Arithmetics of Flatness for Monoids

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Flatness for domains and monoids

Introduction

Let R be a Dedekind domain whose class group is not a torsion group. Then there is a flat overdomain S of R (that is $R \subseteq S \subseteq q(R)$), such that $S \neq T^{-1}R$ for every multiplicatively closed subset T of R.

- Flat overmonoids in the category of monoids behave different: they are always monoids of fractions.

Let R be a domain, M a torsion-free R-module and put $R^\bullet := R \setminus \{0\}$, $M^\bullet := M \setminus \{0\}$.

- M is a factorable R-module if and only if $M^\bullet$ is a flat $R^\bullet$-act and M is atomic.
- If M is a pre-Schreier R-module, then $M^\bullet$ is a flat $R^\bullet$-act; conversely, if R is a pre-Schreier domain and $M^\bullet$ is a flat $R^\bullet$-act, then M is a pre-Schreier R-module.
- If R is a pre-Schreier domain and M a flat R-module, then $M^\bullet$ is a flat $R^\bullet$-act.

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Flatness for Monoids

Acts

In this talk, a *monoid* H is a multiplicatively written commutative and cancellative semigroup with unit element 1.

A non-empty set A is called an *H-act*, if there is a map $H \times A \to A$, $(s,a) \mapsto sa$ such that $1a = a$ and $(st)a = s(ta)$ for all $s, t \in H$ and $a \in A$; a map $\varphi : A \to B$ with H-acts A, B is a *morphism* of H-acts, if $\varphi(sa) = s\varphi(a)$ for all $s \in H$ and $a \in A$.

Let A, B be H-acts. A map $\rho : A \times B \to X$ to a set X is called *H-balanced*, if $\rho(sa, b) = \rho(a, sb)$ for all $s \in H$, $a \in A$ and $b \in B$. A set T together with an H-balanced map $\tau : A \times B \to T$ is called (the) *tensor product* of A and B if for every set X every H-balanced map $\rho : A \times B \to X$ factors uniquely through τ; it is denoted by $A \otimes B$.

FACT: For all $a, a' \in A$, $b, b' \in B$: $a \otimes b = a' \otimes b'$ if and only if there are $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$ and $s_1, \ldots, s_{n+1}, t_1, \ldots, t_n \in H$ such that $a = a_1 s_1$, $s_1 b = t_1 b_1$, $a_i t_i = a_{i+1} s_{i+1}$, $s_{i+1} b_i = t_{i+1} b_{i+1}$ for $i = 1, \ldots, n-1$, and $a_n t_n = a' s_{n+1}$, $s_{n+1} b_n = b'$.

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Flatness for Monoids

Definition

Any H-act A defines a covariant functor $A \otimes -$ from the category of H-acts to the category of sets; A is called

- *flat* if $A \otimes -$ preserves monomorphisms,
- *weakly flat* if $A \otimes -$ preserves all embeddings of ideals into H,
- *principally weakly flat* if $A \otimes -$ preserves all embeddings of principal ideals into H.

Further, A is said to be *torsion-free* if for all $s \in H$ and $a, b \in A$ the equality $sa = sb$ implies $a = b$.

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Flatness for Monoids

Properties

Theorem

Let H be a monoid and A an H-Act.
Then the following conditions are equivalent:
(1) A is flat,
(2) A is weakly flat,
(3) A is principally weakly flat and for all $a, b \in A$ and $s, t \in H$ such that $sa = tb$ there exist $c \in A$ and $u \in Hs \cap Ht$ such that $sa = tb = uc$,
(4) A is torsion-free and for all $a, b \in A$ and $s, t \in H$ such that $sa = tb$ there exist $c \in A$ and $u \in Hs \cap Ht$ such that $sa = tb = uc$,
(5) A is torsion-free and for all ideals I and J of H: $(I \cap J)A = IA \cap JA$.
(6) For all $a, b \in A$ and $s, t \in H$ such that $sa = tb$ there exist $c \in A$ and $u, v \in H$ such that $a = uc$, $b = vc$ and $us = vt$.

- Let T be a submonoid of a monoid H. Then $T^{-1}H$ is a flat H-act.
- Let H be a discrete valuation monoid and A a torsion-free H-Act. Then A is flat.

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Properties

- Let $\varphi : H \to D$ be a morphism of monoids making D a flat H-act. Then for all $u, v \in H$ such that $\varphi(u)|_D \varphi(v)$ there are $w \in \varphi^{-1}(D^\times)$ such that $u|_H vw$.

- In particular, if $q(\varphi) : q(H) \to q(D)$ denotes the canonical morphism induced in the quotient monoids, then $q(\varphi)^{-1}(D) = \varphi^{-1}(D^\times)^{-1}H$.

- Let H, D be monoids such that $H \subseteq D \subseteq q(H)$. The following conditions are equivalent:

(1) D is a flat H-act,
(2) $D = (H \cap D^\times)^{-1}H$,
(3) for every $x \in D$ there are $u \in H \cap D^\times$ such that $ux \in H$,
(4) $(H :_H x)D = D$ for every $x \in D$.

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Let R be a domain, M a torsion-free R-module and put $R^\bullet := R \setminus \{0\}$, $M^\bullet := M \setminus \{0\}$.

For all $r \in R$ and $x, y \in M$ such that $x = ry$, r is called an *R-divisor of x* and *y an M-divisor of x*.

$x \in M^\bullet$ is *irreducible* if every R-divisor of x is a unit of R. M is *factorable* if every $x \in M^\bullet$ has an M-divisor dividing every M-divisor of x.

- M is a factorable R-module if and only if $M^\bullet$ is a flat $R^\bullet$-act and every $x \in M^\bullet$ has an irreducible M-divisor.

M is a *pre-Schreier R-module* if for every $r, s \in H$ and $x, y \in M^\bullet$ such that $rx = sy$ there are $t, u, v \in R$ and $z \in M$ such that $r = tv$, $s = tu$, $x = uz$ and $y = vz$.

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For all $r \in R$ and $x, y \in M$ such that $x = ry$, r is called an *R-divisor of x* and y *an M-divisor of x*.
$x \in M^\bullet$ is *irreducible* if every R-divisor of x is a unit of R. M is *factorable* if every $x \in M^\bullet$ has an M-divisor dividing every M-divisor of x.

- M is a factorable R-module if and only if $M^\bullet$ is a flat $R^\bullet$-act and every $x \in M^\bullet$ has an irreducible M-divisor.

M is a *pre-Schreier R-module* if for every $r, s \in H$ and $x, y \in M^\bullet$ such that $rx = sy$ there are $t, u, v \in R$ and $z \in M$ such that $r = tv$, $s = tu$, $x = uz$ and $y = vz$.

- If M is a pre-Schreier R-module, then $M^\bullet$ is a flat $R^\bullet$-act; conversely, if R is a pre-Schreier domain and $M^\bullet$ is a flat $R^\bullet$-act, then M is a pre-Schreier R-module.
- If R is a pre-Schreier domain and M a flat R-module, then $M^\bullet$ is a flat $R^\bullet$-act.

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