad y^{-1}H \cap F = (y')^{-1}H \cap F$.
- The set of congruence classes $\mathcal{C}(H,F) = \{[y] \mid y \in F\}$ (resp., $\mathcal{C}^*(H,F) = \{[y] \mid y \in (F \setminus F^\times) \cup \{1\}\}$) is the class semigroup (resp., reduced class semigroup) of H in F.
- $\mathcal{C}(H,F) = \{[y] \mid y \in F^\times\} \cup \mathcal{C}^*(H,F)$, and either

$$\{[y] \mid y \in F^\times\} \subset \mathcal{C}^*(H,F) \text{ or } \{[y] \mid y \in F^\times\} \cap \mathcal{C}^*(H,F) = \{[1]\}.$$

C-monoids

- A monoid H is called a C-monoid if H is a submonoid of a factorial monoid F such that $H \cap F^\times = H^\times$ and $\mathcal{C}^*(H, F)$ is finite.

- A domain R is a C-domain if its multiplicative monoid $R^\bullet$ is a C-monoid.

- Reinhart, 2013
  If R is a non-local semilocal Noetherian domain, then $\mathcal{C}_v(\widehat{R})$ and $\widehat{R}/(R : \widehat{R})$ are both finite if and only if R is a C-domain.

ex) If $R = \mathbb{Z}[2i]$, then $\widehat{R} = \mathbb{Z}[i]$ and $(R : \widehat{R}) = 2\mathbb{Z}[i]$, whence R is a non-Krull C-domain.

More generalily, every non-principal order in a number field is a non-Krull C-domain.

C-monoids

- Halter-Koch, 2005
  Every C-monoid is a Mori monoid, and a C-monoid is completely integrally closed if and only if its reduced class semigroup is a group.

$$\rightsquigarrow \left\{ \begin{array}{l} \text{Krull monoids with} \\ \text{finite class group} \end{array} \right\} \subset \{\text{C-monoids}\} \subset \{\text{Mori monoids}\}.$$

Krull monoids are central objects to the study of non-unique factorizations, in particular, the monoid $\mathcal{B}(G)$ of product-one sequences over a finite abelian group G.

- Cziszter-Domokos-Geroldinger, 2016
  The monoid $\mathcal{B}(G)$ is finitely generated C-monoid defined in $\mathcal{F}(G)$.
- Geroldinger-Grynkiewicz-OH-Zhong, 2022
  The following statements are equivalent:
  (a) G is abelian.
  (b) $\mathcal{B}(G)$ is a Krull monoid.
  (c) $\mathcal{B}(G)$ is a transfer Krull monoid.
  (d) $\mathcal{B}(G)$ is a weakly Krull monoid.
  (e) $\mathcal{C}(\mathcal{B}(G), \mathcal{F}(G)) \cong G$.

$\mathcal{L}\big(\mathcal{B}(G)\big)$ has an universal character.

For an atomic monoid H,

$$\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\},$$

where $\mathsf{L}(a)$ is the set of all factorization lengths k.

- Classic
  If H is a Krull monoid with finite class group G such that each class contains a prime divisor, then $\mathcal{L}(H) = \mathcal{L}\big(\mathcal{B}(G)\big)$.

- Baeth-Geroldinger-Grynkiewicz-Smertnig, 2015
  Let R be a hereditary Noetherian prime ring, and H be the monoid of stable isomorphism classes of finitely generated projective right R-modules. Then, there exist a commutative Krull monoid H_0 and a non-trivial commutative monoid D such that

$$H = \big((H_0 \setminus H_0^\times) \times D\big) \cup \big(H_0^\times \times \{1_D\}\big)$$

  is not Krull, but $\mathcal{L}(H) = \mathcal{L}\big(\mathcal{B}(G)\big)$ for some abelian group G.

C-monoids: The class semigroup

- Geroldinger-Zhong, 2019
  If H is a C-monoid, then H is seminormal if and only if its class semigroup is a union of groups.

Let $\mathcal{C}$ be a commutative semigroup, and $e, f \in \mathsf{E}(\mathcal{C})$.

- $\mathcal{C}_e = \{x \in \mathcal{C} \mid x + e = x \text{ and } x + y = e\}$ is a group with identity e, and $\mathcal{C}_e \cap \mathcal{C}_f = \emptyset$ if $e \neq f$.
- $\mathcal{C}$ is a union of groups if and only if $\mathcal{C} = \bigcup_{e \in \mathsf{E}(\mathcal{C})} \mathcal{C}_e$.

Observation

If H is a C-monoid defined in F, then

(a) if $[a] \in \mathsf{E}(\mathcal{C}^*(H, F))$, then $a \in \widehat{H}$.

(b) if H is seminormal, then $\{[x] \mid x \in H\} \subset \mathsf{E}(\mathcal{C}^*(H, F))$.

(c) if H is completely integrally closed, then $[a] \in \mathsf{E}(\mathcal{C}^*(H, F))$ if and only if $[a] = [1]$.

Outline

1. C-monoids

2. Results

Finitely primary monoids

- A monoid H is finitely primary of rank s and exponent α if there exist $s, \alpha \in \mathbb{N}$ such that H is a submonoid of a factorial monoid $F = F^\times \times \mathcal{F}\big(\{p_1, \ldots, p_s\}\big)$ satisfying

$$H \setminus H^\times \subseteq (p_1 \ldots p_s)F \qquad \text{and} \qquad (p_1 \ldots p_s)^\alpha F \subseteq H\,.$$

ex) Every numerical semigroup is a finitely primary monoid.

- For a domain R, the following statements are equivalent:
  (a) R is a root-closed 1-dimensional local Mori domain.
  (b) $R^\bullet$ is a root-closed finitely primary monoid.

Finitely primary monoids: The class semigroup

Theorem

Let H be finitely primary of rank s. Then

$$\widetilde{H} \setminus (\widetilde{H})^\times = H' \setminus (H')^\times = (p_1 \ldots p_s)F\,,$$

and $\widetilde{H}$ is a C-monoid defined in F. Moreover, if H is root-closed, then

$$\mathcal{C}^*(H,F) \cong \mathcal{C}_1 \times \ldots \times \mathcal{C}_s\,,$$

where $\mathcal{C}_i = \left\{[p_i]_H^F, [1]_H^F\right\}$ is a subsemigroup of $\mathcal{C}^*(H,F)$.

Sketch of the proof.

- $[p] \in \mathsf{E}(\mathcal{C}(H,F))$ for every prime $p \in F$, and thus
$$\mathcal{C}^*(H,F) = \{[p_1^{r_1} \cdots p_s^{r_s}] \mid r_i \in \{0,1\} \text{ for all } i \in [1,s]\}.$$
- The map $\theta \colon \mathcal{C}^*(H,F) \to \mathcal{C}_1 \times \cdots \times \mathcal{C}_s$, given by $\theta([x]) = ([p_1^{r_1}], \ldots, [p_s^{r_s}])$, is an semigroup isomorphism.

Weakly Krull Mori monoids

- A monoid H is weakly Krull if

$$H = \bigcap_{\mathfrak{p} \in \mathfrak{X}(H)} H_{\mathfrak{p}} \quad \text{and} \quad \{\mathfrak{p} \in \mathfrak{X}(H) \mid a \in \mathfrak{p}\} \text{ is finite for all } a \in H\,.$$

- A domain R is weakly Krull if and only if $R^\bullet$ is a weakly Krull monoid.

ex) Every Krull monoid is a (root-closed) weakly Krull Mori monoid.

ex) Every 1-dimensional Noetherian domain is weakly Krull.

- If H is a weakly Krull Mori monoid, then

$$\mathcal{I}_v^*(H) \cong \prod_{\mathfrak{p} \in \mathfrak{X}(H)} (H_{\mathfrak{p}})_{\mathsf{red}}\,,$$

given by $\mathfrak{a} \mapsto (a_{\mathfrak{p}} H_{\mathfrak{p}}^\times)_{\mathfrak{p} \in \mathfrak{X}(H)}$ if $\mathfrak{a}_{\mathfrak{p}} = a_{\mathfrak{p}} H_{\mathfrak{p}}$.

Weakly Krull Mori monoids: The class semigroup

Theorem

Let H be a root-closed weakly Krull Mori monoid with $\emptyset \neq \mathfrak{f} = (H : \widehat{H})$ such that $H_\mathfrak{p}$ is finitely primary for each $\mathfrak{p} \in \mathfrak{X}(H)$.

1. *If $\widehat{H}_\mathfrak{p}^\times / H_\mathfrak{p}^\times$ is finite for each $\mathfrak{p} \in \mathfrak{X}(H)$, then $\mathcal{I}_v^*(H)$ is a C-monoid defined in $\widehat{\mathcal{I}_v^*(H)}$, and there exists a semigroup isomorphism*

$$\mathcal{C}^*\left(\mathcal{I}_v^*(H), \widehat{\mathcal{I}_v^*(H)}\right) \cong \prod_{\mathfrak{p} \in P^*} \mathcal{C}^*(H_\mathfrak{p}, \widehat{H}_\mathfrak{p}) \cong \prod_{\mathfrak{p} \in P^*} \left(\mathcal{C}_1 \times \cdots \times \mathcal{C}_{s_\mathfrak{p}}\right),$$

   where, for each $\mathfrak{p} \in P^* = \{\mathfrak{p} \in \mathfrak{X}(H) \mid \mathfrak{f} \subseteq \mathfrak{p}\}$, $s_\mathfrak{p} = |\{\mathfrak{P} \in \mathfrak{X}(\widehat{H}) \mid \mathfrak{P} \cap H = \mathfrak{p}\}|$, $\mathcal{C}_i = \left\{[\mathfrak{P}_i(\mathfrak{p})], [1]\right\}$ for each $i \in [1, s_\mathfrak{p}]$, and $\{\mathfrak{P}_1(\mathfrak{p}), \ldots, \mathfrak{P}_{s_\mathfrak{p}}(\mathfrak{p})\}$ is the set of pairwise non-associated prime elements in $\widehat{H}_\mathfrak{p}$.

2. *Suppose that $\mathcal{C}_v(H)$ is finite.*
   (a) *H_{red} is a C-monoid defined in $F = \mathcal{F}(P) \times \prod_{\mathfrak{p} \in P^*} \widehat{H}_\mathfrak{p} / H_\mathfrak{p}^\times$.*
   (b) *If H_{red} is dense in F, then H is weakly factorial if and only if $\widehat{H}$ is factorial. In this case, we have $\mathcal{C}^*(H_{red}, F) \cong \mathcal{C}^*\left(\mathcal{I}_v^*(H), \widehat{\mathcal{I}_v^*(H)}\right)$.*

Weakly Krull Mori monoids: Root-closed examples

- Angermüller, 1983

  $\mathbb{Z}[\sqrt{d}]$ is root-closed, but not integrally closed if and only of d is squeare-free and $d \equiv 1 \pmod 8$.

- Picavet-L'Hermitte, 2002

  An order in a number field is root-closed, but not integrally closed if and only if $(R \colon \widehat{R})$ is an intersection of maximal ideals P_i of $\widehat{R}$ such that $|\widehat{R}/P_i| = 2$ for each P_i.

J.S. Oh, *On the class semigroup of root-closed weakly Krull Mori monoids*, Semigroup Forum **105** (2022), 517-533.

Thank you for your attention!