Some remarks on Prüfer rings with zero-divisors

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Outline

- Definition of Prüfer domain
  - Many different characterizations
- Five different conditions in rings with zero-divisors
  - Semihereditary rings
  - $\mathrm{w.\,gl.\,dim}(R) \leq 1$
  - Arithmetical rings
  - Gaussian rings
  - Prüfer rings
- Transfer of these conditions in some constructions:
  - Pullbacks
  - Regular homomorphic images

Terminology:

From now on, all rings are <u>commutative</u> unitary rings.

- An ideal $\mathfrak{a}$ of R is called **regular** if it contains a regular element.
  $\mathrm{Reg}(R)$ will denote the subset of regular elements of R.

- $\mathrm{Tot}(R)$ will denote the **total quotient ring** of R.

- An **overring** of R is a ring R' such that $R \subseteq R' \subseteq \mathrm{Tot}(R)$.

- A (fractional) ideal $\mathfrak{a}$ of a ring R is **invertible** if there exists an R-submodule $\mathfrak{b}$ of $\mathrm{Tot}(R)$ such that $\mathfrak{a}\mathfrak{b} = R$.

Prüfer domains

A domain D is called a **Prüfer domain** if it satisfies one of the following equivalent conditions:

1. Every f.g. (2-generated) ideal of D is invertible.
2. Every f.g. ideal of D is projective.
3. Every ideal of D is flat.
4. Every submodule of a flat module is flat.
5. Every f.g. ideal of D is locally principal.
6. $D_\mathfrak{p}$ is a valuation domain for every $\mathfrak{p} \in \mathrm{Spec}(D)$.
7. $D_\mathfrak{p}$ is a chained ring for every $\mathfrak{p} \in \mathrm{Spec}(D)$.
8. $\mathfrak{i} \cap (\mathfrak{j} + \mathfrak{k}) = (\mathfrak{i} \cap \mathfrak{j}) + (\mathfrak{i} \cap \mathfrak{k})$ for any three ideals $\mathfrak{i}, \mathfrak{j}$ and $\mathfrak{k}$ of D.
9. For every $f(X), g(X) \in D[X]$, $c(fg) = c(f)c(g)$.
10. Every overring of D is integrally closed.
11. Every overring of D is flat.

Prüfer domains

Plenty of characterizations:

- 14 on Burbaki
- 9 on Fontana, Huckaba and Papick's book
- 40 on Gilmer's book
- (also 11 on the previous slide and 25 on Wikipedia)

Examples:

- Dedekind domains are Prüfer domains
  - Dedekind = Prüfer + Noetherian
- The ring of entire functions on the complex plane is a Prüfer domain
- The ring of integer-valued polynomials over $\mathbb{Q}$ is a Prüfer domain

$$\mathrm{Int}(\mathbb{Z}) := \{f(X) \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$$

Five Prüfer conditions in rings with zero-divisors

(1) **Semihereditary rings** [Cartan and Eilenberg, 1956]

- Every f.g. ideal of R is projective.
- $\mathrm{Tot}(R)$ is ab. flat and $R_\mathfrak{p}$ is a valuation domain for every $\mathfrak{p} \in \mathrm{Spec}(R)$.

(2) $\mathrm{w.\,gl.\,dim}(R) \leq 1$

- Every ideal of R is flat.
- $R_\mathfrak{p}$ is a valuation domain for every $\mathfrak{p} \in \mathrm{Spec}(R)$.

(3) **Arithmetical rings** [Fuchs, 1949]

- Every f.g. ideal of R is locally principal.
- $R_\mathfrak{p}$ is a chained ring for every $\mathfrak{p} \in \mathrm{Spec}(R)$.
- $\mathfrak{i} \cap (\mathfrak{j} + \mathfrak{k}) = (\mathfrak{i} \cap \mathfrak{j}) + (\mathfrak{i} \cap \mathfrak{k})$ for any three ideals $\mathfrak{i}, \mathfrak{j}$ and $\mathfrak{k}$ of R.

(4) **Gaussian rings** [Tsang, 1965]

- For every $f(X), g(X) \in R[X]$, $c(fg) = c(f)c(g)$.

(5) **Prüfer rings** [Griffin, 1970]

- Every f.g. (2-generated) regular ideal of R is invertible.
- Every overring of R is integrally closed/flat.

Theorem [Bazzoni and Glaz, 2007]:

Let R be a ring. Then, for $i = 1, \dots, 4$:

- R has the Prüfer condition (n) if and only if R has the Prüfer condition $(n+1)$ and $\mathrm{Tot}(R)$ has the Prüfer condition (n).
- R has the Prüfer condition (n) if and only if R is a Prüfer ring and $\mathrm{Tot}(R)$ has the Prüfer condition (n).
- If $\mathrm{Tot}(R)$ is absolutely flat, then all five Prüfer conditions on R are equivalent.

Preservation properties

- Overrings:
  If R is a ring having Prüfer condition (n), then every overring of R has the same Prüfer condition.

- Localizations:
  Prüfer conditions $(1) \div (4)$ are preserved under localizations, while condition (5) is not.

- Quotients:
  - Quotients of Gaussian rings [resp. arithmetical rings] are still Gaussian [resp. arithmetical].
  - The same holds for Prüfer rings if quotients are taken with respect to regular ideals.
  - Prüfer conditions (1) and (2) are, in general, not preserved under homomorphic images (e.g. quotients of valuation domains are not necessarily domains).

Prüfer conditions in pullbacks: Why pullbacks?

"Attaching spectral spaces": given a pullback diagram with β surjective

$$\begin{tikzcd} D \arrow[r, "u"] \arrow[d, "v"] & A \arrow[d, "\alpha"] \\ B \arrow[r, two heads, "\beta"] & C \end{tikzcd}$$

we get a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(D) & \mathrm{Spec}(A) \arrow[l, hook', "u^*"'] \\ \mathrm{Spec}(B) \arrow[u, "v^*"] & \mathrm{Spec}(C) \arrow[l, hook', "\beta^*"'] \arrow[u, "\alpha^*"'] \end{tikzcd}$$

$\mathrm{Spec}(D)$ is homeomorphic to the topological space defined by the disjoint union of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ modulo the equivalence relation generated by $\mathfrak{p} \sim \alpha^*(\mathfrak{p})$, for each $\mathfrak{p} \in \mathrm{Spec}(C)$

Prüfer conditions in pullbacks: Why pullbacks?

Several constructions arises from pullbacks.

- **Amalgamated duplication along an ideal:** Let $\mathfrak{a}$ be an ideal of A:

$$A \bowtie \mathfrak{a} := \{(a, a+x) \mid a \in A, x \in \mathfrak{a}\}$$

$$\begin{tikzcd} A \bowtie \mathfrak{a} \arrow[r] \arrow[d] & A \arrow[d, "\pi"] \\ A \arrow[r, "\pi"] & A/\mathfrak{a}, \end{tikzcd}$$

- **Amalgamated algebras along an ideal:** Consider a ring homomorphism $f : A \longrightarrow B$ and an ideal $\mathfrak{b}$ of B,

$$A \bowtie^f \mathfrak{b} := \{(a, f(a)+b) \mid a \in A, b \in \mathfrak{b}\}$$

$$\begin{tikzcd} A \bowtie^f \mathfrak{b} \arrow[r] \arrow[d] & A \arrow[d, "\pi \circ f"] \\ B \arrow[r, "\pi"] & B/\mathfrak{b} \end{tikzcd}$$

Prüfer conditions in pullbacks: Why pullbacks?

Several constructions arises from pullbacks.

- **Bi-amalgamated algebras:** Let $f : A \longrightarrow B$, $g : A \longrightarrow C$ be ring homomorphisms and let $\mathfrak{b}$ [resp. $\mathfrak{c}$] be an ideal of B [resp. C] satisfying $f^{-1}(\mathfrak{b}) = g^{-1}(\mathfrak{c})$.

$$A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) := \{(f(a) + b, g(a) + c) \mid a \in A, b \in \mathfrak{b}, c \in \mathfrak{c}\}.$$

$$\begin{tikzcd} A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) \arrow[r] \arrow[d] & f(A) + \mathfrak{b} \arrow[d, "\alpha"] \\ g(A) + \mathfrak{c} \arrow[r, "\beta"] & A/\mathfrak{i}_0, \end{tikzcd}$$

Prüfer conditions in pullbacks: Why pullbacks?

Several constructions arises from pullbacks.

- **Constructions of the type $A + XB[X]$ and $A + XB[[X]]$:** Let $A \subseteq B$ be a ring extension and let $X := \{X_1, \ldots, X_n\}$ be a finite set of indeterminates over B. The subring $A + XB[X]$ of $B[X]$ arises from the following pullback diagram

$$\begin{tikzcd} A + XB[X] \arrow[r] \arrow[d, hook] & A \arrow[d, hook, "\iota"] \\ B[X] \arrow[r, "\pi"] & B \end{tikzcd}$$

- **$D + \mathfrak{m}$ construction:** Let $\mathfrak{m}$ be a maximal ideal of a ring T and let D be a subring of T such that $D \cap \mathfrak{m} = (0)$. The ring $D + \mathfrak{m}$ defined by the pullback diagram

$$\begin{tikzcd} D + \mathfrak{m} \arrow[r] \arrow[d, hook] & D \arrow[d, hook, "\iota"] \\ T \arrow[r, "\pi"] & T/\mathfrak{m} \end{tikzcd}$$

Several constructions arises from pullbacks.

- **CPI-extensions ("complete pre-image"):** Let $\mathfrak{p} \in \operatorname{Spec}(A)$. Set $k = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, and let $\pi : A_\mathfrak{p} \to k$ and $\lambda : A \to A_\mathfrak{p}$ be the canonical projection and the localization map respectively. Then $Q(A/\mathfrak{p}) \cong k$, so that $A/\mathfrak{p}$ can be identified as a subring of k.

$$\begin{tikzcd}
\lambda(A) + \mathfrak{p}A_\mathfrak{p} \arrow[r] \arrow[d, hook] & A/\mathfrak{p} \arrow[d, hook] \\
A_\mathfrak{p} \arrow[r, "\pi"] & k
\end{tikzcd}$$

Prüfer conditions in pullbacks:

Let $R \subseteq T$ be a ring extension with non-zero [regular] conductor $\mathfrak{c} = (R : T)$. Then $A := R/\mathfrak{c}$ is a subring of $B := T/\mathfrak{c}$ and the pullback diagram

$$\begin{tikzcd} R = A \times_B T \arrow[r] \arrow[d, hook] & A = R/\mathfrak{c} \arrow[d, hook] \\ T \arrow[r, "\pi"] & B = T/\mathfrak{c} \end{tikzcd}$$

is called a **[regular] conductor square**.

Remark: If $\mathfrak{c}$ is a regular ideal of T, then T is an overring of R.

Theorem [Boynton, 2008]:

Consider a <u>regular</u> conductor square as presented before:

$$\begin{tikzcd} R \arrow[r] \arrow[d, hook] & A \arrow[d, hook] \\ T \arrow[r, "\pi"] & B \end{tikzcd}$$

1. For $n = 1, 2, 3, 4$, R has Prüfer condition (n) if and only if T has Prüfer condition (n), and for each prime ideal $\mathfrak{p}$ of R, $A_{\mathfrak{p}}$ is a Prüfer ring, and $B_{\mathfrak{p}}$ is an overring of $A_{\mathfrak{p}}$.

2. If R is a Prüfer ring, then A and T are Prüfer rings, and $B_{\mathfrak{p}}$ is an overring of $A_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}$ of R. Conversely, for each prime ideal $\mathfrak{p}$ of R, if $A_{\mathfrak{p}}$ and $T_{\mathfrak{p}}$ are Prüfer rings, and $B_{\mathfrak{p}}$ is an overring of $A_{\mathfrak{p}}$, then R is a Prüfer ring.

Conductor squares are particular pullbacks in which the two morphisms are injective and surjective respectively.

Let $\pi : T \to B$ be a surjective ring homomorphism, take a subring A of B and set $R := \pi^{-1}(A)$. Then:

- $\ker(\pi)$ is contained in the conductor $\mathfrak{c} = (R : T)$. In particular, $\ker(\pi)$ is a common ideal of R and T.
- R is canonically isomorphic to the fiber product $A \times_B T$:

$$\begin{tikzcd} R \arrow[r] \arrow[d, hook] & A \arrow[d, hook] \\ T \arrow[r, "\pi"] & B \end{tikzcd}$$

Theorem [— and Finocchiaro, 2019]:

Let $\pi : T \to B$ be a surjective ring homomorphism, where B is an overring of some ring A. Assume that $\ker(\pi)$ is a regular ideal of T. Set $R := \pi^{-1}(A)$.

$$\begin{tikzcd}
R \arrow[r] \arrow[d, hook] & A \arrow[d, hook, "overring"] \\
T \arrow[r, "\pi"] & B
\end{tikzcd}$$

1. R is a Prüfer ring if and only if both T and A are Prüfer rings;
2. R is a Gaussian [resp. arithmetical] ring if and only if both T and A are Gaussian [resp. arithmetical] rings;
3. If both T and A are rings of weak global dimension ≤ 1 [resp. semihereditary rings], then so is R.

Some consequences:

Corollary:

Let R be a ring with total quotient ring $T = \mathrm{Tot}(R)$. Then, for $n = 1, 2, 3$, $R + XT[X]$ has Prüfer condition (n) if and only if R has Prüfer condition (n) and T is absolutely flat.

A *Manis pair* $(A, \mathfrak{p})$ is a pair where A is a ring, $\mathfrak{p}$ is a prime ideal of A and for every $x \in \mathrm{Tot}(A) \setminus A$, there exists $y \in \mathfrak{p}$ such that $xy \in A \setminus \mathfrak{p}$. Given a ring A and a prime ideal $\mathfrak{m}$ of A, A is called a *Prüfer Manis ring* if the following equivalent conditions hold:

- $(A, \mathfrak{m})$ is a Manis pair and A is a Prüfer ring.
- A is a Prüfer ring and $\mathfrak{m}$ is the unique regular maximal ideal of A.
- $(A, \mathfrak{m})$ is a Manis pair and $\mathfrak{m}$ is the unique regular maximal ideal of A.

Corollary:

Let B be a Prüfer Manis ring and let V be a valuation domain with quotient field $B/\mathfrak{m}$, where $\mathfrak{m}$ denotes the unique regular maximal ideal of B. Consider the canonical projection $\pi : B \to B/\mathfrak{m}$. Then $\pi^{-1}(V)$ is a Prüfer Manis ring.

Corollary [Houston and Taylor, 2007]:

Let T be a domain, $\mathfrak{i} \leq T$ and let D be a domain contained in $T/\mathfrak{i}$. Then $R := \pi^{-1}(D)$ is a Prüfer ring if and only if both D and T are Prüfer rings, $\mathfrak{i}$ is a prime ideal of T and $Q(D) = Q(T/\mathfrak{i})$.

$$\begin{tikzcd} R \arrow[r] \arrow[d, hook] & D \arrow[d, hook] \\ T \arrow[r, "\pi"] & T/\mathfrak{i} \end{tikzcd}$$

Corollary [Boisen and Larsen, 1973]:

A Prüfer ring is the homomorphic image of a Prüfer domain if and only if its total quotient ring is the homomorphic image of a Prüfer domain.

Essentially because we can consider the following pullback diagram:

$$\begin{tikzcd} D \arrow[r] \arrow[d, hook] & R \arrow[d, hook] \\ D' \arrow[r] & \mathrm{Tot}(R) \end{tikzcd}$$

The "overring assumption": it cannot be dropped in the "if-parts" of the theorem

(1)

$$\begin{tikzcd} k + Yk(X)[Y]_{(Y)} \arrow[r] \arrow[d, hook] & k \arrow[d, hook] \\ k(X)[Y]_{(Y)} \arrow[r, "\pi"] & k(X) \end{tikzcd}$$

Both k and $k(X)[Y]_{(Y)}$ are (local) Prüfer rings, $\ker(\pi)$ is clearly a regular ideal of $k(X)[Y]_{(Y)}$, but $R := k + Yk(X)[Y]_{(Y)}$ is not a Prüfer ring: it is a local domain, but it is not a valuation domain, since X, X^{-1} are in the quotient field of R but none of them belongs to R.

(2)

$$\begin{tikzcd} \mathbb{Z} + X^2\mathbb{Q}[X] \arrow[r] \arrow[d, hook] & \mathbb{Z} \arrow[d, hook] \\ \mathbb{Z} + X\mathbb{Q}[X] \arrow[r] & \frac{\mathbb{Z}+X\mathbb{Q}[X]}{(X^2)} \end{tikzcd}$$

Both $\mathbb{Z} + X\mathbb{Q}[X]$ and $\mathbb{Z}$ are Prüfer rings, the kernel of the bottom morphism is regular, but $\mathbb{Z} + X^2\mathbb{Q}[X]$ is not a Prüfer ring.

The "overring assumption": can we deduce it if A is Prüfer?

- if R is a local Prüfer ring, then T is a localization of R and T is an overring of R;

  Proposition: R is a local ring with Prüfer condition (n) if and only if T is a local ring with Prüfer condition (n), A is a local Prüfer ring and B is an overring of A.

- we can deduce that B is an overring of A also if $\mathrm{Tot}(A)$ is an absolutely flat ring (so, in particular, if A is a domain).

  Proposition: R has Prüfer condition (n) if and only if both T and A have the same Prüfer condition (n) and B is an overring of A.

A trivial situation:

Theorem [— and Finocchiaro, 2019:]

Consider any pullback diagram

$$\begin{tikzcd} A \times_C B \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

and assume that $\ker(f)$ and $\ker(g)$ are regular ideals of A and B respectively. Then the following conditions are equivalent:

- $A \times_C B$ has Prüfer condition (n);
- A and B have Prüfer condition (n) and $A \times_C B = A \times B$.

Regular homomorphic images

Definition:

Let $f : A \to B$ be a ring morphism. We say that f is a **regular morphism** if $f^{-1}(\mathrm{Reg}(B)) \subseteq \mathrm{Reg}(A)$. We say that a ring B is a **regular homomorphic image** of A if there exists a surjective regular morphism $f : A \to B$.

Theorem [— and Finocchiaro]:

Every regular homomorphic image of a Prüfer ring is a Prüfer ring.

Corollary

(1) Let A be a local Prüfer ring. Then $A/Z(A)$ is a Prüfer domain.

(2) Let R be a Prüfer ring in which every zero-divisor is nilpotent. Then $R/\mathfrak{p}$ is a Prüfer domain for every $\mathfrak{p} \in \mathrm{Spec}(R)$.

Corollary [Bakkari and Mahdou, 2009]:

Let $(T, \mathfrak{m})$ be a local ring of the form $T = k + \mathfrak{m}$, for some field k. Take a subdomain D of k such that $Q(D) = k$ and set $R := D + \mathfrak{m}$. Then R has Prüfer condition (n) if and only if T and D have the same Prüfer condition (n).

$$\begin{tikzcd}
D + \mathfrak{m} \arrow[r, "\pi_0"] \arrow[d, hook] & D \arrow[d, hook, "overring"] \\
k + \mathfrak{m} \arrow[r, "\pi"] & k
\end{tikzcd}$$

Sketch of the proof:

- If $\mathfrak{m} = \ker(\pi)$ is a regular ideal of T, we can apply our first theorem.
- If $\mathfrak{m}$ consists only of zero-divisors, then $k + \mathfrak{m} = Tot(D + \mathfrak{m})$, $\mathfrak{m} = Z(T)$ and π_0 is a regular morphism.

Thank you!

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