Derived set-like constructions in commutative algebra

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A topological introduction

Let X be a topological space.

- A point $x \in X$ is isolated if $\{x\}$ is an open set.
- A point $x \in X$ is a limit point if it is not isolated.
- The derived set of X is the set of the limit points of X.
- We denote the derived set by $\mathcal{D}(X)$.
- $\mathcal{D}(X)$ is always a closed subspace of X.
- $\mathcal{D}(X)$ can be empty (if the space is discrete).
- It may be $\mathcal{D}(X) = X$ (e.g., $X = \mathbb{R}$ with the Euclidean topology).

A topological introduction (2)

- $\mathcal{D}(X)$ is itself a topological space, so we can consider $\mathcal{D}(\mathcal{D}(X))$.
- It need not to be the whole $\mathcal{D}(X)$!
  - For example, if $X = \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$, then $\mathcal{D}(X) = \{0\}$ and thus $\mathcal{D}(\mathcal{D}(X)) = \emptyset$.
- We set $\mathcal{D}^2(X) := \mathcal{D}(\mathcal{D}(X))$.
- In the same way, we define $\mathcal{D}^3(X)$, $\mathcal{D}^4(X)$, $\mathcal{D}^5(X)$, ...

A topological introduction (3)

- Let α be an ordinal. We define

$$\mathcal{D}^\alpha(X) = \begin{cases} \mathcal{D}(\mathcal{D}^\gamma(X)) & \text{if } \alpha = \gamma + 1, \\ \bigcap_{\beta<\alpha} \mathcal{D}^\beta(X) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

- $\{\mathcal{D}^\alpha(X)\}$ is a descending chain of closed subsets of X.
- There is a (minimal) α such that $\mathcal{D}^\alpha(X) = \mathcal{D}^{\alpha+1}(X)$ (and thus $\mathcal{D}^\alpha(X) = \mathcal{D}^\beta(X)$ for all $\beta > \alpha$): it is called the Cantor-Bendixson rank of X.
- If, for this α, we have $\mathcal{D}^\alpha(X) = \emptyset$, we say that X is scattered.
- Equivalently, X is scattered if and only of every open set has an isolated point.

Back to algebra

- In this talk, I will show two algebraic constructions that are analogues to the derived set, and three applications of these constructions.
- Throughout the talk, D will be an integral domain, and K will be its quotient field.
- A D-submodule I of K is a fractional ideal if $dI \subseteq D$ for some $d \neq 0$.
- $\mathcal{F}(D)$ is the set of fractional ideals of D.
- $\mathrm{F}(D)$ is the set of D-submodules of K.

Part I

Jaffard and pre-Jaffard families I: Closure operations

Overrings

- An overring of D is a domain between D and K.
- We denote by $\mathrm{Over}(D)$ the set of all overrings of D.
- A sublocalization of D is an overring in the form $\bigcap\{D_P \mid P \in X\}$ for some family $X \subseteq \mathrm{Spec}(D)$.
- A flat overring of D is an overring that is flat as a D-module.
- Flat overrings (in particular, localizations) are sublocalizations; the converse fails.

Families of overrings

Let Θ be a family of flat overrings.

- Θ is complete if, for every ideal I of D, we have $I = \bigcap\{IT \mid T \in \Theta\}$.
    - Equivalently, if for every $P \in \mathrm{Spec}(D)$ there is a $T \in \Theta$ such that $PT \neq T$.
- Θ is independent if $TT' = K$ for all $T \neq T'$ in K.
    - Equivalently, if for every $P \in \mathrm{Spec}(D)$, $P \neq (0)$ there is at most one $T \in \Theta$ such that $PT \neq T$.
- Θ is locally finite if every $x \in D \setminus \{0\}$ is a unit in all but finitely many $T \in \Theta$.

Jaffard families

Definition

We say that $\Theta \subseteq \mathrm{Over}(D)$ is a Jaffard family if:

- *either $\Theta = \{K\}$ or $K \notin \Theta$;*
- *all $T \in \Theta$ are flat;*
- *Θ is complete;*
- *Θ is independent;*
- *Θ is locally finite.*

- If D is a Dedekind domain, $\Theta := \{D_M \mid M \in \mathrm{Max}(D)\}$ is a Jaffard family.

Why Jaffard families?

- Jaffard families generalize the concept of h-local domains.
    - A domain is *h-local* if $\{D_M \mid M \in \mathrm{Max}(D)\}$ is locally finite and every nonzero prime ideal is contained in only one maximal ideal.
    - $\{D_M \mid M \in \mathrm{Max}(D)\}$ is a Jaffard family if and only if D is h-local.
- If $\{X_\alpha\}$ is a family of D-submodules of K with nonzero intersection and $T \in \Theta$, then
$$\left(\bigcap_{\alpha\in A} X_\alpha\right) T = \bigcap_{\alpha\in A} X_\alpha T.$$
- If $T \in \Theta$, then $(I : J)T = (IT : JT)$ for every D-submodules I, J of K such that $(I : J) \neq (0)$.
- If M is a torsion D-module, then $M \simeq \bigoplus\{M \otimes T \mid T \in \Theta\}$.

Factorization properties

Let Θ be a Jaffard family of D.

- Every ideal I can be uniquely factored as $I = J_1 \cdots J_k$, where:
  - each J_i survives in exactly one $T_i \in \Theta$;
  - $T_i \neq T_j$ for all $i \neq j$.

  For Dedekind domains, we get back prime factorization.
- $\mathcal{F}(D) \simeq \bigoplus\{\mathcal{F}(T) \mid T \in \Theta\}$ (as monoids).
- $\mathrm{Inv}(D) \simeq \bigoplus\{\mathrm{Inv}(T) \mid T \in \Theta\}$ (as groups).
- If $\star$ is a star operation, then there are (uniquely and explicitly determined) star operations $\star_T$ on each T such that

$$I^\star = \bigcap_{T \in \Theta} (IT)^{\star_T}.$$

  In particular, $\mathrm{Star}(D) \simeq \prod\{\mathrm{Star}(T) \mid T \in \Theta\}$.

Topological aspects of Jaffard families

- The Zariski topology on $\mathrm{Over}(D)$ is generated by the sets

$$\mathcal{B}(x) := \{T \in \mathrm{Over}(D) \mid x \in T\}.$$

  It is related to the Zariski tppology on the spectrum.
- The inverse topology is generated by the complements of the $\mathcal{B}(x)$, and is related to properties of representations of D.
- Let Θ be a Jaffard family.
  - Θ is compact in the Zariski topology.
  - In the inverse topology, Θ is a discrete space.
  - Thus, all elements of Θ are isolated in Θ^{inv}.
  - This "explains" the fact that the representation has good properties: each point (=overring) is "sufficiently separated" from the rest of Θ.
  - What about non-discrete spaces?

Pre-Jaffard families

Definition

We say that $\Theta \subseteq \mathrm{Over}(D)$ is a Jaffard family if:

- *either $\Theta = \{K\}$ or $K \notin \Theta$;*
- *all $T \in \Theta$ are flat;*
- *Θ is complete;*
- *Θ is independent;*
- *Θ is locally finite.*

Pre-Jaffard families

Definition

We say that $\Theta \subseteq \mathrm{Over}(D)$ is a pre-Jaffard family if:

- *either $\Theta = \{K\}$ or $K \notin \Theta$;*
- *all $T \in \Theta$ are flat;*
- *Θ is complete;*
- *Θ is independent;*
- *Θ is compact, with respect to the Zariski topology.*

- The typical example is $\{D_M \mid M \in \mathrm{Max}(D)\}$, where D is a one-dimensional domain.
- Θ is Hausdorff, with respect to the inverse topology.
- Problem: is the last condition redundant?
- We want to give an algebraic notion of isolated point.

Jaffard overrings

Let T be a flat overring of D.

- We say that T is a Jaffard overring of D if it belongs to a Jaffard family of D.
- Define $T^\perp := \bigcap\{D_P \mid P = (0) \text{ or } P \in \mathrm{Spec}(D) \setminus \Sigma(T)\}$.
  - Here $\Sigma(T) := \{P \in \mathrm{Spec}(D) \mid T \subseteq D_P\}$.
- $T^\perp$ is again a sublocalization, and $\Sigma(T) \cup \Sigma(T^\perp) = \mathrm{Spec}(D)$.
  - $\{T, T^\perp\}$ is always complete.
- The following are equivalent:
  - T is a Jaffard overring;
  - $T \cdot T^\perp = K$;
  - if $P \neq (0)$ is a prime ideal of D, then $PT = T$ or $PT^\perp = T^\perp$;
  - $\{T, T^\perp\}$ is independent.

Jaffard overrings (2)

Let Θ be a pre-Jaffard family.

- Θ is a Jaffard family if and only if every $T \in \Theta$ is a Jaffard overring.
- T is a Jaffard overring if and only if $\Theta \setminus \{T\}$ is compact in the Zariski topology.
- If $T \in \Theta$ is a Jaffard overring, then T is isolated.
- The converse does not hold: T may be isolated, but not a Jaffard overring.
- The problem is that an overring of T may be a limit point of $\Theta \setminus \{T\}$ in the space of all overrings.

The derived sequence

Let Θ be a pre-Jaffard family.

- We denote by $\mathcal{N}(\Theta)$ the set of elements of Θ that are not Jaffard overrings of D.
- $\mathcal{N}(\Theta)$ is a closed set of Θ: we want to use in place of $\mathcal{D}(X)$.
- $\mathcal{N}(\Theta)$ is not a pre-Jaffard family of D: we have to take the overring

$$T_1 := \bigcap \{T \mid T \in \mathcal{N}(\Theta)\}.$$

- $\mathcal{N}(\Theta)$ is a pre-Jaffard family of T_1.
- Thus we can define $\mathcal{N}(\mathcal{N}(\Theta))$ as the elements of $\mathcal{N}(\Theta)$ that are not Jaffard overrings of T_1.

The derived sequence (2)

We define recursively:

- $\mathcal{N}^0(\Theta) := \Theta$ and $T_0 := D$.
- $\mathcal{N}^1(\Theta) = \mathcal{N}(\Theta)$ and $T_1 := \bigcap\{T \mid T \in \mathcal{N}^1(\Theta)\}$.
- We always define $T_\alpha := \bigcap\{T \mid T \in \mathcal{N}^\alpha(\Theta)\}$.
- For ordinals $\alpha > 1$:
  - if $\alpha = \gamma + 1$ is a successor ordinal, $\mathcal{N}^\alpha(\Theta) = \mathcal{N}(\mathcal{N}^\gamma(\Theta))$;
  - if α is a limit ordinal, $\mathcal{N}^\alpha(\Theta) := \bigcap_{\beta<\alpha} \mathcal{N}^\beta(\Theta)$.
- We obtain a decreasing sequence of subsets of Θ,

$$\Theta = \mathcal{N}^0(\Theta) \supseteq \mathcal{N}^1(\Theta) \supseteq \mathcal{N}^2(\Theta) \supseteq \cdots \supseteq \mathcal{N}^\alpha(\Theta) \supseteq \cdots$$

  and an increasing sequence of overrings of D (the derived sequence with respect to Θ):

$$D = T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots \subseteq T_\alpha \subseteq \cdots$$

The derived sequence (3)

- The Jaffard degree of a pre-Jaffard family Θ is the minimal ordinal α such that $T_{\alpha+1} = T_\alpha$. Equivalently, such that $\mathcal{N}^\alpha(\Theta) = \mathcal{N}^{\alpha+1}(\Theta)$.
- We call T_α the dull limit of Θ.
- The dull limit is the point at which we cannot go further: no element of Θ_α is a Jaffard overring of T_α.
- Let Θ be a pre-Jaffard family with Jaffard degree α. We say that:
  - Θ is sharp if $T_\alpha = K$ (equivalently, if $\mathcal{N}^\alpha(\Theta) = \emptyset$);
  - Θ is dull if $T_\alpha \neq K$ (equivalently, if $\mathcal{N}^\alpha(\Theta) \neq \emptyset$).
- The terminology comes from the theory of one-dimensional Prüfer domains: D is ultimately sharp if and only if $\Theta := \{D_M \mid M \in \mathrm{Max}(D)\}$ is sharp.

Examples

- If $\Theta = \{K\}$ (and so $D = K$) then Θ is sharp with Jaffard degree 0.
- If Θ is a Jaffard family and $D \neq K$, then $\mathcal{N}^1(\Theta) = \emptyset$. Thus, Θ is sharp with Jaffard degree 1.
- If Θ is a pre-Jaffard family with a single S that is not a Jaffard overring, then $\mathcal{N}^1(\Theta) = \{S\}$ and $\mathcal{N}^2(\Theta) = \emptyset$: hence, Θ is sharp with Jaffard degree 2. In this case, we say that Θ is a weak Jaffard family pointed at S.
- Let D be the ring of algebraic integers and $\Theta = \{D_M \mid M \in \mathrm{Max}(D)\}$. Then, no $T \in \Theta$ is a Jafffard overring, so that $\mathcal{N}(\Theta) = \Theta$ and $T_1 = T_0$. Hence, Θ is dull with Jaffard degree 0.

Stable semistar operations

- A stable semistar operation is a map $\star : \mathrm{F}(D) \longrightarrow \mathrm{F}(D)$ such that:
  - $\star$ is a closure operation;
  - $x \cdot I^\star = (xI)^\star$ for all $x \in K$, $I \in \mathrm{F}(D)$;
  - $(I \cap J)^\star = I^\star \cap J^\star$ for all $I, J \in \mathrm{F}(D)$.
- If Θ is a Jaffard family, then $I^\star = \bigcap_{T \in \Theta} (IT)^\star$ for every I, and thus there is a natural isomorphism

$$\mathrm{SStar}_{\mathrm{stab}}(D) \simeq \prod_{T \in \Theta} \mathrm{SStar}_{\mathrm{stab}}(T)$$

- We call a family Θ stable-preserving if this factorization holds for every stable semistar operation.
- Let D be an almost Dedekind domain with exactly one maximal ideal that is not finitely generated. Then, $\{D_M \mid M \in \mathrm{Max}(D)\}$ is not a Jaffard family, but it is stable-preserving.

Length functions

- A singular length function is a map $\ell : \mathrm{Mod}(D) \longrightarrow \{0, \infty\}$ such that
  - $\ell(0) = 0$;
  - if $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ is exact, then $\ell(M) = \ell(N) + \ell(P)$;
  - $\ell(M) = \sup\{\ell(N) \mid N \leq M \text{ is finitely generated}\}$.
- There is a natural correspondence between singular length functions and stable semistar operations.
- If $\star$ correspond to ℓ, the factorization of $\star$ corresponds to the factorization
$$\ell = \sum_{T \in \Theta} \ell \otimes T,$$
where $(\ell \otimes T)(M) = \ell(M \otimes T)$.
- Singular length functions factorize exactly if Θ is stable-preserving.

How to use the derived sequence

Let Θ be a pre-Jaffard family.

- Let Λ_1 be the union of
  - all Jaffard overrings of Θ (i.e., $\Theta \setminus \mathcal{N}(\Theta)$);
  - $T_1 := \bigcap\{T \mid T \in \mathcal{N}(\Theta)\}$, the first element of the derived sequence.
- We are concentrating all bad points of Θ in T_1.
- Λ_1 is a weak Jaffard family and thus it is stable-preserving.
- If $\star$ is any stable semistar operation, we have

$$I^\star = \bigcap_{T \in \Lambda_1} (IT)^\star = \bigcap_{T \in \Theta \setminus \mathcal{N}(\Theta)} (IT)^\star \cap (IT_1)^\star$$

- Now we do the same for T_1.

How to use the derived sequence (2)

- For all α we consider

$$\Lambda_\alpha := (\Theta \setminus \mathcal{N}^\alpha(\Theta)) \cup \{T_\alpha\}.$$

- By induction, every Λ_α is stable-preserving:

$$I^\star = \bigcap_{T\in\Lambda_\alpha} (IT)^\star = \bigcap_{T\in\Theta\setminus\mathcal{N}^\alpha(\Theta)} (IT)^\star \cap (IT_\alpha)^\star.$$

- In particular, this holds if α is the Jaffard degree of Θ.
- If Θ is sharp, $T_\alpha = K$ can be eliminated, and Θ is stable-preserving.

Dimension 1

- If D has dimension 1, consider $\Theta := \{D_M \mid M \in \mathrm{Max}(D)\}$.
- In this case, D_M is a Jaffard overring if and only if M is an isolated point of $\mathrm{Max}(D)^{\mathrm{inv}}$ (i.e., $\mathrm{Max}(D)$ endowed with the inverse topology).
- Therefore, the derived sequence of Θ corresponds exactly to the derived sequence of $\mathrm{Max}(D)^{\mathrm{inv}}$.
- Θ is sharp if and only if $\mathrm{Max}(D)^{\mathrm{inv}}$ is a scattered space; in this case, the stable semistar operations have the form

$$I^\star = \bigcap_{M \in \mathrm{Max}(D)} (ID_M)^{\star_M}$$

  where each $\star_M$ is a stable semistar operation on D_M.
- If Θ is dull, then there will be stable operations that cannot be written in this way.

Part II

Jaffard and pre-Jaffard families II: The Picard group

The Picard group

- The Picard group of the domain D is the quotient between the group $\mathrm{Inv}(D)$ of all invertible ideals of D and the subgroup of the principal ideals.
- Equivalently, it is the group of all projective modules of rank 1 (modulo isomorphism), with the tensor product as operation.
- The Picard group is a global property: if D is local, then $\mathrm{Pic}(D) = (0)$.
- If D is a Dedekind domain, then $\mathrm{Pic}(D) = (0)$ if and only if D is a principal ideal domain.

Integer-valued polynomials

Let $\mathrm{Int}(D) := \{f \in K[X] \mid f(D) \subseteq D\}$.

- If D is Dedekind, there is an exact sequence

$$0 \longrightarrow \mathrm{Pic}(D) \longrightarrow \mathrm{Pic}(\mathrm{Int}(D)) \longrightarrow \bigoplus_{M \in \mathrm{Max}(D)} \mathrm{Pic}(\mathrm{Int}(D_M)) \longrightarrow 0.$$

- Moreover, $\mathrm{Pic}(\mathrm{Int}(D_M))$ is known (it can be expressed as a quotient of a group of continuous functions).
- The same exact sequence holds for one-dimensional Noetherian domains.

Jaffard families and the Picard group

$$0 \longrightarrow \mathrm{Pic}(D) \longrightarrow \mathrm{Pic}(\mathrm{Int}(D)) \longrightarrow \bigoplus_{M \in \mathrm{Max}(D)} \mathrm{Pic}(\mathrm{Int}(D_M)) \longrightarrow 0.$$

Jaffard families and the Picard group

$$0 \longrightarrow \mathrm{Pic}(D) \longrightarrow \mathrm{Pic}(\mathrm{Int}(D)) \longrightarrow \bigoplus_{T\in\Theta} \mathrm{Pic}(\mathrm{Int}(T)) \longrightarrow 0.$$

- We want to substitute $\{D_M \mid M \in \mathrm{Max}(D)\}$ with a Jaffard family Θ.
- In general, the kernel is wrong (take for example $\Theta = \{D\}$).

Jaffard families and the Picard group

$$0 \longrightarrow \mathrm{Pic}(D,\Theta) \longrightarrow \mathrm{Pic}(\mathrm{Int}(D)) \longrightarrow \bigoplus_{T\in\Theta} \mathrm{Pic}(\mathrm{Int}(T)) \longrightarrow 0.$$

- We want to substitute $\{D_M \mid M \in \mathrm{Max}(D)\}$ with a Jaffard family Θ.
- In general, the kernel is wrong (take for example $\Theta = \{D\}$).
- This can be resolved using, instead of $\mathrm{Pic}(D)$, the subgroup

$$\mathrm{Pic}(D,\Theta) := \{[I] \in \mathrm{Pic}(D) \mid IT \text{ is principal for all } T \in \Theta\}.$$

- If $\Theta = \{D_M \mid M \in \mathrm{Max}(D)\}$, then $\mathrm{Pic}(D,\Theta) = \mathrm{Pic}(D)$.
- If $\Theta = \{D\}$, then $\mathrm{Pic}(D,\Theta) = (0)$.

- $\mathrm{Pic}(D, \Theta)$ is not easy to find.
- Using a little bit of homological algebra, we can transform the previous sequence into

$$0 \longrightarrow \mathrm{Pic}(D) \longrightarrow \mathrm{Pic}(\mathrm{Int}(D)) \longrightarrow \bigoplus_{T \in \Theta} \frac{\mathrm{Pic}(\mathrm{Int}(T))}{\mathrm{Pic}(T)} \longrightarrow 0.$$

- Better,

$$\frac{\mathrm{Pic}(\mathrm{Int}(D))}{\mathrm{Pic}(D)} \simeq \bigoplus_{T \in \Theta} \frac{\mathrm{Pic}(\mathrm{Int}(T))}{\mathrm{Pic}(T)}$$

Jaffard families and the Picard group (2)

- $\mathrm{Pic}(D,\Theta)$ is not easy to find.
- Using a little bit of homological algebra, we can transform the previous sequence into

$$0 \longrightarrow \mathrm{Pic}(D) \longrightarrow \mathrm{Pic}(\mathrm{Int}(D)) \longrightarrow \bigoplus_{T\in\Theta} \frac{\mathrm{Pic}(\mathrm{Int}(T))}{\mathrm{Pic}(T)} \longrightarrow 0.$$

- Better,

$$\mathcal{P}(D) \simeq \bigoplus_{T\in\Theta} \mathcal{P}(T)$$

where $\mathcal{P}(A) := \mathrm{Pic}(\mathrm{Int}(A))/\mathrm{Pic}(A)$.

Weak Jaffard families

- Let Θ be a pre-Jaffard family.
- We concentrate $\mathcal{N}(\Theta)$ into the first step of the derived sequence, T_1.
- We don't get an exact sequence with $\mathrm{Pic}(\mathrm{Int}(D))$.
- However, using a reasoning similar to the one for Jaffard families, we obtain an exact sequence

$$0 \longrightarrow \bigoplus_{T\in\Theta\setminus\mathcal{N}(\Theta)} \mathcal{P}(T) \longrightarrow \mathcal{P}(D) \longrightarrow \frac{\mathrm{Pic}(\mathrm{Int}(D)T_1)}{\mathrm{Pic}(T_1)} \longrightarrow 0.$$

- What about T_α instead of T_1?

Pre-Jaffard families

- Extending the previous results to pre-Jaffard properties runs into two problems.
  - We use the fact that $\mathrm{Int}(D)T = \mathrm{Int}(T)$ if T is a Jaffard overring, but the equality does not hold for arbitrary flat overrings.
  - We need some ways to split exact sequences.
- These are not solvable in general: we need to add new hypothesis.
- Let Θ be a pre-Jaffard family and α an ordinal; suppose that
  - $\mathrm{Int}(D)T = \mathrm{Int}(T)$ if $T \in \Theta \setminus \mathcal{N}^\alpha(\Theta)$ or $T = T_\gamma$ with $\gamma < \alpha$;
  - $\mathcal{P}(T)$ is a free group for each $T \in \Theta \setminus \mathcal{N}^\alpha(\Theta)$.

  Then, there is an exact sequence

$$0 \longrightarrow \bigoplus_{T \in \Theta \setminus \mathcal{N}^\alpha(\Theta)} \mathcal{P}(T) \longrightarrow \mathcal{P}(D) \longrightarrow \frac{\mathrm{Pic}(\mathrm{Int}(D)T_\alpha)}{\mathrm{Pic}(T_\alpha)} \longrightarrow 0.$$

- If $T_\alpha = K$, then $\mathcal{P}(D) \simeq \bigoplus_{T \in \Theta} \mathcal{P}(T)$.

Pre-Jaffard families (2)

- The previous hypothesis hold in some interesting cases.
- $\mathrm{Int}(D)T = \mathrm{Int}(T)$ holds if $\mathrm{Int}(D)$ behaves well under localizations.
    - This happens for some almost Dedekind domains (that can be characterized).
- If V is a valuation domain, then $\mathcal{P}(V) = \mathrm{Pic}(\mathrm{Int}(V))$ is free.
- In this case, we have an exact sequence

$$0 \longrightarrow \bigoplus_{T \in \Theta \setminus \mathcal{N}^{\alpha}(\Theta)} \mathcal{P}(T) \longrightarrow \mathcal{P}(D) \longrightarrow \mathcal{P}(T_\alpha) \longrightarrow 0.$$

- If $\mathrm{Max}(D)^{\mathrm{inv}}$ is scattered, then

$$\mathcal{P}(D) \simeq \bigoplus_{M \in \mathrm{Max}(D)} \mathcal{P}(D_M).$$

Beyond integer-valued polynomials

- The results for $\mathrm{Int}(D)$ actually holds for other constructions.
- Let R be one of the following: $D[X]$, $\mathrm{Int}(E,D)$, $\mathbb{B}_x(D)$ (the Bhargava ring with respect to x).
- Define $\mathrm{LPic}(R,D)$ as the quotient $\mathrm{Pic}(R)/\mathrm{Pic}(D)$.
- If Θ is a Jaffard family of D, then

$$\mathrm{LPic}(R,D) \simeq \bigoplus_{T\in\Theta} \mathrm{LPic}(RT,T).$$

- The same decomposition holds if Θ is a sharp pre-Jaffard family of D and each $\mathrm{LPic}(RT,T)$ is free.

Part III

Almost Dedekind domains

Almost Dedekind domains

- An almost Dedekind domain in an integral domain D such that D_M is a DVR for all $M \in \mathrm{Max}(D)$.
- An almost Dedekind domain is Prüfer and one-dimensional.
- We use $\mathcal{M}$ to denote $\mathrm{Max}(D)$ with the inverse topology.

Radical factorization

- An ideal I of a domain D has radical factorization if we can write $I = J_1 \cdots J_n$ for some radical ideals J_i.
- If every ideal of D has radical factorization, then D is an SP-domain.
- Every SP-domain is almost Dedekind, but not all almost Dedekind domains are SP-domains.
- The following are equivalent for an almost Dedekind domain D:
  - D is an SP-domain;
  - the radical of every finitely generated ideal is finitely generated;
  - D has no critical maximal ideal (more on them later).

The map associated to an ideal

Let D be an almost Dedekind domain.

- To every fractional ideal I we can associate a map

$$\begin{aligned}\nu_I\colon \mathcal{M} &\longrightarrow \mathbb{Z},\\ M &\longmapsto v_M(I),\end{aligned}$$

  where v_M is the valuation associated to D_M and $v_M(I) := \inf\{v_M(x) \mid x \in I\}$.
- If I is finitely generated (=invertible), then ν_I is of a function of compact support.
  - If $f : X \longrightarrow \mathbb{Z}$, $\mathrm{supp}(f)$ is the closure of $\{x \in X \mid f(x) \neq 0\}$.
- In general, it is not continuous.
- Indeed, in general, ν_I is not bounded, while every continuous function with compact support is bounded.

Continuity

Let $\mathrm{Inv}(D)$ be the group of invertible ideals of D.

- For finitely generated ideals, radical factorization corresponds to continuity of ν_I.
  - If $I = \mathrm{rad}(J)$ for some finitely generated ideal J, then $V(I)$ is clopen in $\mathcal{M}$ and thus ν_I is continuous.
- [Huebo-Kwegna–Olberding–Reinhart] If D is an SP-domain with nonzero Jacobson radical, then $\mathrm{Inv}(D) \simeq \mathcal{C}(\mathcal{M}, \mathbb{Z})$.
- If D is an SP-domain, then $\mathrm{Inv}(D) \simeq \mathcal{C}_c(\mathcal{M}, \mathbb{Z})$.
- In particular, $\mathrm{Inv}(D)$ is a free group.
  - It is a subgroup of the group of bounded functions, which is free.

Critical maximal ideals

- Let M be a maximal ideal of an almost Dedekind domain D. Then, M is critical if it does not contain a finitely generated radical ideal.
- We denote by $\mathrm{Crit}(D)$ the set of critical ideals of D.
- A finitely generated ideal I has radical factorization if and only if $V(I) \cap \mathrm{Crit}(D) = \emptyset$.
- D is an SP-domain if and only if $\mathrm{Crit}(D)$ is empty.
- In general, $\mathrm{Crit}(D)$ is a closed subset of $\mathcal{M}$, and $\mathrm{Crit}(D) \subseteq \mathcal{D}(\mathcal{M})$.
- We can do a derived-set like construction.

Critical maximal ideals (2)

- Let $\mathrm{Crit}(D)$ be the set of critical maximal ideals.
- Then, $T_1 := \bigcap\{D_P \mid P \in \mathrm{Crit}(D)\}$ is an almost Dedekind domain whose maximal ideals are the extensions of the elements of $\mathrm{Crit}(D)$.
- We can construct $\mathrm{Crit}(T_1)$.
- We set $\mathrm{Crit}_2(D) := \{P \in \mathcal{M} \mid PT_1 \in \mathrm{Crit}(T_1)\}$.
- More generally:
  - $T_\alpha := \bigcap\{D_P \mid P \in \mathrm{Crit}_\alpha(D)\}$;
  - if $\alpha = \gamma + 1$, then $\mathrm{Crit}_\alpha(D) := \{P \in \mathcal{M} \mid PT_\gamma \in \mathrm{Crit}(T_\gamma)\}$;
  - if α is a limit ordinal, then $\mathrm{Crit}_\alpha(D) = \bigcap_{\beta<\alpha} \mathrm{Crit}_\beta(D)$.
- Structurally, this is the same as the construction of $\mathcal{D}^\alpha(X)$ or $\mathcal{N}^\alpha(\Theta)$.

Exact sequences

- We have an exact sequence
$$0 \longrightarrow \Delta_1 \longrightarrow \mathrm{Inv}(D) \longrightarrow \mathrm{Inv}(T_1) \longrightarrow 0,$$
where the map $\mathrm{Inv}(D) \longrightarrow \mathrm{Inv}(T_1)$ is the extension.
- $\Delta_1 = \{I \in \mathrm{Inv}(D) \mid IT_1 = T_1\} = \langle\{I \mid V(I) \cap \mathrm{Crit}(D) = \emptyset\}\rangle$.
- The proper ideals in Δ_1 are exactly the ones having radical factorization.
- Therefore, $\Delta_1 \simeq \mathcal{C}_c(\mathcal{M} \setminus \mathrm{Crit}(D), \mathbb{Z})$.
- More generally, for every α we have an exact sequence
$$0 \longrightarrow \Delta_\alpha \longrightarrow \mathrm{Inv}(D) \longrightarrow \mathrm{Inv}(T_\alpha) \longrightarrow 0.$$
where Δ_α is generated by the proper ideals I such that $V(I) \cap \mathrm{Crit}_\alpha(D) = \emptyset$.

Exact sequences (2)

- The map $\mathrm{Inv}(D) \longrightarrow \mathrm{Inv}(T_\alpha)$ is the composition of the step-wise maps $\mathrm{Inv}(T_\beta) \longrightarrow \mathrm{Inv}(T_{\beta+1})$.
- So, the kernel Δ_α is the "composition" of these kernels.
- By the case $\alpha = 1$, they are isomorphic to $\mathcal{C}_c(X_\beta, \mathbb{Z})$ (where $X_\beta = \mathrm{Crit}_\beta(D) \setminus \mathrm{Crit}_{\beta+1}(D)$).
- In particular, they are free groups.
- This allows to split some sequence of kernels: we obtain that

$$\Delta_\alpha \simeq \bigoplus_{\beta<\alpha} \ker(\mathrm{Inv}(T_\beta) \longrightarrow \mathrm{Inv}(T_{\beta+1})) \simeq \bigoplus_{\beta<\alpha} \mathcal{C}_c(X_\beta, \mathbb{Z})$$

or, in other words, an exact sequence

$$0 \longrightarrow \bigoplus_{\beta<\alpha} \mathcal{C}_c(X_\beta, \mathbb{Z}) \longrightarrow \mathrm{Inv}(D) \longrightarrow \mathrm{Inv}(T_\alpha) \longrightarrow 0.$$

SP-scattered domains

- There is a (minimal) ordinal α such that $\mathrm{Crit}_\alpha(D) = \mathrm{Crit}_{\alpha+1}(D)$.
- We call α the SP-rank of D.
- If, for this α, we have $\mathrm{Crit}_\alpha(D) = \emptyset$, we say that D is SP-scattered.
- In this case, $T_\alpha = K$, and the exact sequence becomes

$$0 \longrightarrow \bigoplus_{\beta<\alpha} \mathcal{C}_c(X_\beta, \mathbb{Z}) \longrightarrow \mathrm{Inv}(D) \longrightarrow 0 \longrightarrow 0.$$

- So, $\mathrm{Inv}(D) \simeq \bigoplus_{\beta<\alpha} \mathcal{C}_c(X_\beta, \mathbb{Z})$.
- In particular, $\mathrm{Inv}(D)$ is free.

Anti-SP domains

- The worst case for the previous construction is when $\mathrm{Crit}(D) = \mathcal{M}$.
- This is equivalent to saying that all finitely generated ideals are unbounded.
- This is impossible (by Baire category theorem), and thus all almost Dedekind domains are SP-scattered.
- In particular, $\mathrm{Inv}(D)$ is free for every almost Dedekind domain D, and there is always a bounded finitely generated ideal.
- $\mathcal{M} \setminus \mathrm{Crit}(D)$ is always dense in $\mathcal{M}$: "almost all" maximal ideals are non-critical.

Length functions

Let D be an almost Dedekind domain.

- Let ℓ be a singular length function on D, and let $\tau(I) := \ell(D/I)$.
- Like $\ell \otimes T$, we can define $(\tau \otimes T)(I) = \ell(T/IT)$.
- If $\mathcal{M}$ is scattered, then $\ell = \sum \ell \otimes D_M$ and thus $\tau = \sum \tau \otimes D_M$.
- In this case, any stable semistar operation is in the form $I \mapsto \bigcap\{ID_P \mid P \in \Delta\}$ for some $\Delta \subseteq \mathcal{M}$.
- In particular, $\tau(I) = \tau(\mathrm{rad}(I))$.
- For ideals, this means that stable semistar operations are radical: $1 \in I^\star$ if and only if $1 \in \mathrm{rad}(I)^\star$.

Length functions (2)

- If D is an SP-domain, then every ideal contains a power of its radical.
- Since $\tau(I) = \tau(I^n)$, it follows that the equality $\tau(I) = \tau(\mathrm{rad}(I))$ holds also for SP-domains.
- Using the sequence $\{T_\alpha\}$, for every ordinal α, we have

$$\tau(I) = \tau(\mathrm{rad}(I)) + (\tau \otimes T_\alpha)(I).$$

- Choosing α to be the SP-rank, $T_\alpha = K$ and $\tau(I) = \tau(\mathrm{rad}(I))$ for all almost Dedekind domains.
- Every stable semistar operation is the supremum of a family of $s_\Delta : I \mapsto \bigcap\{ID_P \mid P \in \Delta\}$.
- Problem: is there a more explicit way to write them?

Thank you for your attention!

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Invertible ideals and pre-Jaffard families

- A similar reasoning can be done with the derived sequence of a pre-Jaffard family (for example, for arbitrary one-dimensional Prüfer domains).
- The first exact sequence becomes

$$0 \longrightarrow \bigoplus_{A \in \Theta \setminus \mathcal{N}(\Theta)} \mathrm{Inv}(A) \longrightarrow \mathrm{Inv}(D) \longrightarrow \mathrm{Inv}(T_1) \longrightarrow 0$$

- Similarly, we have $0 \longrightarrow \Delta_\alpha \longrightarrow \mathrm{Inv}(D) \longrightarrow \mathrm{Inv}(T_\alpha) \longrightarrow 0$.
- The proof of the splitting $\Delta_\alpha \simeq \bigoplus \mathrm{Inv}(A)$, however, works only if the $\mathrm{Inv}(A)$ are free or divisible.
- What happens in the general case?

Anti-SP domains

Let D be an almost Dedekind domain.

- We say that D is anti-SP if $\mathrm{Crit}(D) = \mathsf{Max}(D)$.
- In this case, the procedure outlined above stops at the first step.
- If D is anti-SP, ν_I is unbounded for all finitely generated ideals I, and $Y_n := \nu_I^{-1}((n, +\infty))$ is dense in $\mathcal{M}$ and in $V(I)$.
- Since $V(I)$ is compact Hausdorff, also $\bigcap_n Y_n$ is dense.
- However, $\bigcap_n Y_n = \emptyset$.
- Thus, there are no anti-SP domains, and all almost Dedekind domains are SP-scattered.