Polynomial Dedekind Domains

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Polynomial Dedekind domains

A Dedekind domain D is a one dimensional, integrally closed Noetherian domain. The class group of D is the abelian group $\mathrm{Cl}(D) = \mathrm{Fr}(D)/\mathcal{P}(D)$: it measures how far is D from being a UFD (or, equivalently, a PID), since D UFD$\Leftrightarrow \mathrm{Cl}(D) = (0)$.

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We are interested in Dedekind domains D such that $\mathbb{Z}[X] \subset D \subseteq \mathbb{Q}[X]$ (**Polynomial Dedekind Domains**). We show that such a D:

- can be realized as a ring of integer-valued polynomials;
- $\mathrm{Cl}(D) = \bigoplus_{n\in\mathbb{N}} G_n$, G_n finitely generated abelian groups.

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- can be realized as a ring of integer-valued polynomials;
- $\mathrm{Cl}(D) = \bigoplus_{n\in\mathbb{N}} G_n$, G_n finitely generated abelian groups.

Conversely, every such a group occurs as the class group of a Polynomial Dedekind domain.

Example: We may represent $\mathbb{Q}[X]$ as follows:

$$\mathbb{Q}[X] = \bigcap_{q \in \mathcal{P}^{\mathrm{irr}}} \mathbb{Q}[X]_{(q)}$$

where $\mathcal{P}^{\mathrm{irr}}$ is the set of irreducible polynomials over $\mathbb{Q}$. It is well-known that $\mathbb{Q}[X]_{(q)}, q \in \mathcal{P}^{\mathrm{irr}}$, are the DVRs of $\mathbb{Q}(X)$ containing $\mathbb{Q}$ $(+\mathbb{Q}[\frac{1}{X}]_{(\frac{1}{X})})$.

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Idea: Find non-trivial Polynomial Dedekind domains by intersecting $\mathbb{Q}[X]$ with DVRs which are residually algebraic over $\mathbb{Z}_{(p)}$ (that is, the extension of the residue fields is algebraic) for some prime $p \in \mathbb{Z}$; it is well-known that we may disregard residually transcendental extensions of $\mathbb{Z}_{(p)}$.

Problem

Describe the DVRs W of $\mathbb{Q}(X)$ which are residually algebraic extensions of $\mathbb{Z}_{(p)}$, $p \in \mathbb{Z}$ prime.

Non-trivial example of Polynomial Dedekind domain

Theorem (Eakin-Heinzer, 1973)

Let $p_1, \ldots, p_n \in \mathbb{Z}$ be primes and for each $i = 1, \ldots, n$, let $\{W_{i,j}\}_{j=1}^{m_i}$ be finitely many DVRs of $\mathbb{Q}(X)$ which are residually algebraic extensions of $\mathbb{Z}_{(p_i)}$.
Then the following is a Dedekind domain:

$$D = \bigcap_{i=1}^{n} \bigcap_{j=1}^{m_i} W_{i,j} \cap \mathbb{Q}[X].$$

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Corollary (E.-H., 1973)

Let G be a finitely generated abelian group. Then there exists a Dedekind domain D, $\mathbb{Z}[X] \subset D \subseteq \mathbb{Q}[X]$ with class group G.

Notation

For $p \in \mathbb{P}$, we set:

- $\mathbb{Z}_{(p)}$: the localization of $\mathbb{Z}$ at $p\mathbb{Z}$.
- $\mathbb{Q}_p, \mathbb{Z}_p$: the field of p-adic numbers and the ring of p-adic integers, respectively.
- $\overline{\mathbb{Q}_p}, \overline{\mathbb{Z}_p}$: a fixed algebraic closure of $\mathbb{Q}_p$ and the absolute integral closure of $\mathbb{Z}_p$, respectively.
- $\mathbb{C}_p$, $\mathbb{O}_p$: the completion of $\overline{\mathbb{Q}_p}$ and $\overline{\mathbb{Z}_p}$, respectively.
- $v = v_p$ denotes the unique extension of the p-adic valuation on $\mathbb{Q}_p$ to $\mathbb{C}_p$.

DVRs of $\mathbb{Q}(X)$ r.a. over $\mathbb{Z}_{(p)}$

Theorem (P. 2023)

If W is a DVR of $\mathbb{Q}(X)$ which is a residually algebraic extension of $\mathbb{Z}_{(p)}$ for some $p \in \mathbb{P}$; then there exists $\alpha \in \mathbb{C}_p$, transcendental over $\mathbb{Q}$, such that

$$W = \mathbb{Z}_{(p),\alpha} = \{\phi \in \mathbb{Q}(X) \mid \phi(\alpha) \in \mathbb{O}_p\}$$

$\alpha \in \overline{\mathbb{Q}_p}$ if and only if the residue field extension $W/M \supseteq \mathbb{Z}/p\mathbb{Z}$ is finite.

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For $\alpha \in \mathbb{C}_p$, it is not true in general that $\mathbb{Z}_{(p),\alpha}$ is a DVR!

Theorem (P. 2023)

Let k be an algebraic extension of $\mathbb{F}_p$ and Γ a totally ordered group such that $\mathbb{Z} \subseteq \Gamma \subseteq \mathbb{Q}$. Then there exists $\alpha \in \mathbb{C}_p$, transcendental over $\mathbb{Q}$, such that $\mathbb{Z}_{(p),\alpha}$ has residue field k and value group Γ.

Elements of $\mathbb{C}_p$ of bounded ramification

For $\alpha \in \mathbb{C}_p$ we consider the extension $\mathbb{Q}_p(\alpha)$ of $\mathbb{Q}_p$, which is transcendental precisely when $\alpha \notin \overline{\mathbb{Q}_p}$.
We set e_α to be the ramification index of $\mathbb{O}_p \cap \mathbb{Q}_p(\alpha)$ over $\mathbb{Z}_p$.
We consider

$$\mathbb{C}_p^{\mathrm{br}} \doteqdot \{\alpha \in \mathbb{C}_p \mid e_\alpha \in \mathbb{N}\}$$

Theorem (P. 2023)

$\mathbb{C}_p^{br}$ is a field, $\overline{\mathbb{Q}_p} \subset \mathbb{C}_p^{br} \subset \mathbb{C}_p$ and we have

$$\mathbb{C}_p^{br} = \bigcup_{[K:\mathbb{Q}_p]<\infty} \widehat{K^{unr}}$$

where the union is over the set of all the finite extensions K of $\mathbb{Q}_p$ and K^{unr} is the maximal unramified extension of K inside $\overline{\mathbb{Q}_p}$.

Eakin-Heinzer's construction revisited

In Eakin-Heinzer's construction, for each i, j there exists some $\alpha_{i,j} \in \mathbb{C}_{p_i}^{\mathrm{br}}$ such that

$$W_{i,j} = \mathbb{Z}_{(p_i),\alpha_{i,j}} = \{\phi \in \mathbb{Q}(X) \mid \phi(\alpha_{i,j}) \in \mathbb{O}_{p_i}\}$$

and so their Dedekind domain is equal to:

$$\begin{aligned} D = & \bigcap_{\substack{i=1,\dots,n \\ j=1,\dots,m_i}} \mathbb{Z}_{(p_i),\alpha_{i,j}} \cap \mathbb{Q}[X] = \\ & = \{f \in \mathbb{Q}[X] \mid v_{p_i}(f(\alpha_{i,j})) \geq 0, \forall i = 1, \dots, n, j = 1, \dots, m_i\} = \\ & = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O}) \end{aligned}$$

where $\underline{E} = \prod_{i=1}^{n} E_i$, $E_i = \{\alpha_{i,j} \mid j = 1, \dots, m_i\} \subset \mathbb{O}_{p_i}^{\mathrm{br}}$ and $\mathcal{O} = \prod_p \mathbb{O}_p$.
These are polynomials which are simultaneously integer-valued on different finite subsets of $\mathbb{C}_{p_i}$, for $i = 1, \dots, n$.

Representation as intersection of DVRs

Given a subset $\underline{E} = \prod_p E_p$ of $\mathcal{O} = \prod_p \mathbb{O}_p$ we have:

$$\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O}) = \bigcap_{p \in \mathbb{P}} \bigcap_{\alpha_p \in E_p} \mathbb{Z}_{(p),\alpha_p} \cap \bigcap_{q \in \mathcal{P}^{\mathrm{irr}}} \mathbb{Q}[X]_{(q)}$$

where we recall that

$$\mathbb{Z}_{(p),\alpha_p} = \{\phi \in \mathbb{Q}(X) \mid \phi(\alpha_p) \in \mathbb{O}_p\}$$

Lemma

$\mathbb{Z}_{(p),\alpha_p}$ is a DVR if and only if $\alpha_p \in \mathbb{C}_p^{br}$ and α_p is transcendental over $\mathbb{Q}$.

Lemma

Let $p \in \mathbb{P}$ and $\mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p) = \{f \in \mathbb{Q}[X] \mid f(E_p) \subseteq \mathbb{O}_p\}$. Then

$$(\mathbb{Z} \setminus p\mathbb{Z})^{-1}\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O}) = \mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p)$$

Local case

For $E_p \subseteq \mathbb{O}_p$,
$\mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p) = \{f \in \mathbb{Q}[X] \mid f(E_p) \subseteq \mathbb{O}_p\} = \bigcap_{\alpha_p \in E_p} \mathbb{Z}_{(p),\alpha_p} \cap \mathbb{Q}[X]$.

Proposition

Let E_p be a subset of $\mathbb{O}_p$. Then $\mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p)$ is a Dedekind domain if and only if E_p is a finite subset of $\mathbb{O}_p^{br}$ of transcendental elements over $\mathbb{Q}$.

Moreover, if $E_p = \{\alpha_1, \ldots, \alpha_n\}$ with the α_i's pairwise non-conjugate over $\mathbb{Q}_p$ and e is the g.c.d. of the ramification indexes of $\mathbb{Q}_p(\alpha_i)/\mathbb{Q}_p$ for $i = 1, \ldots, n$, then $Cl(\mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p))$ is isomorphic to $\mathbb{Z}/e\mathbb{Z} \oplus \mathbb{Z}^{n-1}$.

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In particular, $\mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p)$ is a PID if and only if E_p contains at most one element which is transcendental over $\mathbb{Q}$ and unramified over $\mathbb{Q}_p$.

Note that $E_p = \emptyset \Leftrightarrow \mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p) = \mathbb{Q}[X]$.

Towards the global case

In general, if $\mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p)$ is Dedekind for each $p \in \mathbb{P}$ and $\underline{E} = \prod_p E_p$, the ring $R = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O}) = \bigcap_{p \in \mathbb{P}} \mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p)$ may not be Dedekind!

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Example: $E_p = \{\alpha_p\}$ with $v_p(\alpha_p) > 0, \forall p \in \mathbb{P} \Rightarrow X \in pR, \forall p \in \mathbb{P}$

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Definition

We say that $\underline{E} = \prod_p E_p \subset \mathcal{O} = \prod_p \mathbb{O}_p$ is *polynomially factorizable* if, for each $g \in \mathbb{Z}[X]$ and $\alpha = (\alpha_p) \in \underline{E}$, there exist $n, d \in \mathbb{Z}$, $n, d \geq 1$ such that $\frac{g(\alpha)^n}{d}$ is a unit of $\mathcal{O}$, that is, $v_p(\frac{g(\alpha_p)^n}{d}) = 0$, $\forall p \in \mathbb{P}$.

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In general, if $\mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p)$ is Dedekind for each $p \in \mathbb{P}$ and $\underline{E} = \prod_p E_p$, the ring $R = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O}) = \bigcap_{p\in\mathbb{P}} \mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p)$ may not be Dedekind!
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Example

$\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ is not polynomially factorizable: for each $q \in \mathbb{Z}[X]$, there exist infinitely many $p \in \mathbb{P}$ for which there exists $n \in \mathbb{Z}$ such that $q(n)$ is divisible by p.

Polynomially factorizable sets

Lemma

Let $\underline{E} = \prod_p E_p \subset \mathcal{O}$*, where* E_p *is a finite subset of* $\mathbb{O}_p$ *of transcendental elements over* $\mathbb{Q}$*.*
Then $\underline{E}$ *is polynomially factorizable if and only if, for each (irreducible)* $g \in \mathbb{Z}[X]$ *the following set is finite:*

$$\mathbb{P}_{g,\underline{E}} = \{p \in \mathbb{P} \mid \exists \alpha_p \in E_p, v_p(g(\alpha_p)) > 0\}$$

Global case

Theorem

Let $\underline{E} = \prod_p E_p \subset \mathcal{O} = \prod_p \mathbb{O}_p$ be a subset. Then $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O}) = \{f \in \mathbb{Q}[X] \mid f(\alpha) \in \mathcal{O}, \forall \alpha \in \underline{E}\}$ is a Dedekind domain if and only if $E_p \subset \mathbb{O}_p^{br}$ is a finite set of transcendental elements over $\mathbb{Q}$ for each prime p and $\underline{E}$ is polynomially factorizable.
In this case, $Cl(\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O}))$ is the direct sum of a countable family of finitely generated abelian groups.

Polynomial Dedekind domains

Recall that $\mathcal{O} = \prod_p \mathbb{O}_p$, $\mathbb{O}_p$ completion of $\overline{\mathbb{Z}_p}$; $\mathbb{C}_p^{\text{br}}$ =elements of $\mathbb{C}_p$ of bounded ramification.

Theorem

Let R be a Dedekind domain such that $\mathbb{Z}[X] \subset R \subseteq \mathbb{Q}[X]$. Then $R = \text{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O})$, for some subset $\underline{E} = \prod_p E_p \subset \mathcal{O}^{br}$ such that E_p is a finite set of transcendental elements over $\mathbb{Q}$ for each prime p and $\underline{E}$ is polynomially factorizable.

Corollary

Let R be a PID such that $\mathbb{Z}[X] \subset R \subset \mathbb{Q}[X]$. Then $R = \text{Int}_{\mathbb{Q}}(\{\alpha\}, \mathcal{O})$, for some $\alpha = (\alpha_p) \in \mathcal{O}^{br}$ such that, for each $p \in \mathbb{P}$, α_p is transcendental over $\mathbb{Q}$, α_p is unramified over $\mathbb{Q}_p$ and $\{\alpha\}$ is polynomially factorizable.

We get "finite residue fields of prime characteristic" if $E_p \subset \overline{\mathbb{Z}_p}, \forall p \in \mathbb{P}$.

Chang's construction revisited

Let $\{G_i\}_{i\in I}$ be a countable family of finitely generated abelian groups. For each $i \in I$ we have

$$G_i \cong \mathbb{Z}^{m_i} \oplus \mathbb{Z}/n_{i,1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_{i,k_i}\mathbb{Z}$$

We partition $\mathbb{P} = \bigcup_{i\in I} \mathbb{P}_i$ where $\mathbb{P}_i = \{p_i, q_{i,1}, \ldots, q_{i,k_i}\}$ and for each $i \in I$ we fix the following $1 + k_i$ sets:

i) $E_{p_i} = \{\alpha_{p_i,1}, \ldots, \alpha_{p_i,m_i+1}\} \subset \mathbb{Z}_{p_i}$, $\alpha_{p_i,j}$ transcendental over $\mathbb{Q}$.

ii) $E_{q_{i,j}} = \{\alpha_{q_{i,j}}\} \subset \overline{\mathbb{Z}_{q_{i,j}}}$ such that $\alpha_{q_{i,j}}$ is transcendental over $\mathbb{Q}$ and satisfies $\alpha_{q_{i,j}}^{n_{i,j}} = \tilde{q}_{i,j}$, where $v_{q_{i,j}}(\tilde{q}_{i,j}) = 1$.

We set $\underline{E}_i = E_{p_i} \times \prod_{j=1}^{k_i} E_{q_{i,j}}$ and

$$R_i = \mathrm{Int}_{\mathbb{Q}}(E_{p_i}, \mathbb{Z}_{p_i}) \cap \bigcap_{j=1}^{k_i} \mathrm{Int}_{\mathbb{Q}}(E_{q_{i,j}}, \overline{\mathbb{Z}}_{q_{i,j}}) = \mathrm{Int}_{\mathbb{Q}}(\underline{E}_i, \overline{\widehat{\mathbb{Z}}})$$

By Eakin-Heinzer's result, R_i is a Dedekind domain with class group isomorphic to G_i.

Realization Theorem for Polynomial Dedekind domains

We set

$$R = \bigcap_{i\in I} R_i = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}})$$

where $\underline{E} = \prod_i \underline{E}_i$. In order for R to be Dedekind, $\underline{E}$ must be polynomially factorizable, that is, $\mathbb{P}_{g,\underline{E}} = \{p \in \mathbb{P} \mid \exists \alpha_p \in E_p, v_p(g(\alpha_p)) > 0\}$ finite for each $g \in \mathbb{Z}[X]$.

By a suitable alteration of $\alpha_p \in E_p$, as $p \in \mathbb{P}$, we may achieve this property.

Theorem (P. 2023)

Let G be a direct sum of a countable family $\{G_i\}_{i\in I}$ of finitely generated abelian groups.

Then there exists a Dedekind domain D, $\mathbb{Z}[X] \subset D \subseteq \mathbb{Q}[X]$ with class group isomorphic to G.

Thank you!

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