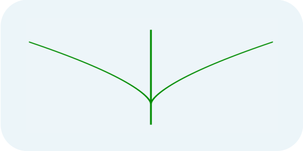
July 13, 2023

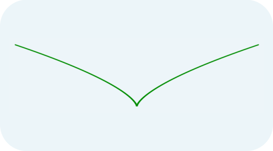
Associated primes of powers of monomial ideals

Bounding the copersistence index

$$180 = 2^2 \cdot 3^2 \cdot 5 \dashrightarrow \quad 2 \qquad 3 \qquad 5$$



$X^3 - XY^3$

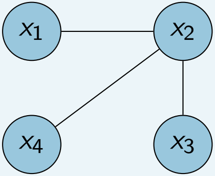


$X^2 - Y^3$

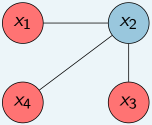
X

$$\mathrm{Ass}(\mathbb{Z}/180\mathbb{Z}) = \{2\mathbb{Z}, 3\mathbb{Z}, 5\mathbb{Z}\}$$

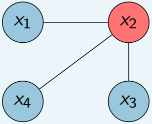
$$\mathrm{Ass}(K[X,Y]/(X^3 - XY^3)) = \{(X^2 - Y^3), (X)\}$$



$I = (x_1x_2, x_2x_3, x_2x_4)$



(x_1, x_3, x_4)



(x_2)

edge ideals - - → vertex covers

$$\mathrm{Ass}(R/I) = \{(x_2), (x_1, x_3, x_4)\}$$

Definition

R ring, $I \subseteq R$ ideal

$$\mathrm{Ass}(R/I) \coloneqq \{P \in \mathrm{Spec}(R) \mid P = I : w \text{ for some } w \in R\}.$$

"associated primes of I in R"

Definition

R Noetherian ring, $I \subseteq R$ ideal. Let $I = Q_1 \cap \cdots \cap Q_m$ be an irredundant primary decomposition of I. Then

$$\mathrm{Ass}(R/I) \coloneqq \left\{ \sqrt{Q_1}, \ldots, \sqrt{Q_m} \right\}.$$

In the following: I monomial ideal in $R = K[X_1, \ldots, X_r]$.

For $P \in \mathrm{Ass}(R/I)$ it holds that

- P is a monomial ideal,
- there exists a monomial $X^a := X_1^{a_1} \cdots X_r^{a_r}$ such that

$$P = I : X^a.$$

$I = (xy, yz, xz)$

- $I : x = (y, z)$
- $I : y = (x, z)$
- $I : z = (x, y)$

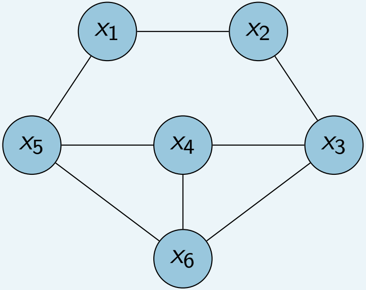
$$\mathrm{Ass}(R/I) \subseteq \left\{ \begin{array}{ccc} (x) & (y) & (z) \\ (x,y) \checkmark & (x,z) \checkmark & (y,z) \checkmark \\ & (x,y,z) & \end{array} \right\}$$

$I^2 = (x^2y^2, xy^2z, x^2yz, y^2z^2, xyz^2, x^2z^2)$

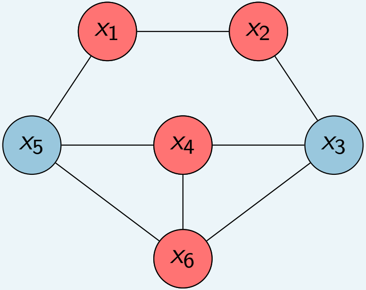
- $I^2 : x^2y = (y, z)$
- $I^2 : y^2x = (x, z)$
- $I^2 : z^2y = (x, y)$
- $I^2 : xyz = (x, y, z)$

$$\mathrm{Ass}(R/I^2) \subseteq \left\{ \begin{array}{ccc} (x) & (y) & (z) \\ (x,y) & (x,z) & (y,z) \\ & (x,y,z) & \end{array} \right\}$$

The set of associated primes of an ideal changes when looking at its powers.

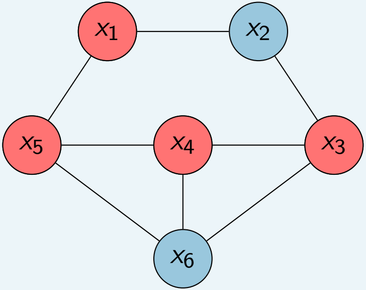


$J = ($ minimal vertex covers $)$

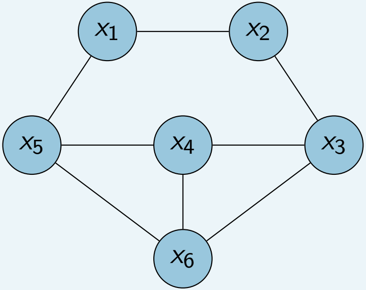


$J = ($ minimal vertex covers $)$

$= (x_1 x_2 x_4 x_6$

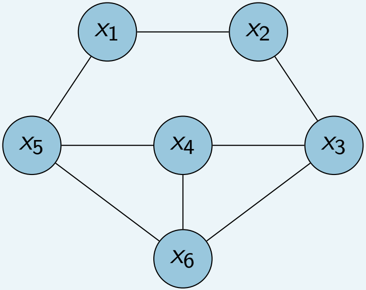


$$J = (\text{ minimal vertex covers })$$
$$= (x_1x_2x_4x_6, x_1x_3x_4x_5, \dots)$$



$J = ($ minimal vertex covers $)$

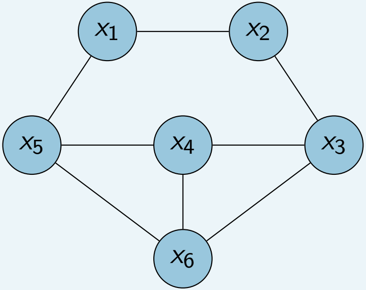
$$\mathrm{Ass}(R/J\ ) = \left\{ \begin{array}{ll} (x_1, x_5) & (x_1, x_2) \\ (x_3, x_4) & (x_2, x_3) \\ (x_4, x_5) & (x_3, x_6) \\ (x_5, x_6) & (x_4, x_6) \end{array} \right\}$$



$J = ($ minimal vertex covers $)$

$$\mathrm{Ass}(R/J^2) = \mathrm{Ass}(R/J) \;\cup$$

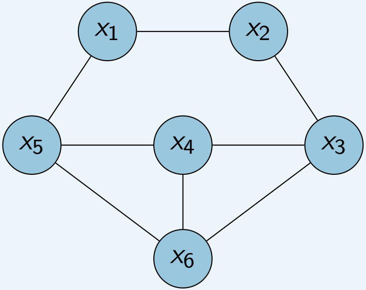
$$\left\{ (x_4, x_5, x_6) \; (x_3, x_4, x_6) \; (x_1, x_2, x_3, x_4, x_6) \; (x_1, x_2, x_3, x_5, x_6) \right\}$$



$J = ($ minimal vertex covers $)$

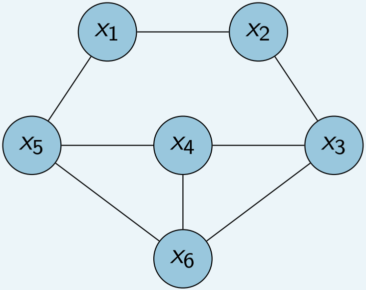
$$\mathrm{Ass}(R/J^3) = \mathrm{Ass}(R/J^2) \quad \cup$$

$$\Big\{ (x_1, x_2, x_3, x_4, x_5, x_6) \Big\}$$



$J = ($ minimal vertex covers $)$

$$\mathrm{Ass}(R/J^4) = \mathrm{Ass}(R/J^3)$$



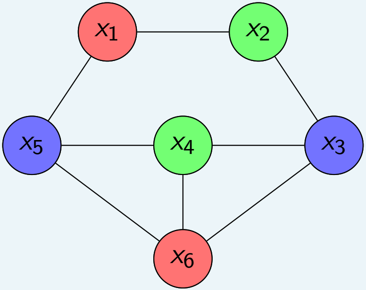
$J = ($ minimal vertex covers $)$

$\mathrm{Ass}(R/J^n) = \mathrm{Ass}(R/J^3)$

for all $n \geq 3$

Proposition (Francisco, Ha, Tuyl, 2011)

If $(\mathrm{Ass}(R/J^n))_{n\in\mathbb{N}}$ *is constant after* $N \in \mathbb{N}$*, then* $\chi(G) \leq N + 1$



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If $(\mathrm{Ass}(R/J^n))_{n\in\mathbb{N}}$ *is constant after* $N \in \mathbb{N}$, *then* $\chi(G) \leq N+1$

Changes of $\mathrm{Ass}(R/I^n)$ in n?

- Brodmann, 1979: $(\mathrm{Ass}(R/I^n))_{n\in\mathbb{N}}$ stabilizes

Definition

stability index of I: smallest $\mathrm{B}^I_= \in \mathbb{N}$ such that for all $n \geq \mathrm{B}^I_=$

$$\mathrm{Ass}(R/I^n) = \mathrm{Ass}(R/I^{\mathrm{B}^I_=})$$

Some known results about the changes of $\mathrm{Ass}(R/I^n)$

- edge ideals [Martínez-Bernal, Morey, Villarreal, 2012]
- cover ideals of perfect graphs [Francisco, Hà, Tuyl, 2011]
- ideals with all powers integrally closed [Ratliff, 1984]

$(\mathrm{Ass}(R/I^n))_{n\in\mathbb{N}}$ is increasing

- ideals can be constructed with
  - $(\mathrm{Ass}(R/I^n))_{n\in\mathbb{N}}$ not increasing [Kaiser, Stehlík, Škrekovski, 2012]
  - $(\mathrm{Ass}(R/I^n))_{n\in\mathbb{N}}$ not monotone [McAdam, Eakin, 1979]
  - $\mathrm{B}^I_=$ arbitrarily large [Hà, Nguyen, Trung, Trung, 2021]

- conjecture [J. Herzog]: if I square-free, $\mathrm{B}^I_= \leq r-1$

- upper bound for $\mathrm{B}^I_=$ of monomial ideals

I monomial ideal in $K[X_1, \ldots, X_r]$

- r – number of variables
- s – number of generators
- d – maximal total degree of the generators

Theorem (Hoa, 2006)

$(\mathrm{Ass}(R/I^n))_{n\in\mathbb{N}}$ *is*

- *increasing for* $n \geq s^{r+3}(s+r)^4 d^2 \left(2d^2\right)^{s^2-s+1}$,
- *decreasing for* $n \geq d(rs+s+d)\left(\sqrt{r}\right)^{r+1}(\sqrt{2}d)^{(r+1)(s-1)}$.

Example

$I = (a^6, b^6, a^5b, ab^5, ca^4b^4, a^4xy^2, b^4x^2y) \subseteq K[a, b, c, x, y]$

- $r = 5,\ s = 7,\ d = 9$
- upper bound $\approx 10^{108}$
- stability index: 4

persistence index of I: smallest integer $\mathrm{B}^I_{\subseteq}$ such that

$$\mathrm{Ass}(R/I^n) \subseteq \mathrm{Ass}(R/I^{n+1}) \text{ for all } n \geq \mathrm{B}^I_{\subseteq}.$$

copersistence index of I: smallest integer $\mathrm{B}^I_{\supseteq}$ such that

$$\mathrm{Ass}(R/I^n) \supseteq \mathrm{Ass}(R/I^{n+1}) \text{ for all } n \geq \mathrm{B}^I_{\supseteq}.$$

$$\mathrm{B}^I_{=} = \max\{\mathrm{B}^I_{\subseteq}, \mathrm{B}^I_{\supseteq}\}$$

Theorem (Heuberger, R., Rissner, 2023)

I monomial ideal in $K[X_1, \ldots, X_r]$

- r – *number of variables*
- s – *number of generators of I*
- d – *maximal total degree of the generators*

$A\boldsymbol{x} \leq \boldsymbol{b}$ *system of inequalities (fulfilling properties explained on the next slides);*

$\sigma\colon \mathbb{N}^3 \to \mathbb{N}$ *such that*

- $\sigma(d, r, s) \geq \Delta(A \mid \boldsymbol{b})(\mathrm{size}(A) + 1)$ *and*
- σ *is non-decreasing in d, r and s;*

Then

$$\mathrm{B}^{I}_{\supseteq} \leq \sigma(d, r, s).$$

$$I = (X^{a_1}, \ldots, X^{a_s})$$

$$\begin{pmatrix} \begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & & & \begin{matrix} -I_r \\ 0 \; \cdots \; 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \\ & \begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & & \begin{matrix} -I_r \\ 0 \; \cdots \; 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \\ & & \ddots & \vdots & \\ & & \begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & \begin{matrix} -I_r \\ 0 \; \cdots \; 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \end{pmatrix} \begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{1r} \\ \vdots \\ \alpha_{s1} \\ \vdots \\ \alpha_{sr} \\ h_1 \\ \vdots \\ h_r \\ n \end{pmatrix} \leq \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$I = (X^{a_1}, \ldots, X^{a_s})$$

$$\begin{pmatrix}
\begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & & & & \begin{matrix} -I_r \\ 0 \; \cdots \; 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \\
& \begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & & & \begin{matrix} -I_r \\ 0 \; \cdots \; 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \\
& & \ddots & & \vdots & \\
& & & \begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & \begin{matrix} -I_r \\ 0 \; \cdots \; 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix}
\end{pmatrix}
\begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{1r} \\ \vdots \\ \alpha_{s1} \\ \vdots \\ \alpha_{sr} \\ h_1 \\ \vdots \\ h_r \\ n \end{pmatrix}
\leq
\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$\alpha_{11} + \cdots + \alpha_{1r} = n, \; \alpha_{11} a_1 + \cdots + \alpha_{1r} a_r \leq (1, 0, \ldots, 0) + h$$

$$I = (X^{a_1}, \ldots, X^{a_s})$$

$$\begin{pmatrix}
a_1 \cdots a_s & & & & -I_r & \\
-1 \cdots -1 & & & & 0 \cdots 0 & 1 \\
& a_1 \cdots a_s & & & -I_r & \\
& -1 \cdots -1 & & & 0 \cdots 0 & 1 \\
& & \ddots & & \vdots & \\
& & & a_1 \cdots a_s & -I_r & \\
& & & -1 \cdots -1 & 0 \cdots 0 & 1
\end{pmatrix}
\begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{1r} \\ \vdots \\ \alpha_{s1} \\ \vdots \\ \alpha_{sr} \\ h_1 \\ \vdots \\ h_r \\ n \end{pmatrix}
\leq
\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$\alpha_{11} + \cdots + \alpha_{1r} = n, \; \alpha_{11} a_1 + \cdots + \alpha_{1r} a_r \leq (1, 0, \ldots, 0) + h$$

$$I = (X^{a_1}, \dots, X^{a_s})$$

$$\begin{pmatrix}
\begin{matrix} a_1 & \cdots & a_s \\ -1 & \cdots & -1 \end{matrix} & & & & \begin{matrix} -I_r \\ 0 \cdots 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \\
& \begin{matrix} a_1 & \cdots & a_s \\ -1 & \cdots & -1 \end{matrix} & & & \begin{matrix} -I_r \\ 0 \cdots 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \\
& & \ddots & & \vdots & \\
& & & \begin{matrix} a_1 & \cdots & a_s \\ -1 & \cdots & -1 \end{matrix} & \begin{matrix} -I_r \\ 0 \cdots 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix}
\end{pmatrix}
\begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{1r} \\ \vdots \\ \alpha_{s1} \\ \vdots \\ \alpha_{sr} \\ h_1 \\ \vdots \\ h_r \\ n \end{pmatrix}
\leq
\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$\alpha_{11} + \cdots + \alpha_{1r} = n, \ \alpha_{11} a_1 + \cdots + \alpha_{1r} a_r \leq (1, 0, \dots, 0) + h$$

$I = (X^{a_1}, \dots, X^{a_s})$

$$
\begin{pmatrix}
\boxed{\begin{matrix} a_1 & \cdots & a_s \\ -1 & \cdots & -1 \end{matrix}} & & & \boxed{\begin{matrix} & -I_r & \\ 0 & \cdots & 0 \end{matrix}} & \boxed{\begin{matrix} \\ 1 \end{matrix}} \\
& \boxed{\begin{matrix} a_1 & \cdots & a_s \\ -1 & \cdots & -1 \end{matrix}} & & \boxed{\begin{matrix} & -I_r & \\ 0 & \cdots & 0 \end{matrix}} & \boxed{\begin{matrix} \\ 1 \end{matrix}} \\
& & \ddots & \vdots & \\
& & \boxed{\begin{matrix} a_1 & \cdots & a_s \\ -1 & \cdots & -1 \end{matrix}} & \boxed{\begin{matrix} & -I_r & \\ 0 & \cdots & 0 \end{matrix}} & \boxed{\begin{matrix} \\ 1 \end{matrix}}
\end{pmatrix}
\begin{pmatrix}
\boxed{\begin{matrix} \alpha_{11} \\ \vdots \\ \alpha_{1r} \end{matrix}} \\ \vdots \\ \boxed{\begin{matrix} \alpha_{s1} \\ \vdots \\ \alpha_{sr} \end{matrix}} \\ \boxed{\begin{matrix} h_1 \\ \vdots \\ h_r \end{matrix}} \\ \boxed{n}
\end{pmatrix}
\leq
\begin{pmatrix}
1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1
\end{pmatrix}
$$

$$\alpha_{11} + \cdots + \alpha_{1r} = n, \ \alpha_{11} a_1 + \cdots + \alpha_{1r} a_r \leq (1, 0, \dots, 0) + h$$

$$I = (X^{a_1}, \dots, X^{a_s})$$

$$\begin{pmatrix}
\begin{matrix} a_1 & \cdots & a_s \\ -1 & \cdots & -1 \end{matrix} & & & & \begin{matrix} -I_r \\ 0 \cdots 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \\
& \begin{matrix} a_1 & \cdots & a_s \\ -1 & \cdots & -1 \end{matrix} & & & \begin{matrix} -I_r \\ 0 \cdots 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \\
& & \ddots & & \vdots & \\
& & & \begin{matrix} a_1 & \cdots & a_s \\ -1 & \cdots & -1 \end{matrix} & \begin{matrix} -I_r \\ 0 \cdots 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix}
\end{pmatrix}
\begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{1r} \\ \vdots \\ \alpha_{s1} \\ \vdots \\ \alpha_{sr} \\ h_1 \\ \vdots \\ h_r \\ n \end{pmatrix}
\leq
\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$\alpha_{11} + \cdots + \alpha_{1r} = n, \quad \alpha_{11} a_1 + \cdots + \alpha_{1r} a_r \leq (1, 0, \dots, 0) + h$$

$$I = (X^{a_1}, \dots, X^{a_s})$$

$$\begin{pmatrix}
\boxed{\begin{matrix} \mathbf{a}_1 & \cdots & \mathbf{a}_s \\ -1 & \cdots & -1 \end{matrix}} & & & \boxed{\begin{matrix} -I_r \\ 0 \cdots 0 \end{matrix}} & \boxed{\begin{matrix} \\ 1 \end{matrix}} \\
& \boxed{\begin{matrix} \mathbf{a}_1 & \cdots & \mathbf{a}_s \\ -1 & \cdots & -1 \end{matrix}} & & \boxed{\begin{matrix} -I_r \\ 0 \cdots 0 \end{matrix}} & \boxed{\begin{matrix} \\ 1 \end{matrix}} \\
& & \ddots & \vdots & \\
& & \boxed{\begin{matrix} \mathbf{a}_1 & \cdots & \mathbf{a}_s \\ -1 & \cdots & -1 \end{matrix}} & \boxed{\begin{matrix} -I_r \\ 0 \cdots 0 \end{matrix}} & \boxed{\begin{matrix} \\ 1 \end{matrix}}
\end{pmatrix}
\begin{pmatrix}
\boxed{\begin{matrix} \alpha_{11} \\ \vdots \\ \alpha_{1r} \end{matrix}} \\ \vdots \\ \boxed{\begin{matrix} \alpha_{s1} \\ \vdots \\ \alpha_{sr} \end{matrix}} \\ \boxed{\begin{matrix} h_1 \\ \vdots \\ h_r \end{matrix}} \\ \boxed{n}
\end{pmatrix}
\leq
\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$\alpha_{11} + \cdots + \alpha_{1r} = n, \; X^{\alpha_{11} a_1 + \cdots + \alpha_{1r} a_r} \mid X_1 \cdot X^h$$

$$I = (X^{a_1}, \ldots, X^{a_s})$$

$$\begin{pmatrix}
\begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & & & & \begin{matrix} & -I_r & \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \\
& \begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & & & \begin{matrix} & -I_r & \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \\
& & \ddots & & \vdots & \\
& & & \begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & \begin{matrix} & -I_r & \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix}
\end{pmatrix}
\begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{1r} \\ \vdots \\ \alpha_{s1} \\ \vdots \\ \alpha_{sr} \\ h_1 \\ \vdots \\ h_r \\ n \end{pmatrix}
\leq
\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$X_1 \cdot X^h \in I^n,$$

$$I = (X^{a_1}, \ldots, X^{a_s})$$

$$\begin{pmatrix}
\begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & & & & \begin{matrix} -I_r \\ 0 \; \cdots \; 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \\
& \begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & & & \begin{matrix} -I_r \\ 0 \; \cdots \; 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \\
& & \ddots & & \vdots & \\
& & & \begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & \begin{matrix} -I_r \\ 0 \; \cdots \; 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix}
\end{pmatrix}
\begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{1r} \\ \vdots \\ \alpha_{s1} \\ \vdots \\ \alpha_{sr} \\ h_1 \\ \vdots \\ h_r \\ n \end{pmatrix}
\leq
\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$X_1 \cdot X^h \in I^n, \ \ldots, \ X_r \cdot X^h \in I^n$$

$$I = (X^{a_1}, \ldots, X^{a_s})$$

$$\begin{pmatrix}
\begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & & & & \begin{matrix} -I_r \\ 0 \cdots 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \\
& \begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & & & \begin{matrix} -I_r \\ 0 \cdots 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \\
& & \ddots & & \vdots & \\
& & & \begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & \begin{matrix} -I_r \\ 0 \cdots 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix}
\end{pmatrix}
\begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{1r} \\ \vdots \\ \alpha_{s1} \\ \vdots \\ \alpha_{sr} \\ h_1 \\ \vdots \\ h_r \\ n \end{pmatrix}
\leq
\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$X_1 \cdot X^h \in I^n, \ \ldots, \ X_r \cdot X^h \in I^n \quad \Longrightarrow \quad X^h \in I^n : (X_1, \ldots, X_r)$$

$$I = (X^{a_1}, \ldots, X^{a_s})$$

$$\begin{pmatrix} \begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & & & & \begin{matrix} -I_r \\ 0 \ \cdots \ 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \\ & \begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & & & \begin{matrix} -I_r \\ 0 \ \cdots \ 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \\ & & \ddots & & \vdots & \\ & & & \begin{matrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_s \\ -1 & \cdots & -1 \end{matrix} & \begin{matrix} -I_r \\ 0 \ \cdots \ 0 \end{matrix} & \begin{matrix} \\ 1 \end{matrix} \end{pmatrix} \begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{1r} \\ \vdots \\ \alpha_{s1} \\ \vdots \\ \alpha_{sr} \\ h_1 \\ \vdots \\ h_r \\ n \end{pmatrix} \leq \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$X_1 \cdot X^h \in I^n, \ \ldots, \ X_r \cdot X^h \in I^n \implies X^h \in I^n : (X_1, \ldots, X_r)$$

Proposition (Folklore)

$(X_1, \ldots, X_r) \in \mathrm{Ass}(R/I^n)$ *if and only if* $\exists X^h \in I^n : (X_1, \ldots, X_r) \setminus I^n$

$I = (X^{a_1}, \ldots, X^{a_s})$

$$\begin{pmatrix} a_1 \cdots a_s & & & & -I_r & \\ -1 \cdots -1 & & & & 0 \cdots 0 & 1 \\ & a_1 \cdots a_s & & & -I_r & \\ & -1 \cdots -1 & & & 0 \cdots 0 & 1 \\ & & \ddots & & \vdots & \\ & & & a_1 \cdots a_s & -I_r & \\ & & & -1 \cdots -1 & 0 \cdots 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{1r} \\ \vdots \\ \alpha_{s1} \\ \vdots \\ \alpha_{sr} \\ h_1 \\ \vdots \\ h_r \\ n \end{pmatrix} \leq \begin{pmatrix} {\color{red}0} \\ 0 \\ \vdots \\ 0 \\ 0 \\ {\color{red}0} \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ {\color{red}0} \end{pmatrix}$$

$$X_1 \cdot X^h \in I^n, \ \ldots, \ X_r \cdot X^h \in I^n \implies X^h \in I^n : (X_1, \ldots, X_r)$$

Proposition (Folklore)

$(X_1, \ldots, X_r) \in \mathrm{Ass}(R/I^n)$ *if and only if* $\exists X^h \in I^n : (X_1, \ldots, X_r) \setminus I^n$

Theorem (Hoa, 2006)

$$\mathrm{B}^{I}_{\supseteq} \leq d(rs+s+d)\left(\sqrt{r}\right)^{r+1}(\sqrt{2}d)^{(r+1)(s-1)}$$
$$=: \sigma_1(d,s,r)$$

- $(X_1,\ldots,X_r) \in \mathrm{Ass}(R/I^n) \iff I^n : (X_1,\ldots,X_r)/I^n \neq 0$

Theorem (Heuberger, R., Rissner, 2023)

$$\mathrm{B}^{I}_{\supseteq} \leq (rs+r+2)(\sqrt{r})^{r+2}(d+1)^{rs}$$
$$=: \sigma_2(d,s,r)$$

Todo's and open questions

Can the bound be further reduced by

- using a different characterization of $(X_1, \ldots, X_r) \in \mathrm{Ass}(R/I^n)$?
- changing the structure of the matrix?
- finding better estimates on $\Delta(A \mid b)$?

Square-free monomial ideals:

- A has entries in $\{0, 1, -1\}$
- Can we get close to known bounds for edge ideals?
- If yes, can this be adapted to general square-free ideals (edge ideals of hypergraphs)?

Thank you!