Determinantal zeros and factorization of noncommutative polynomials

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Rings and Factorizations (Graz, July 2023)

Outline

(1) Motivation

(2) Determinantal zeros of nc polynomials

(3) Factorization in free algebra

(4) ~~Nullstellensatz~~ Singulärstellensatz

(5) Free Bertini's irreducibility

Hilbert's Nullstellensatz

Geometry vs Algebra

$\underline{x} = (x_1, \ldots, x_d)$

Hilbert's Nullstellensatz: let $f_1, \ldots, f_\ell, g \in \mathbb{C}[\underline{x}]$. Then

$$f_1(\underline{\alpha}) = \cdots = f_\ell(\underline{\alpha}) = 0 \implies g(\underline{\alpha}) = 0 \qquad \text{for all } \underline{\alpha} \in \mathbb{C}^d$$

if and only if

$$g^r = p_1 \cdot f_1 + \cdots + p_\ell \cdot f_\ell \qquad \text{for some } p \in \mathbb{C}[\underline{x}] \text{ and } r \in \mathbb{N}.$$

Cornerstone of algebraic geometry:

solutions of polynomial equations vs ideals

Today: a noncommutative Nullstellensatz

To talk about Nullstellensatz, one needs to say what are

1. functions
2. points (evaluations) in affine space
3. zero sets
4. algebraic counterpart

Noncommutative polynomials

Let $\underline{x} = (x_1, \ldots, x_d)$ be freely noncommuting variables. Elements of the free algebra $\mathbb{C}\langle\underline{x}\rangle$ are **nc polynomials**. We can evaluate them at points in $\mathrm{M}_n(\mathbb{C})^d$. For example, if

$$f = x_1^3 x_2 x_1 x_2 + x_1 x_2 - x_2 x_1 + 2x_1 - 3$$

and $\underline{X} = (X_1, X_2) \in \mathrm{M}_n(\mathbb{C})^2$, then

$$f(\underline{X}) = X_1^3 X_2 X_1 X_2 + X_1 X_2 - X_2 X_1 + 2X_1 - 3I_n \quad \in \mathrm{M}_n(\mathbb{C}).$$

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$$\begin{array}{rcl} \text{polynomials} & \leftrightsquigarrow & \text{evaluations on } \mathbb{C}^d \\ \text{nc polynomials} & \leftrightsquigarrow & \text{evaluations on } \bigcup_{n\in\mathbb{N}} \mathrm{M}_n(\mathbb{C})^d \end{array}$$

Why all n? No nonzero nc polynomial vanishes on all matrices; for each fixed n, there are polynomials vanishing on $\mathrm{M}_n(\mathbb{C})^d$.

Dimension-free "zero sets" of an nc polynomial

Let $f_1, \ldots, f_\ell, g \in \mathbb{C}<\underline{x}>$. There are four popular choices.

Dimension-free "zero sets" of an nc polynomial

(1) nc zero set, "true" zeros

$$Z(f_1,\dots,f_\ell) = \bigcup_n \{\underline{X} \in \mathrm{M}_n(\mathbb{C})^d : f_i(\underline{X}) = 0 \ \forall i\}$$

Amitsur's Nullstellensatz[57] **for fixed** n:

$$Z(f_1,\dots,f_\ell) \cap \mathrm{M}_n(\mathbb{C})^d \subseteq Z(g) \cap \mathrm{M}_n(\mathbb{C})^d \implies g^r \in (f_1,\dots,f_\ell) + \mathrm{PI}_n$$

In general, can't draw conclusions for all n at once!
$g = 1,\ f_1 = x_1x_2 - x_2x_1 - 1$

If $(f_1,\dots,f_\ell)$ is either homogeneous Salomon-Shalit-Shamovich[18] or rationally resolvable Klep-Vinnikov-V[17]:
$Z(f_1,\dots,f_\ell) \subseteq Z(g) \iff g \in (f_1,\dots,f_\ell)$

Dimension-free "zero sets" of an nc polynomial

(2) directed zero set, directional zeros

$$Z_{\mathrm{dir}}(f_1, \ldots, f_\ell) = \bigcup_n \{(\underline{X}, v) \in \mathrm{M}_n(\mathbb{C})^d \times \mathbb{C}^n : f_i(\underline{X})v = 0 \ \forall i\}$$

Bergman's Nullstellensatz[04]:

$$Z_{\mathrm{dir}}(f_1, \ldots, f_\ell) \subseteq Z_{\mathrm{dir}}(g) \iff g \in \mathbb{C}\langle\underline{x}\rangle \cdot f_1 + \cdots + \mathbb{C}\langle\underline{x}\rangle \cdot f_\ell$$

Dimension-free "zero sets" of an nc polynomial

(3) trace zero set, tracial zeros

$$Z_{\mathrm{tr}}(f_1,\dots,f_\ell)=\bigcup_n\{\underline{X}\in \mathrm{M}_n(\mathbb{C})^d : \mathrm{tr}\, f_i(\underline{X})=0\ \forall i\}$$

Brešar-Klep-Špenko Nullstellensatz[11,13]:

$$Z_{\mathrm{tr}}(f_1,\dots,f_\ell)\subseteq Z_{\mathrm{tr}}(g) \iff g \text{ or } 1 \text{ is contained in}$$

$$\mathbb{C}\cdot f_1+\cdots+\mathbb{C}\cdot f_\ell+[\mathbb{C}<\underline{x}>,\mathbb{C}<\underline{x}>]$$

Dimension-free "zero sets" of an nc polynomial

(4) free locus, determinantal zeros

$$\mathscr{Z}(f_1, \dots, f_\ell) = \bigcup_n \{\underline{X} \in \mathrm{M}_n(\mathbb{C})^d : f_i(\underline{X}) \text{ is singular } \forall i\}$$

Why do?

(A) Matrix inequalities:

$$\{(X_1, X_2) \colon X_1, X_2 \text{ hermitian}, I - X_2^2 - X_1 X_2^2 X_1 \succeq 0\}$$

The "Zariski closure of the boundary" is

$$\{(X_1, X_2) \colon \det(I - X_2^2 - X_1 X_2^2 X_1) = 0\}$$

Why do?

propaganda

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(B) NC rational expressions:

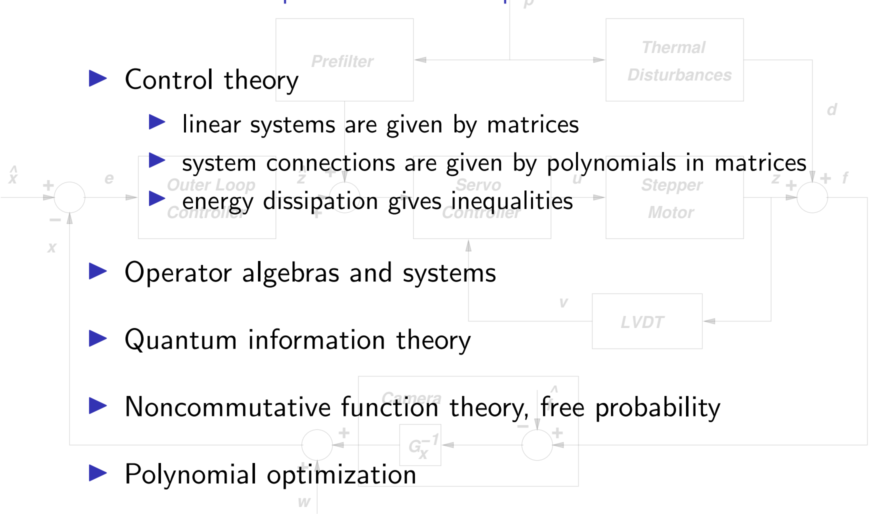
$$(X_1 - X_2 X_4^{-1} X_3)^{-1}$$

its "full" domain is

$$\{(X_1, X_2, X_3, X_4)\colon \det \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \neq 0\}$$

Who cares? propaganda

about dim-free matrix inequalities & rational expressions



- Control theory
  - linear systems are given by matrices
  - system connections are given by polynomials in matrices
  - energy dissipation gives inequalities
- Operator algebras and systems
- Quantum information theory
- Noncommutative function theory, free probability
- Polynomial optimization
- Computational complexity

Free locus

For $f \in \mathbb{C}\langle\underline{x}\rangle$ we define its **free locus** (Klep-V17) as

$$\mathscr{Z}(f) = \bigcup_{n\in\mathbb{N}} \mathscr{Z}_n(f), \quad \mathscr{Z}_n(f) = \{\underline{X} \in \mathrm{M}_n(\mathbb{C})^d : \det f(\underline{X}) = 0\}.$$

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- $\mathscr{Z}_n(f)$ is a (possibly degenerate) hypersurface in $\mathrm{M}_n(\mathbb{C})^d$, invariant under simultaneous conjugation:
  $\underline{X} \in \mathscr{Z}_n(f) \implies P\underline{X}P^{-1} \in \mathscr{Z}_n(f)$ for $P \in \mathrm{GL}_n(\mathbb{C})$
- $\underline{X} \in \mathscr{Z}(f) \implies \left(\begin{smallmatrix} \underline{X} & \star \\ 0 & \star \end{smallmatrix}\right) \in \mathscr{Z}(f)$.
- $\mathscr{Z}(f_1 \cdots f_\ell) = \mathscr{Z}(f_1) \cup \cdots \cup \mathscr{Z}(f_\ell)$
- $\mathscr{Z}(f_1) \cap \cdots \cap \mathscr{Z}(f_\ell) \subseteq \mathscr{Z}(g) \implies \mathscr{Z}(f_j) \subseteq \mathscr{Z}(g)$ for some j

  (surprising?)

Factorization in free algebra

Opus of P. M. Cohn

Every nc polynomial admits a complete factorization into irreducible factors.

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Opus of P. M. Cohn

Every nc polynomial admits a complete factorization into irreducible factors. **Uniqueness?**

$$(x_1x_2+1)(x_3x_2x_1+x_3+x_1) = (x_1x_2x_3+x_1+x_3)(x_2x_1+1)$$

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$f,g\in\mathbb{C}\langle\underline{x}\rangle$ are **stably associated** if

$$\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}=P\begin{pmatrix} f & 0\\ 0 & 1\end{pmatrix}Q \qquad \text{for some } P,Q\in \mathrm{GL}_2(\mathbb{C}\langle\underline{x}\rangle).$$

E.g.

$$\begin{pmatrix} 1+x_1x_2 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix} x_1 & 1+x_1x_2\\ -1 & -x_2\end{pmatrix}\begin{pmatrix} 1+x_2x_1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} x_2 & -1\\ 1+x_1x_2 & x_1\end{pmatrix}$$

Factorization continued

$\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = P \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} Q$

Stable association is an equivalence relation

It preserves irreducibility

Equivalence class of a homogeneous $f \in \mathbb{C}\langle\underline{x}\rangle$ is $\mathbb{C}^* \cdot f$

Bergman[99]: equivalence classes are finite mod $\mathbb{C}^*$

Cohn[73]: irreducible factors in a complete factorization of an nc polynomial are unique up to stable association

$(x_1x_2+1)(x_3x_2x_1+x_3+x_1)=(x_1x_2x_3+x_1+x_3)(x_2x_1+1)$

more can be said about admissible swaps etc.

Factorization continued

$\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = P \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} Q$

Most relevant today:

f, g stably associated $\implies \mathscr{Z}(f) = \mathscr{Z}(g)$

E.g. $I + X_1X_2$ is singular if and only if $I + X_2X_1$ is singular.

Irreducibility theorem

Theorem (Helton-Klep-V[18,22])

Let $f \in \mathbb{C}\langle\underline{x}\rangle$ *be irreducible. Then* $\mathscr{Z}_n(f)$ *is a reduced irreducible hypersurface for all but finitely many* $n \in \mathbb{N}$.

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Example: $f = (1 - x_1)^2 - x_2^2$ is irreducible in $\mathbb{C}<\underline{x}>$,

$\mathscr{Z}_1(f) = \{1 - \xi_1 - \xi_2 = 0\} \cup \{1 - \xi_1 + \xi_2 = 0\}$
is a union of two lines in $\mathbb{C}^2$,

$\mathscr{Z}_2(f)$ is an irreducible hypersurface in $M_2(\mathbb{C})^2$.

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How large can n be so that $\mathscr{Z}_n(f)$ splits even though f is irreducible?

Known upper bound is doubly exponential in $\deg f$.

Singulärstellensatz

~~Nullstellensatz~~

Theorem (Helton-Klep-V[18,22])

(i) *Let $f, g \in \mathbb{C}\langle\underline{x}\rangle$ be irreducible. Then $\mathscr{Z}(f) = \mathscr{Z}(g)$ if and only if f and g are stably associated.*

(ii) *Let $f, g \in \mathbb{C}\langle\underline{x}\rangle$. Then $\mathscr{Z}(f) \subseteq \mathscr{Z}(g)$ if and only if every irreducible factor of f is stably associated to a factor of g.*

nc zero sets $\leftrightsquigarrow$ ideals

directed nc zero sets $\leftrightsquigarrow$ left ideals

free loci $\leftrightsquigarrow$ factorization

Ingredients of the proof

- Linearization from automata thy Higman, Schützenberger

$$a + bc \rightsquigarrow \begin{pmatrix} a & b \\ c & -1 \end{pmatrix}$$

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$$f(\underline{X}) \rightsquigarrow L(\underline{X}) = A_0 \otimes I + A_1 \otimes X_1 + \cdots + A_d \otimes X_d, \quad A_i \in \mathrm{M}_\ell(\mathbb{C})$$

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- Ampliations from NC function theory Voiculescu, Vinnikov
  $\mathscr{Z}_n(f)$ for all n $\rightsquigarrow$ $\mathscr{Z}(f)$?

Real vs Complex

Back towards matrix inequalities

Algebraic geometry: zero sets of complex polynomials in $\mathbb{C}^d$.
Real algebraic geometry: zero sets of real polynomials in $\mathbb{R}^d$.

real = complex fixed by complex conjugation.

On $\mathbb{C}\langle \underline{x} \rangle$ there is a natural involution $*$: $\mathbb{R}$-linear antihomomorphism given by $x_j^* = x_j$ and $\alpha^* = \bar{\alpha}$ for $\alpha \in \mathbb{C}$.

real nc polynomials: $f \in \mathbb{C}\langle \underline{x} \rangle$, $f = f^*$.
real points: $\mathrm{H}_n(\mathbb{C})^d$, tuples of hermitian matrices.

Real free locus:

$$\mathscr{Z}^{\mathrm{re}}(f) = \bigcup_n \mathscr{Z}_n^{\mathrm{re}}(f), \qquad \mathscr{Z}_n^{\mathrm{re}}(f) = \mathscr{Z}_n(f) \cap \mathrm{H}_n(\mathbb{C})^d.$$

Real Singulärstellensatz

Bad example: $f = x_1^2 + x_2^2$ and $g = x_1$.

Then $\mathscr{Z}^{\mathrm{re}}(f) \subseteq \mathscr{Z}^{\mathrm{re}}(g)$ but $\mathscr{Z}(f) \not\subseteq \mathscr{Z}(g)$.

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$f = f^*$ is unsignatured if one of the following equivalent conditions hold:

- there are $\underline{X}, \underline{Y}$ such that $f(\underline{X}), f(\underline{Y})$ are invertible with distinct signatures;
- there are $\underline{X}, \underline{Y}$ such that $f(\underline{X}) \succ 0 \succ f(\underline{Y})$;
- neither f or $-f$ equals $s_1 s_1^* + \cdots + s_\ell s_\ell^*$ for some $s_j \in \mathbb{C}\langle \underline{x} \rangle$.

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Theorem (Helton-Klep-V[22])

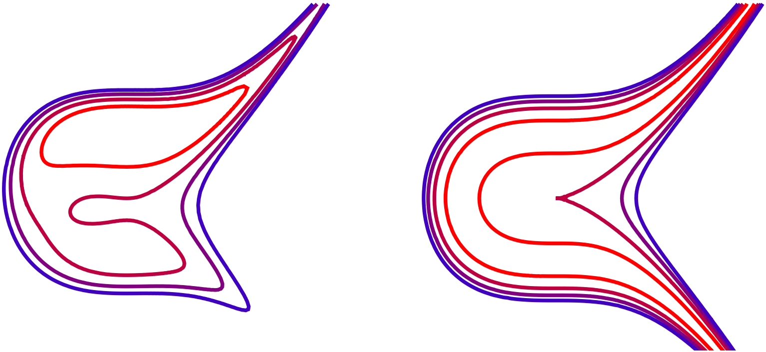
Let $f, g \in \mathbb{C}<\underline{x}>$. If $f = f^*$ is irreducible and unsignatured, then $\mathscr{Z}^{\mathrm{re}}(f) \subseteq \mathscr{Z}^{\mathrm{re}}(g)$ iff f is stably associated to a factor of g.

Some applications

- Helton-Klep-McCullough-V[21]: poly-time algorithm deciding whether a free semialgebraic set is convex
- Augat-Helton-Klep-McCullough[18]: classification of bianalytic maps between convex free semialgebraic sets
- V[19,20]: stability and quasi-convexity of nc polynomials
- Jury-Martin-Shamovich[21]: Blaschke–singular–outer factorization, Clarke measures in free analysis
- Arvind-Joglekar[22]: factorization in free algebra
- Arora-Augat-Jury-Sargent[22]: optimal approximants in Fock space

Bertini's theorem

The simplest case - level sets of a polynomial



$$(x_1^3 - 2x_2^2 + \tfrac{4}{3})(x_1^3 - 2x_2^2) + \tfrac{1}{2}(x_1^2 - x_2) \qquad (x_1^3 - 2x_2^2 + \tfrac{4}{3})(x_1^3 - 2x_2^2)$$

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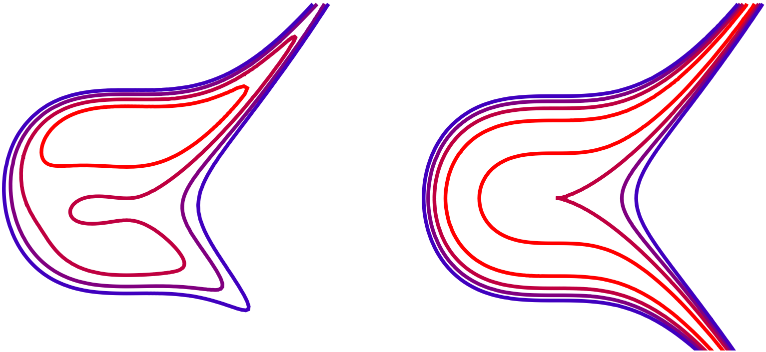

$(x_1^3 - 2x_2^2 + \frac{4}{3})(x_1^3 - 2x_2^2) + \frac{1}{2}(x_1^2 - x_2)$ $\qquad$ $(x_1^3 - 2x_2^2 + \frac{4}{3})(x_1^3 - 2x_2^2)$

Bertini: let $f \in \mathbb{C}[\underline{x}]$. Then either the level sets $\{f = \lambda\}$ are irreducible hypersurfaces for all but finitely many $\lambda \in \mathbb{C}$, or $f = p \circ q$ for some $q \in \mathbb{C}[\underline{x}]$ and $p \in \mathbb{C}[t]$ of degree at least 2.

Eigenlevel sets and free Bertini's theorem

$f \in \mathbb{C}\langle \underline{x} \rangle$ is **composite** if there are $g \in \mathbb{C}\langle \underline{x} \rangle$ and $p \in \mathbb{C}[t]$ with $\deg p > 1$ such that $f = p \circ g$.

An **eigenlevel set** of $f \in \mathbb{C}\langle \underline{x} \rangle$ for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ is

$$\left\{ \underline{X} \in \mathrm{M}_n(\mathbb{C})^d : \lambda \text{ is an eigenvalue of } f(\underline{X}) \right\} = \mathscr{Z}_n(f - \lambda).$$

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Theorem (V[20])

For $f \in \mathbb{C}\langle\underline{x}\rangle$, the following are equivalent:

(i) *f is not composite;*

(ii) *all but finitely many eigenlevel sets of f are irreducible.*

.... *how many n, λ?*

Polynomials with the same eigenvalues

Theorem (V[20])

Let $f, g \in \mathbb{C}<\underline{x}>$. Then the spectra of $f(\underline{X})$ and $g(\underline{X})$ coincide for every matrix tuple $\underline{X}$ if and only if

$$fa = ag$$

for some nonzero $a \in \mathbb{C}<\underline{x}>$.

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for some nonzero $a \in \mathbb{C}\!<\!\underline{x}\!>$. .... deg a?

E.g.

$$f = x_1 + x_2 + x_1 x_2^2$$
$$g = x_1 + x_2 + x_2^2 x_1$$
$$a = 1 + x_1^2 + x_1 x_2 + x_2 x_1 + x_1 x_2^2 x_1$$

satisfy $fa = ag$.

Some open questions

- Bounds

  If f is irreducible, for which n is $\mathscr{L}_n(f)$ irreducible?

  If $f - \lambda$ factors for $\deg(f)$ different λ, is f composite?

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  Bounds on $\deg a$?

  Are equivalence classes finite?

  How to construct whole classes?

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- Low-rank values of nc polynomials

  If $\operatorname{rk} f = \operatorname{rk} g$ pointwise, are f and g stably associated?

  Geometry of $\{\underline{X} \colon \operatorname{rk} f(\underline{X}) \text{ is small}\}$

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- Bertini for nc rational expressions

End credits

Things to take home

- nc polynomial inequalities and equations
  from control, quantum, operator algebras, optimization...
- free locus of an nc polynomial: $\{\det f = 0\}$
- "persistent" irreducible components $\leftrightsquigarrow$ irreducible factors
- inclusion of free loci $\leftrightsquigarrow$ factorization in free algebra
- Bertini: eigenlevel sets detect composition

Thank you!