Strong Types of Atomicity

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Outline

General Notation

General notation we will use throughout this talk:

- $\mathbb{N} := \{1, 2, 3, \dots\}$,
- $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, \dots\}$,
- $\mathbb{P}$ denotes the set of primes, and
- $\mathbb{F}_q$ denotes the field of q elements.

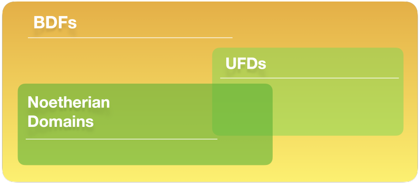
A Convention and some Preliminaries

A monoid is a cancellative commutative semigroup with an identity element.

Definition: Let M be a (multiplicative) monoid.

- We let $M^\times$ denote the group of units (i.e., invertible elements) of M.
- M is called reduced if $M^\times$ is the trivial group.
- M can be universally embedded into an abelian group $\mathrm{gp}(M)$, which is often called the Grothendieck group of M.
- M is torsion-free if $\mathrm{gp}(M)$ is a torsion-free group.
- The rank of M is the rank of the abelian group $\mathrm{gp}(M)$.
- $a \in M \setminus M^\times$ is an atom (or an irreducible) if for any $b, c \in M$ the equality $a = bc$ implies that either $b \in M^\times$ or $c \in M^\times$.
- We let $\mathcal{A}(M)$ denote the set of atoms of M.
- An element of M is atomic if it is a unit or it factors into atoms.
- A subset I of M is an ideal if $IM := \{bm \mid b \in I \text{ and } m \in M\} \subseteq I$.
- An ideal of the form bM for some $b \in M$ is called principal.

Beyond UFDs and Noetherian Domains



- **Unique Factorization Domains:** Gauss, Kummer, Dedekind...
- **Noetherian Domains:** Hilbert, Noether, Krull...

Definitions: Let M be a monoid.

- If $b = a_1 \cdots a_\ell$ for some atoms $a_1, \dots, a_\ell$ in M, then ℓ is a length of b.
- M is a bounded factorization monoid (BFM) if every element of M has a nonempty finite set of lengths.
- An integral domain R is a BFD if R^* is a BFM.

Examples of BFDs:

- UFDs and Noetherian domains.
- Mori domains.
- $\mathbb{Q}[M]$ with $M = (\{0\} \cup \mathbb{R}_{\geq 1}, +)$.

Beyond BFDs: The ACCP

Remark: The arithmetic of lengths of BFMs/BFDs has been well studied.

- Several classes of BFMs/BFDs have been proved to have a well-structured system of sets of lengths (Geroldinger 1988, Freiman-Geroldinger 2000, Geroldinger-Kainrath 2010).
- Several classes of BFMs/BFDs have been proved to have full systems of sets of lengths (Kainrath 1999, Frisch-Nakato-Rissner 2019, Ajran-Gotti 2023).

Definition: A monoid/domain satisfies the ACCP if every ascending chain of principal ideals stabilizes.

Examples of ACCP Domains:

- Every BFM/BFD satisfies the ACCP.
- $R[x]$ satisfies the ACCP if R does.
- $\mathbb{Q}\left[x^{1/p} \mid p \in \mathbb{P}\right]$ is an ACCP domain that is not a BFD.

Proposition (Cohn 1968)

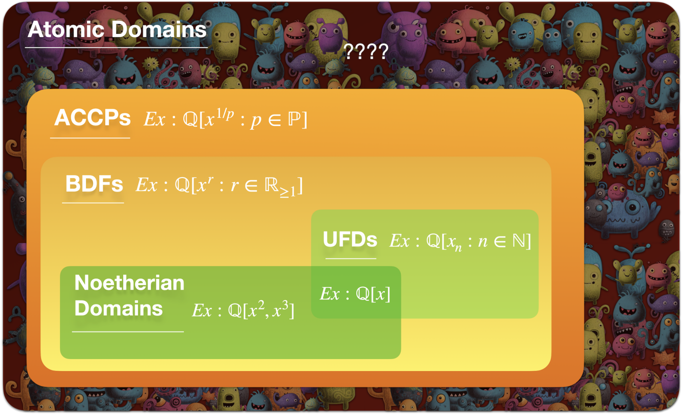
In an ACCP domain every nonzero nonunit factors into atoms.

The Lands of Atomicity

Definition (Atomicity: Cohn 1968)

A monoid/domain is called atomic if each nonzero nonunit factors into atoms.

Wildlands of Atomicity: The class of atomic monoids/domains that do not satisfy the ACCP (it's inhabited by beautiful creatures... and scary monsters).



The Right Wrong Assertion

Cohn's Assertion: ~~An integral domain is atomic **iff** it satisfies the ACCP.~~

Despite of being wrong, this assertion has stimulated several constructions of non-ACCP atomic domains: magical creatures and scary monsters inside the wildlands of atomicity.

- **1974 Grams:** the first counterexample
- **1982 Zaks:** two more constructions (one of them suggested by Cohn)
- **1993 Roitman:** further (stronger) incidental constructions

Further constructions have also been provided more recently.

- **2019 Boynton-Coykendall:** a pullback construction
- **2022 G.-Li:** a finite-dimensional monoid algebra
- **2023 Bell-Brown-Nazemian-Smertnig:** a non-commutative ring
- **2023 Bu-G.-Li-Zhao:** a one-dimensional monoid algebra

Into The Wildlands of Atomicity: Strong Atomicity

Definition (Strong Atomicity: Anderson-Anderson-Zafrullah 1990)

- A monoid M is strongly atomic if for all $b, c \in M$ there exists an atomic common divisor d of b and c in M such that every common divisor of b/d and c/d in M is a unit.
- An integral domain is strongly atomic if its multiplicative monoid is strongly atomic.

Remarks:

- Every strongly atomic monoid/domain is atomic.
- Every ACCP monoid/domain is strongly atomic.

Theorem (Roitman 1993)

There exists an atomic domain that is not strongly atomic.

Theorem (G.-Li 2022)

There exists a strongly atomic domain that is not ACCP.

Grams' Domain Is Strongly Atomic

- Let F be a field.
- Let $(p_n)_{n \ge 0}$ be a strictly increasing sequence of primes.
- Consider the additive monoid $M := \left\langle \frac{1}{p_0^n p_n} \mid n \in \mathbb{N} \right\rangle$.
- Let $F[M]$ be the monoid algebra of M over F.
- $S := \{f \in F[M] \mid f(0) \neq 0\}$ is a multiplicative subset of $F[M]$.

Remark: Neither $F[M]$ nor $F[M]_S$ satisfies the ACCP.

Theorem (Grams 1974)

$F[M]_S$ is an atomic domain.

Theorem (G.-Li 2022)

$F[M]_S$ is a strongly atomic domain.

Atomic Domains/Monoids not Strongly Atomic

Examples:

- **Roitman 1993:** An atomic domain not strongly atomic.
- **G.-Vulakh 2022:** A rank-2 atomic monoid not strongly atomic.
- **CrowdMath 2023:** A rank-1 atomic monoid not strongly atomic.

Definition: For each $k \in \mathbb{N}$, a domain/monoid is a k-MCD if every subset of size at most k has a maximal common divisor.

Remarks:

- Every domain/monoid is 1-MCD.
- A monoid is strongly atomic if and only if it is both atomic and 2-MCD.

Theorem (Roitman 1993)

For each $k \in \mathbb{N}$, there exists an atomic domain that is k-MCD but not $(k+1)$-MCD.

Theorem (G.-Rabinovitz 2023)

For each $k \in \mathbb{N}$, there exists an atomic rank-1 monoid that is k-MCD but not $(k+1)$-MCD.

Hereditary Atomicity: Monoids

Definition

A monoid M is hereditarily atomic if every submonoid of M is atomic.

Examples:

- Every numerical monoid is hereditarily atomic.
- Every reduced Krull monoid is hereditarily atomic.
- The additive monoid $\langle \frac{1}{p} \mid p \in \mathbb{P} \rangle$ is hereditarily atomic.

Proposition: If M is a monoid satisfying the ACCP, then every submonoid N of M with $N^\times = N \cap M^\times$ satisfies the ACCP.

Corollary

Every reduced monoid that satisfies the ACCP is hereditarily atomic.

Hereditary Atomicity: Monoids (cont.)

Theorem

1. **G.-Vulakh 2022:** *Every torsion-free hereditarily atomic monoid satisfies the ACCP.*
2. **G.-Li 2023:** *Every hereditarily atomic monoid satisfies the ACCP.*

Corollary: A reduced monoid is hereditarily atomic if and only if it satisfies the ACCP.

Example: Set $M = (\mathbb{Z} \times \mathbb{N}_0, +)$, which is a submonoid of $\mathbb{Z}^2$.

1. Since $M/M^\times$ is isomorphic to $(\mathbb{N}_0, +)$, the monoid M satisfies the ACCP.
2. The submonoid $N := (\mathbb{N}_0 \times \{0\}) \sqcup (\mathbb{Z} \times \mathbb{N})$ of M is the nonnegative cone of $(\mathbb{Z}^2, +)$ under the lexicographical order $\preceq$.
3. Hence $\mathcal{A}(N) = \big\{ \min_{\preceq}(N \setminus \{(0,0)\}) \big\} = \{(1,0)\}$, and so N is not atomic.
4. Thus, M satisfies the ACCP but is not hereditarily atomic.

Hereditary Atomicity: Abelian Groups

Examples:

- $(\mathbb{Z}, +)$ is hereditarily atomic.
- $(\mathbb{Q}, +)$ is not hereditarily atomic: its submonoid $\mathbb{Q}_{\geq 0}$ is not atomic.

Theorem (G. 2023)

Let G be an abelian group, and let T be the torsion subgroup of G. Then G is hereditarily atomic if and only if G/T is cyclic.

Corollary: $(\mathbb{Z}^2, +)$ is not a hereditarily atomic group.

Magic Beasts Inside $(\mathbb{Z}^2, +)$:

- A non-atomic monoid with nonempty set of atoms.
- An antimatter monoid that is not a subgroup.
- An atomic monoid that does not satisfy the ACCP (G. 2023).
- An ACCP monoid that is not a BFM (Tirador 2023).

Hereditary Atomicity: Integral Domains

Definition

An integral domain R is hereditarily atomic if every subring of R is atomic.

Examples:

- $\mathbb{Z}$ is hereditarily atomic.
- $\mathbb{F}_2[x]$ is hereditarily atomic.
- $\mathbb{Q}[x]$ is not hereditarily atomic: its subring $\mathbb{Z} + x\mathbb{Q}[x]$ is not atomic.

Proposition (Coykendall-G.-Hasenauer 2022)

- *For a field F, the ring $F[x]$ is hereditarily atomic if and only if F is an algebraic extension of $\mathbb{F}_p$ for some $p \in \mathbb{P}$.*
- *If R is an integral domain, then $R[[x]]$ is not hereditarily atomic.*

Proposition (G. 2023)

Let R be an integral domain, and let G be a nontrivial abelian group. Then $R[G]$ is hereditarily atomic if and only if R is an algebraic extension of $\mathbb{F}_p$ for some $p \in \mathbb{P}$ and G is the infinite cyclic group.

Hereditary Atomicity: Fields

Examples:

- $\mathbb{F}_q$ is hereditarily atomic.
- $\mathbb{Q}$ is hereditarily atomic.
- $\mathbb{Q}(x)$ is not hereditarily atomic: its subring $\mathbb{Z} + x\mathbb{Q}[x]$ is not atomic.

Theorem (Coykendall-G.-Hasenauer 2023)

Let F be a field.

- *If $\mathrm{char}(F) = 0$, then F is hereditarily atomic if and only if F is an algebraic extension of $\mathbb{Q}$ such that $\overline{\mathbb{Z}}_F$ is a Dedekind domain.*
- *If $\mathrm{char}(F) = p \in \mathbb{P}$, then F is hereditarily atomic if and only if the transcendental degree of F over $\mathbb{F}_p$ is at most 1 and $\overline{\mathbb{F}_p[x]}_F$ is a Dedekind domain for every $x \in F$.*

Related Open Questions

Question (1)

Let M be the Grams' monoid.

- *Is $\mathbb{Q}[M]$ atomic?*
- *Is $\mathbb{Q}[M]$ strongly atomic?*

Question (2)

- *Does every hereditarily atomic domain satisfy the ACCP?*
- *Is every hereditarily atomic domain strongly atomic?*

Definition: An integral domain is overatomic if all its overrings are atomic.

Question (3)

- *Does every overatomic domain satisfy the ACCP?*
- *Is every overatomic domain strongly atomic?*

References

D. D. Anderson, D. F. Anderson, and M. Zafrullah: *Factorizations in integral domains*, J. Pure Appl. Algebra **69** (1990) 1–19.

J. G. Boynton and J. Coykendall: *An example of an atomic pullback without the ACCP*, J. Pure Appl. Algebra **223** (2019) 619–625.

S. T. Chapman, F. Gotti, and M. Gotti: *Factorization invariants of Puiseux monoids generated by geometric sequences*, Comm. Algebra **48** (2020) 380–396.

P. M. Cohn: *Bezout rings and and their subrings*, Math. Proc. Cambridge Philos. Soc. **64** (1968) 251–264.

J. Coykendall and F. Gotti: *On the atomicity of monoid algebras*, J. Algebra **539** (2019) 138–151.

CrowdMath 2023: *Arithmetic of power monoids*, https://artofproblemsolving.com/polymath/mitprimes2023/f

G. Freiman and A. Geroldinger: *An addition theorem and its arithmetical application*, J. Number Theory **85** (1) (2000) 59–73.

References

- S. Frisch, S. Nakato, and R. Rissner: *Sets of lengths of factorizations of integer-valued polynomials on Dedekind domains with finite residue fields*, J. Algebra **528** (2019) 231–249.
- A. Geroldinger: *Über nicht-eindeutige Zerlegungen in irreduzible Elemente*, Math. Z. **197** (1988) 505–529.
- A. Geroldinger and F. Kainrath: *On the arithmetic of tame monoids with applications to Krull monoids and Mori domains*, J. Pure Appl. Algebra **214** (2010) 2199–2218.
- R. Gilmer: *Commutative Semigroup Rings*, Chicago Lectures in Mathematics, The University of Chicago Press, London, 1984.
- F. Gotti and B. Li: *Divisibility and a weak ascending chain condition on principal ideals*. Submitted. Preprint available on arXiv: https://arxiv.org/abs/2212.06213
- F. Gotti and J. Vulakh: *On the atomic structure of torsion-free monoids*, Semigroup Forum (to appear).

References

A. Grams: *Atomic rings and the ascending chain condition for principal ideals*, Proc. Cambridge Philos. Soc., **75** (1974) 321–329.

F. Kainrath: *Factorization in Krull monoids with infinite class group*, Colloq. Math. **80** (1999) 23–30.

M. Roitman: *Polynomial extensions of atomic domains*, J. Pure Appl. Algebra **87** (1993) 187–199.

A. Zaks: *Atomic rings without a.c.c. on principal ideals*, J. Algebra **80** (1982) 223–231.

End of Presentation

THANK YOU!