Leavitt Path Algebras & Talented monoids via Lie Brackets and Adjacency Matrices

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- W. Bock, C.G. Canto, I.R. Campos, D. Martin Barquero, C. Martin Gonzalez and Sebandal, A. (2022). **Algebraic Entropy and A Complete Classification of Path Algebras over Finite Graphs by Growth**. preprint

- Bock, W., Sebandal, A. (2022). **An Adjacency Matrix Perspective of Talented monoids and Leavitt path algebras**. To appear in *Linear Algebra and its Applications*

- Bock, W., Sebandal, A., & Vilela, J. (2022). **A talented monoid view on Lie bracket algebras over Leavitt path algebras**. *Journal of Algebra and Its Applications*

- Hazrat, R., Sebandal, A., & Vilela, J. P. (2022). **Graphs with disjoint cycles classification via the talented monoid**. *Journal of Algebra*

- Sebandal, A., & Vilela, J. (2022). **The Jordan–Hölder theorem for monoids with group action.** *Journal of Algebra and Its Applications*

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Leavitt path algebra

$L_K(E)$

Directed graph

E

Talented monoid

T_E

$$\begin{tikzcd} & \text{Leavitt path algebra} \\ & L_K(E) \arrow[d, dashed, leftrightarrow] \\ \text{Directed graph} \quad E \arrow[ur] \arrow[r] & \text{Talented monoid} \quad T_E \end{tikzcd}$$

Graded Classification Conjecture

For finite graphs E and F:

$$T_E \cong T_F \iff Gr\text{-}L_K(E) \approx_{gr} Gr\text{-}L_K(F)$$

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$\mathbb{Z}$-isomorphism of talented monoids

Graded equivalence of categories of graded modules over the Leavitt path algebra

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$\mathbb{Z}$-isomorphism of talented monoids

Graded equivalence of categories of graded modules over the Leavitt path algebra

Graded Classification Theorem:

Polycephaly Graphs (Hazrat, circa 2013) ✓

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Leavitt path algebra

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T_E

Adjacency Matrix

$\mathrm{Adj}(E)$

Leavitt path algebra

$L_K(E)$

Lie Bracket algebra

$[L_K(E), L_K(E)]$

Directed graph

E

Talented monoid

T_E

Adjacency Matrix

$\mathrm{Adj}(E)$

Leavitt path algebra

$L_K(E)$

Abrams, Pino (2005)
Ara, Moreno, Pardo (2007)

Directed graph

E

Talented monoid

T_E

Hazrat (2013)
Hazrat, Li (2020)

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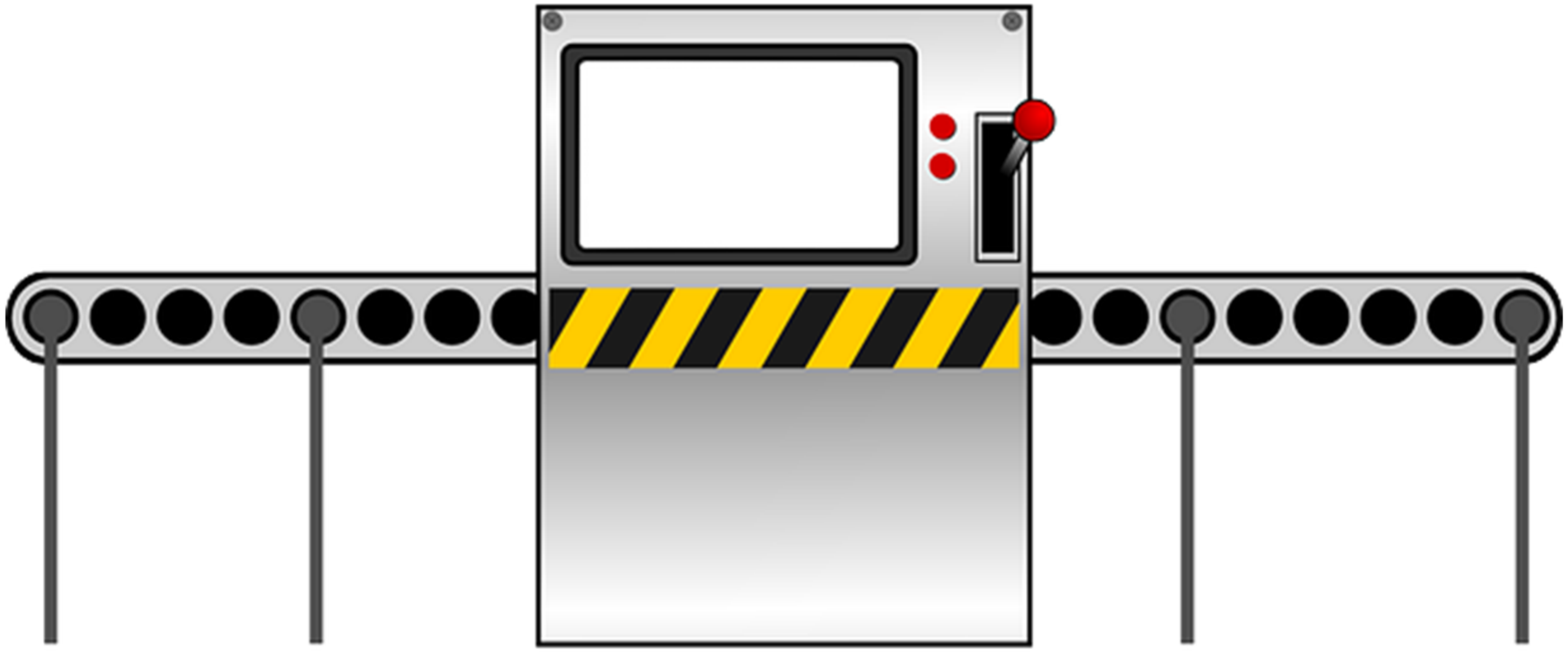
Beaver

Cat

Ant

The Mad Veterinarian



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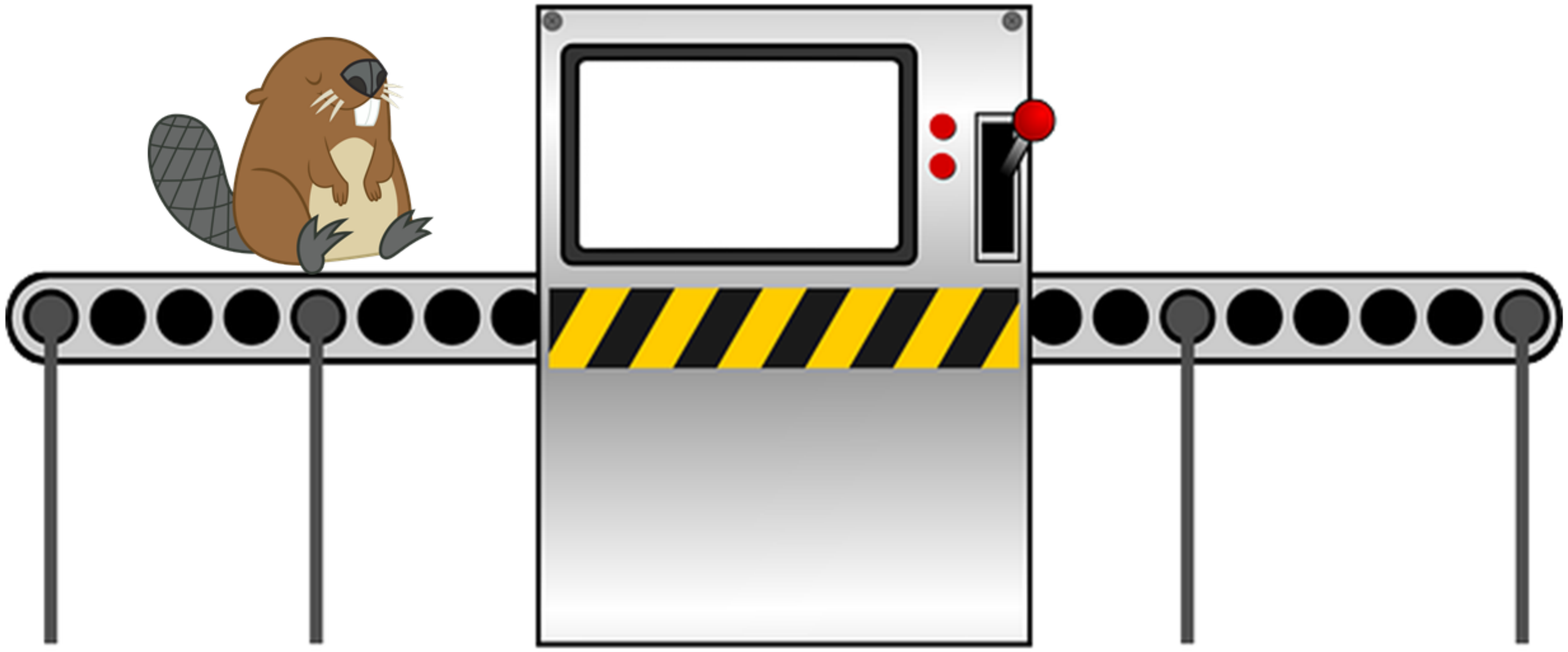

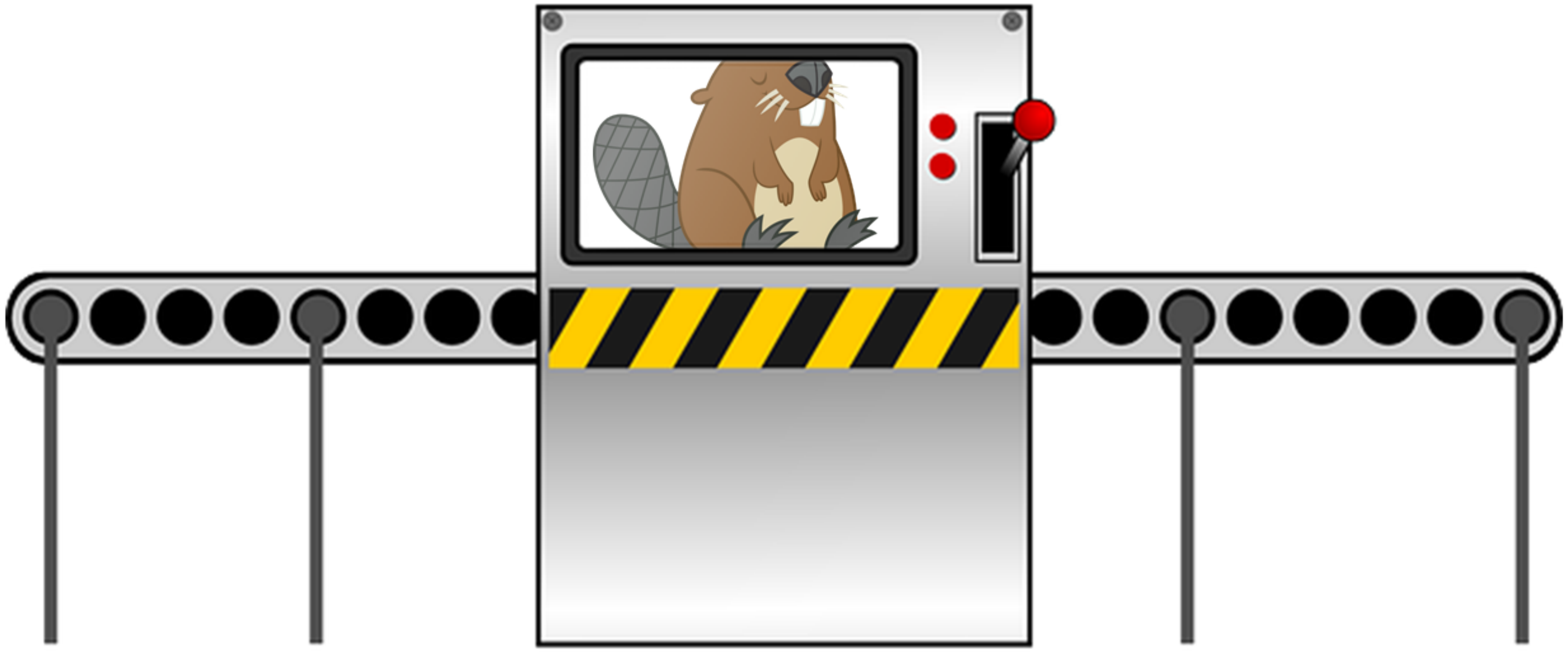

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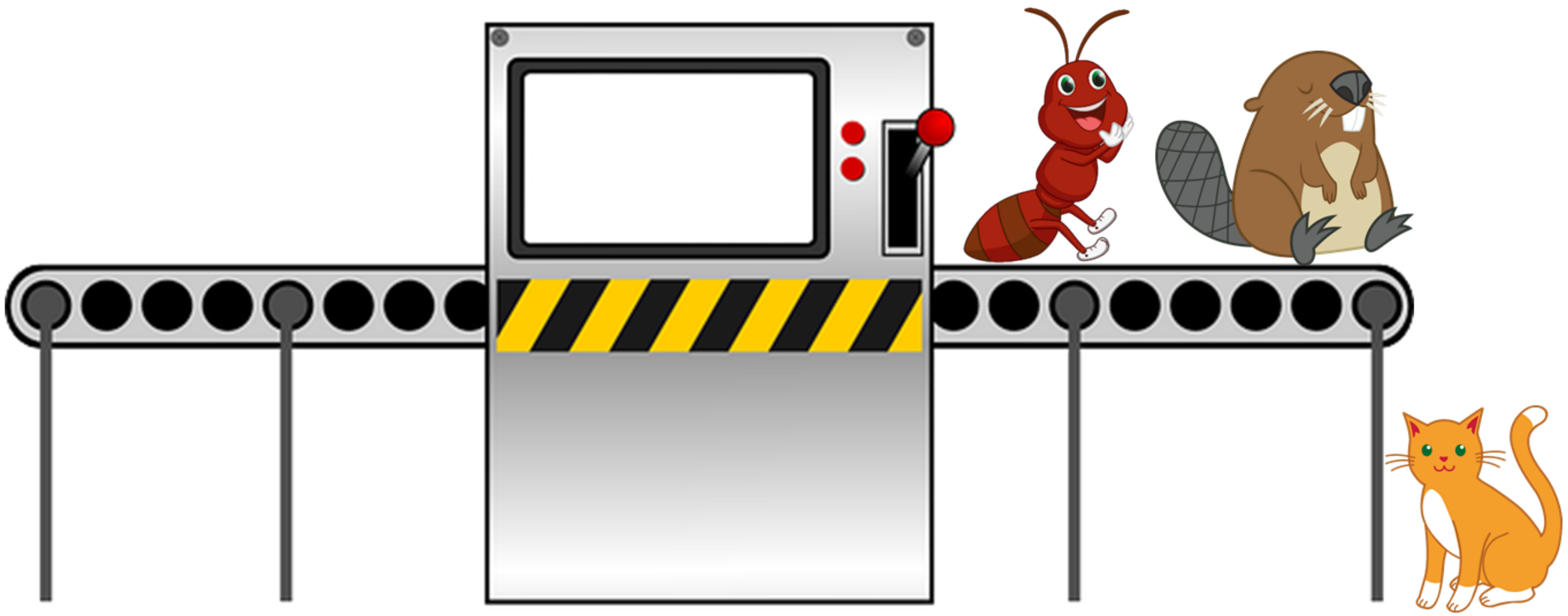

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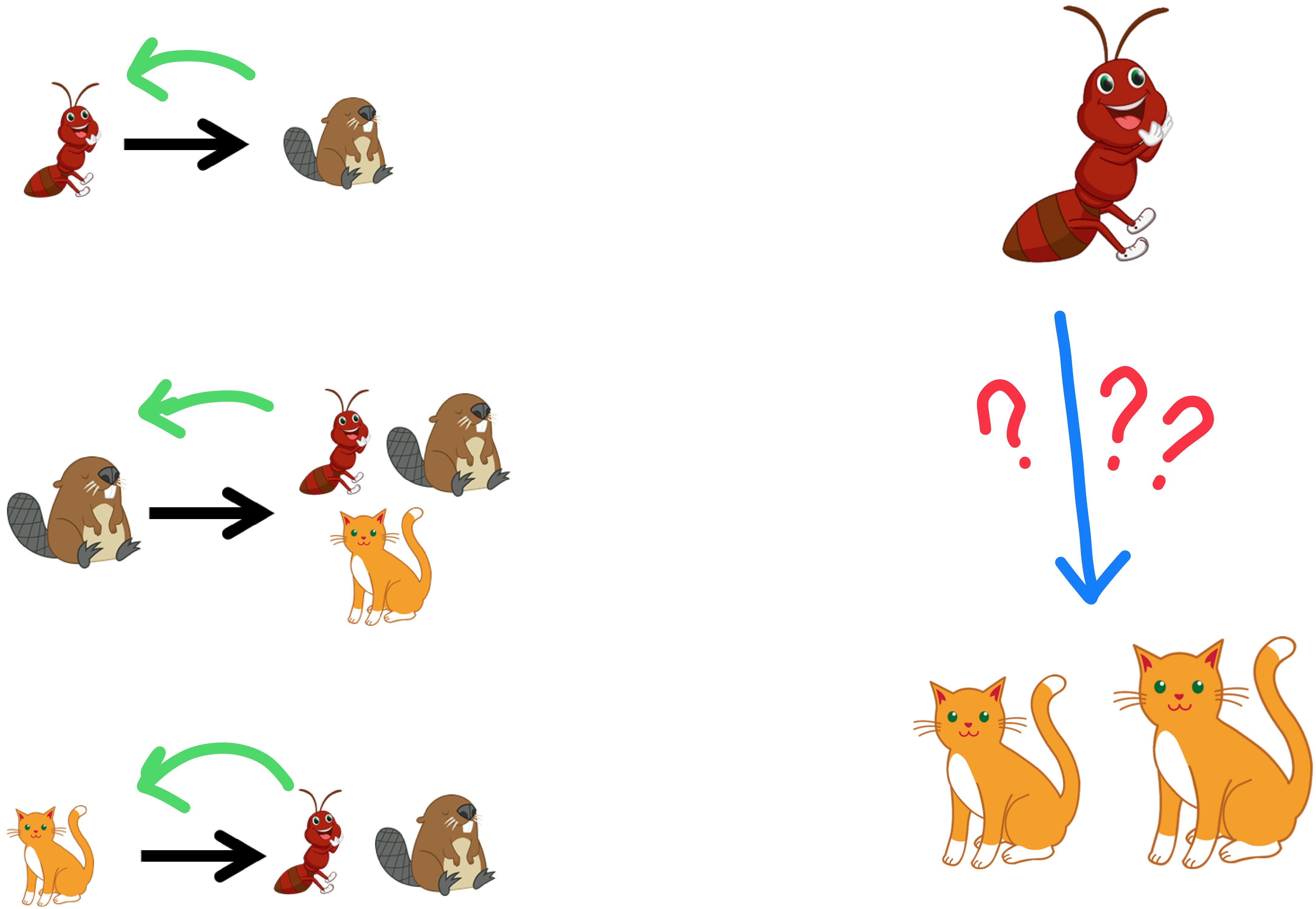

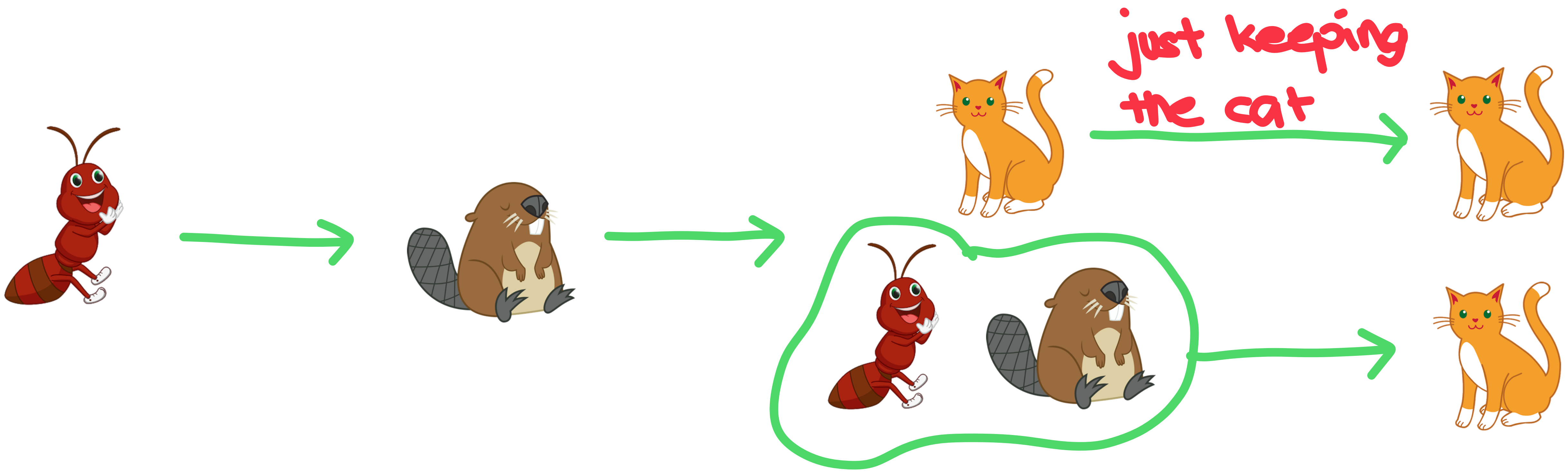
just keeping the cat

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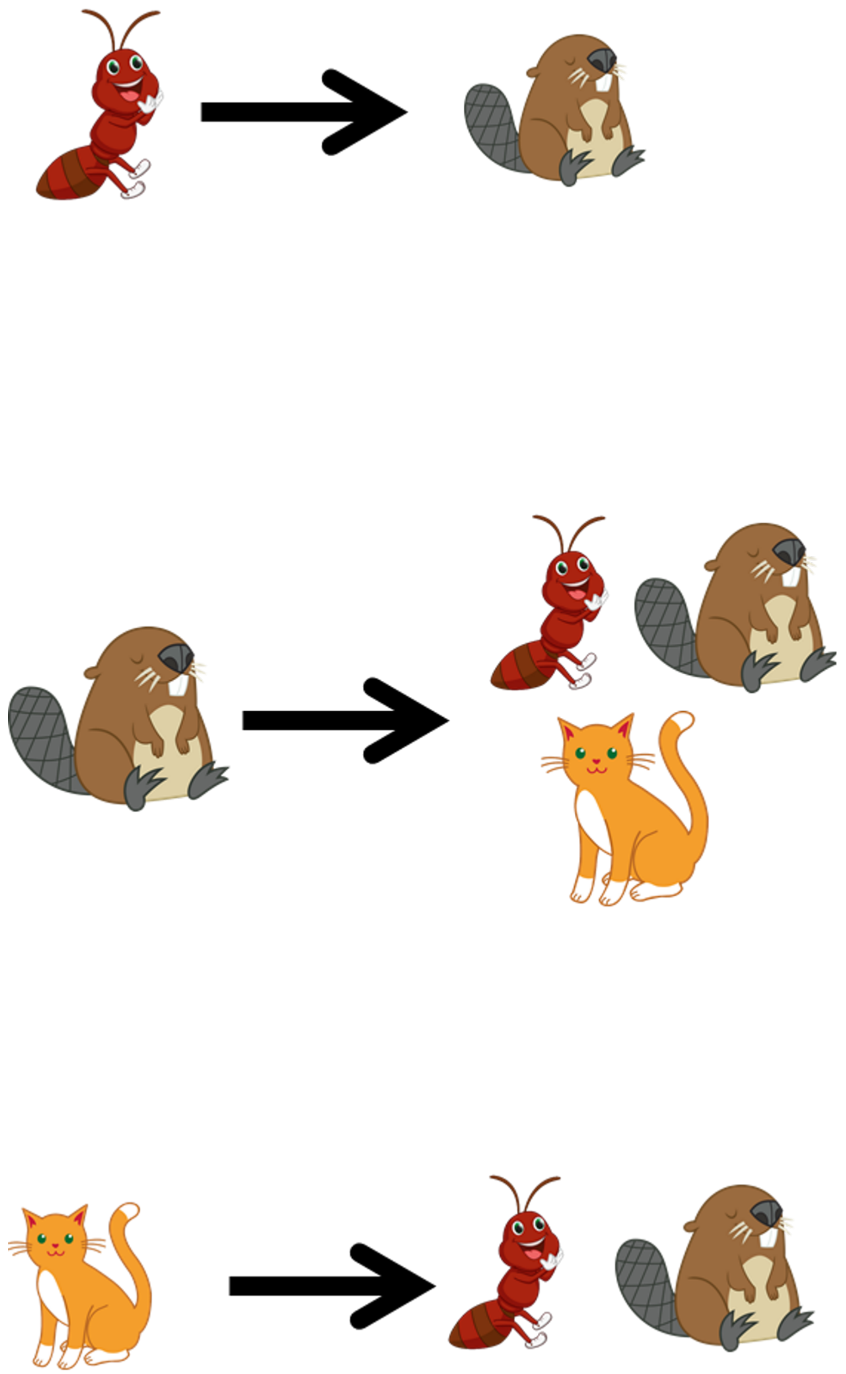

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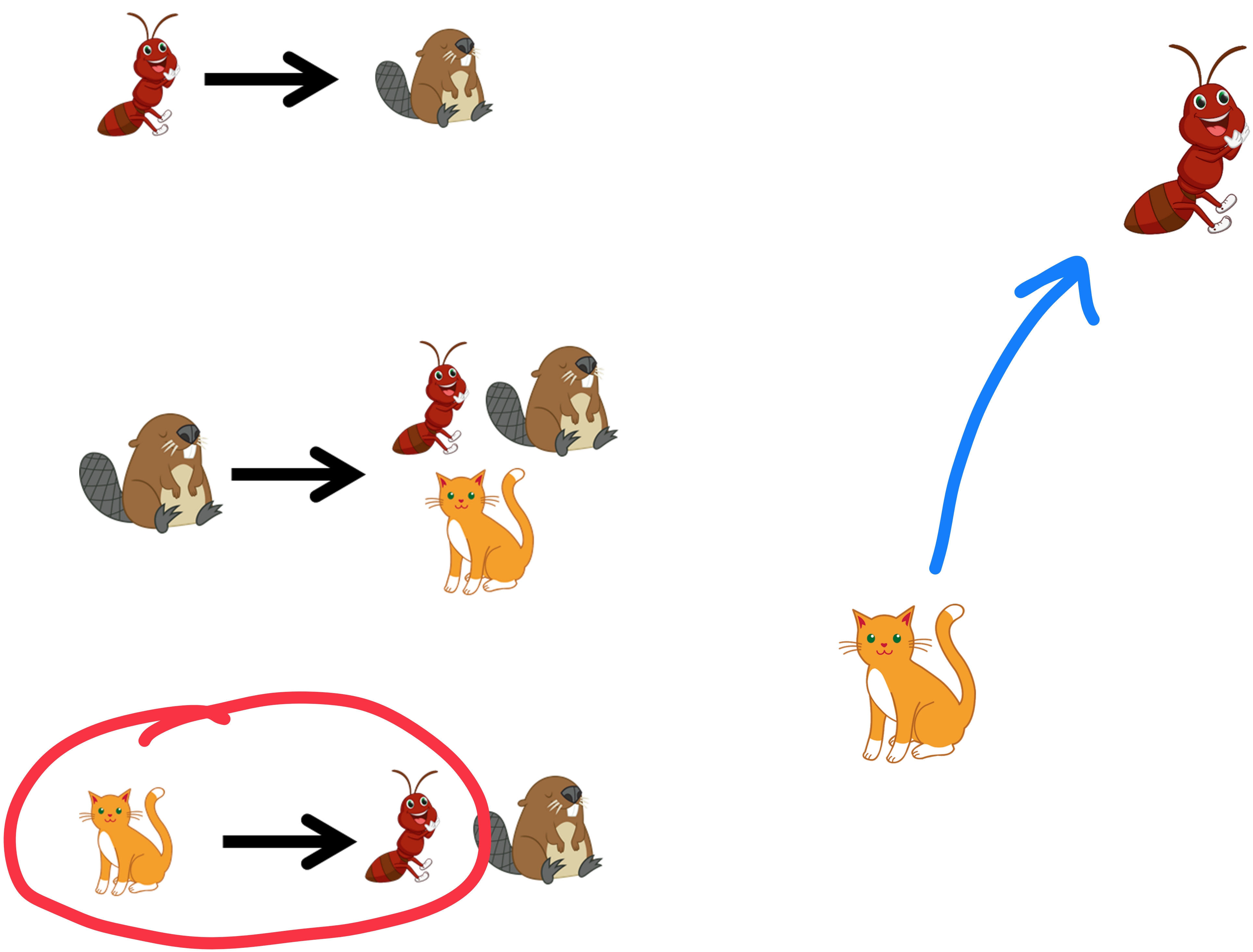

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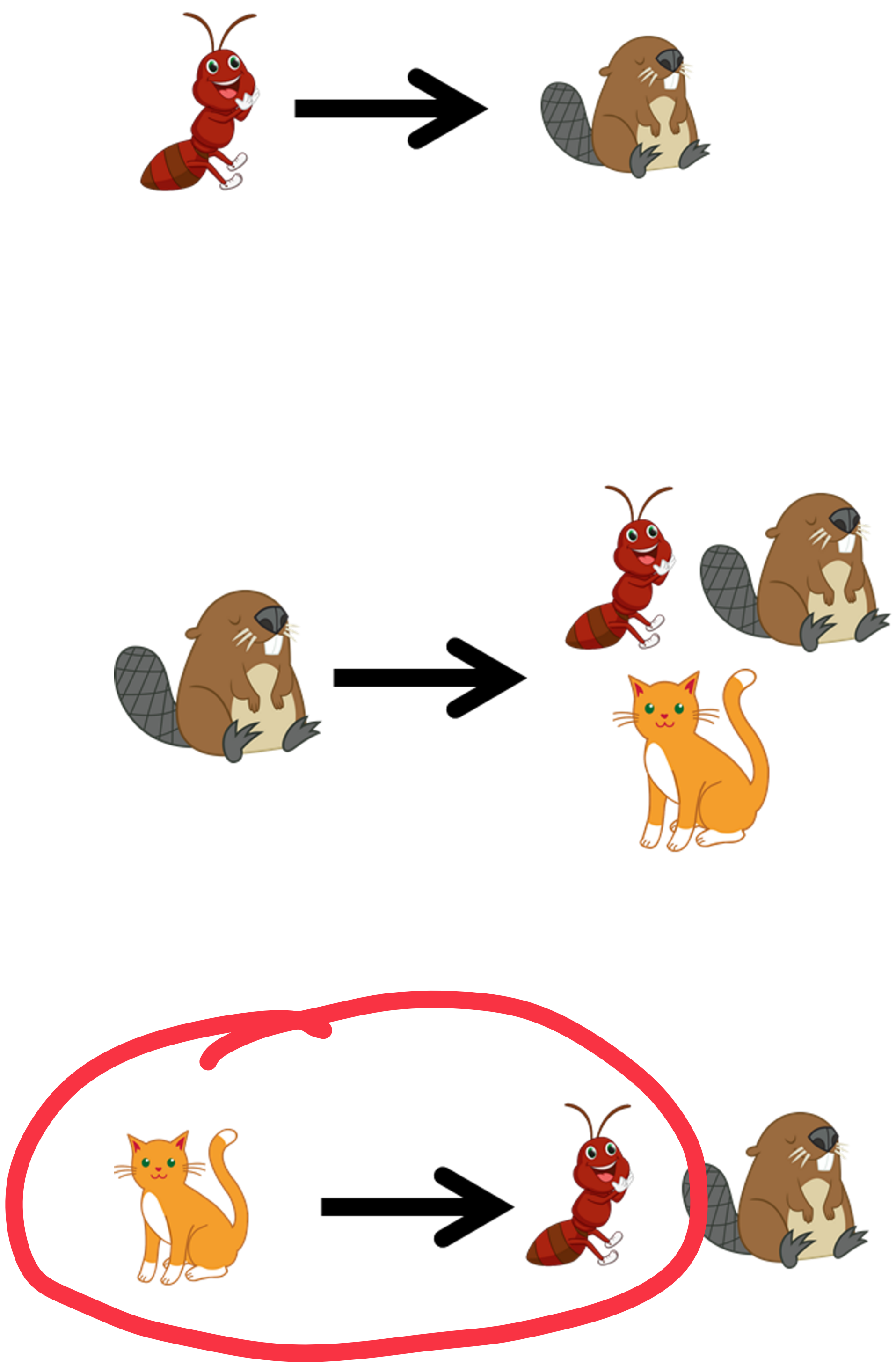
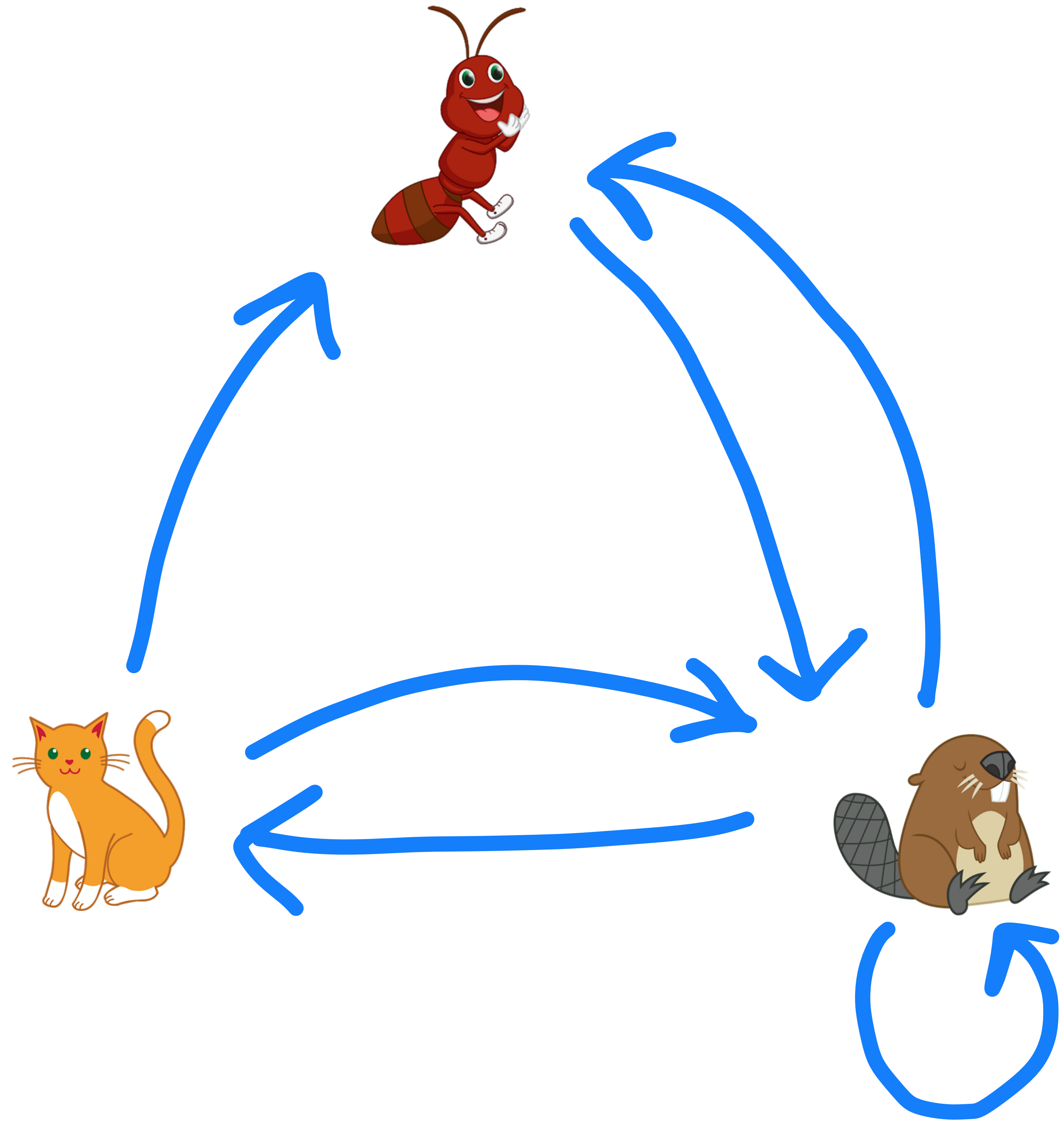
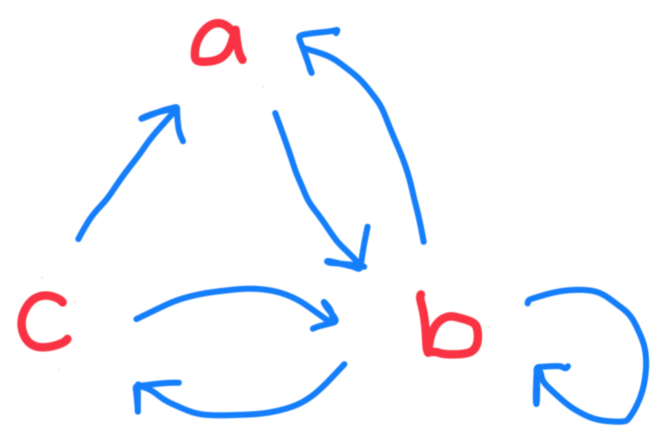




Graph Monoid

Abrams, Sklar (2010)



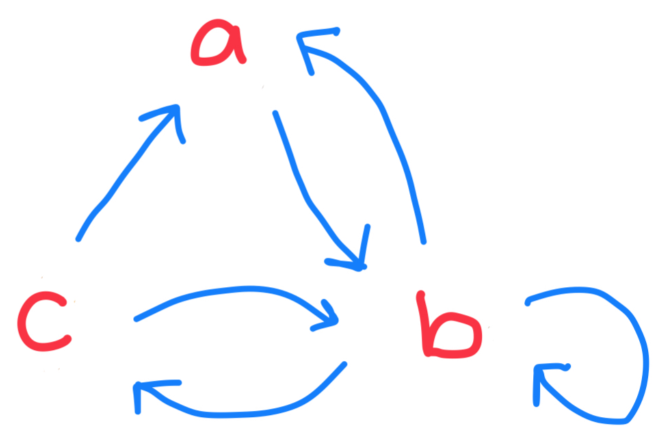
$$a = b$$

$$b = a + b + c$$

$$c = a + b$$

Graph Monoid

Abrams, Sklar (2010)

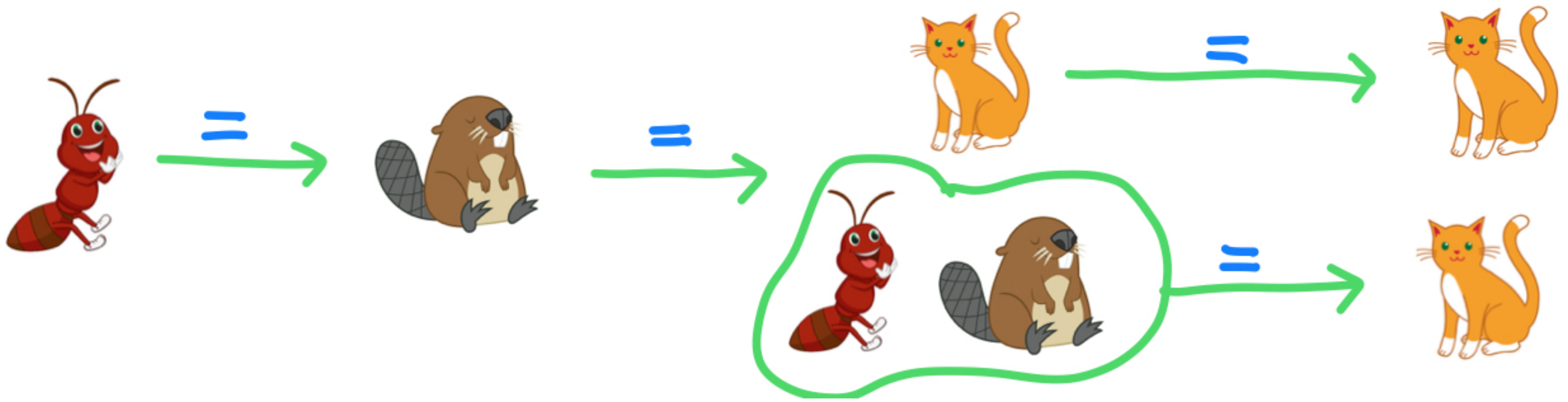


$$a = b$$
$$b = a + b + c$$
$$c = a + b$$

Let E be a row-finite directed graph. The *graph monoid* of E, denoted by M_E, is the abelian monoid generated by $\{v : v \in E^0\}$, subject to

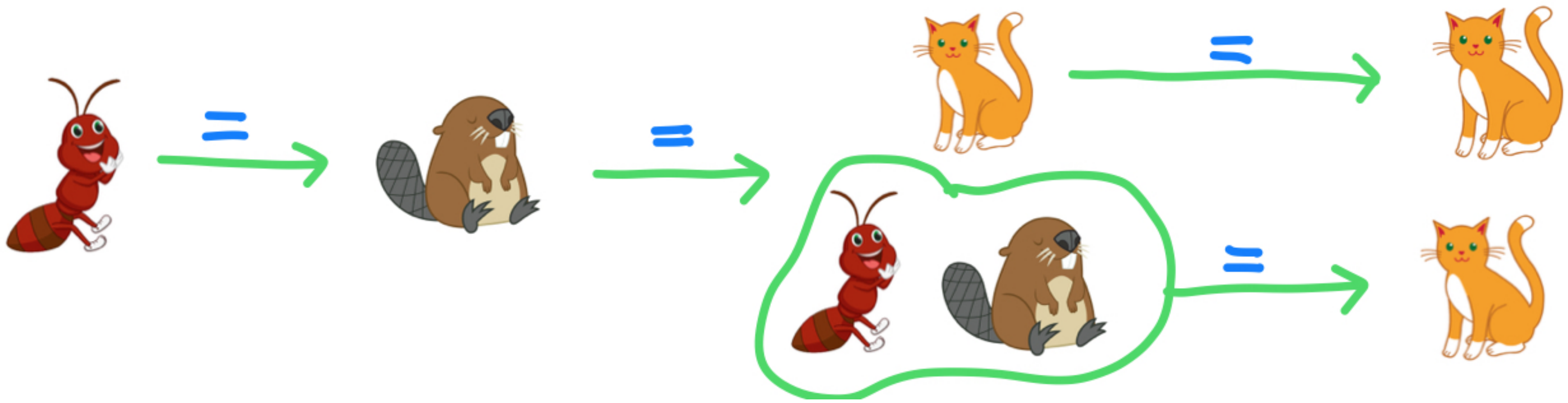
$$v = \sum_{e \in s^{-1}(v)} r(e)$$

for every $v \in E^0$ that is not a sink.



In M_E :

$$a = b = a + b + c = c + c = 2c$$



In M_E :

$$a = b = a + b + c = c + c = 2c$$

$$M_E \quad = \quad \mathcal{V}(L_K(E))$$

– monoid of finitely generated projective module of $L_K(E)$

[Ara, Moreno, Pardo (2007)]

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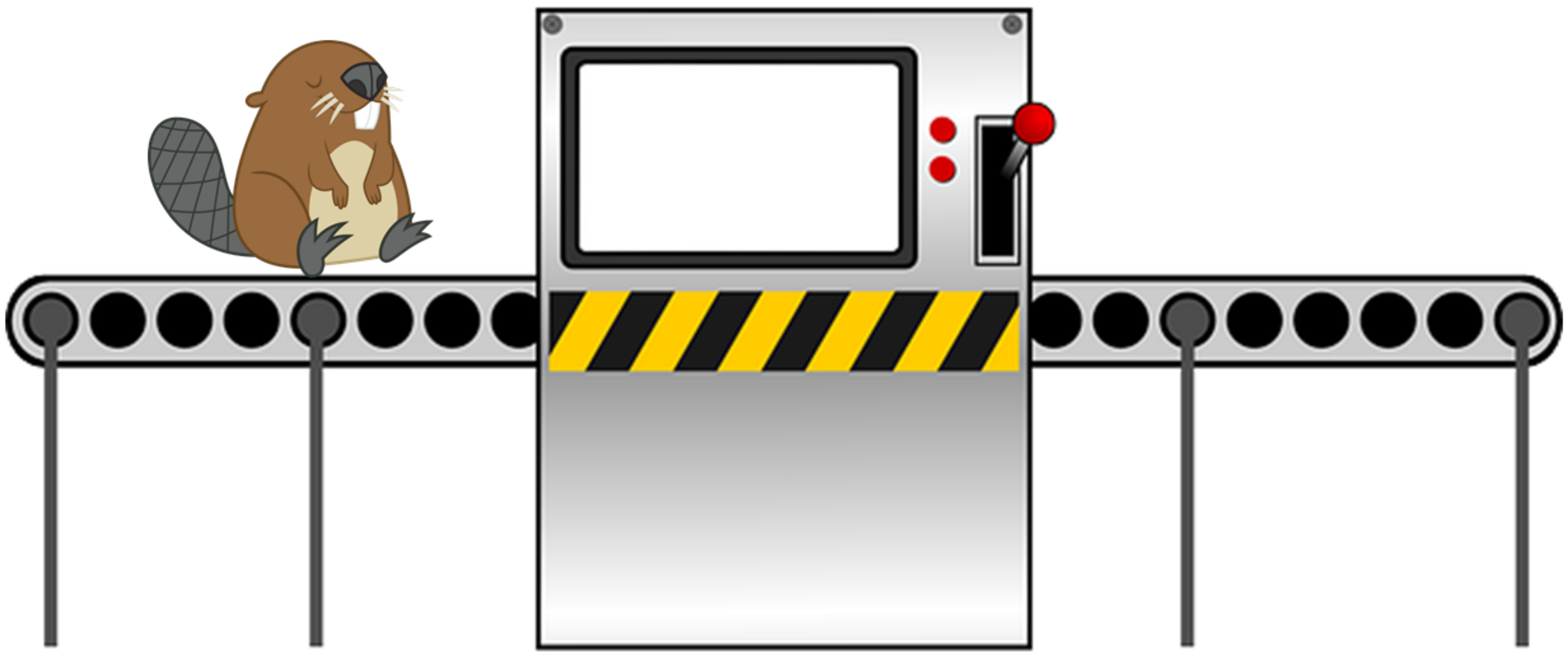
age n



Hazrat and Li (2019) modified the puzzle as follows:

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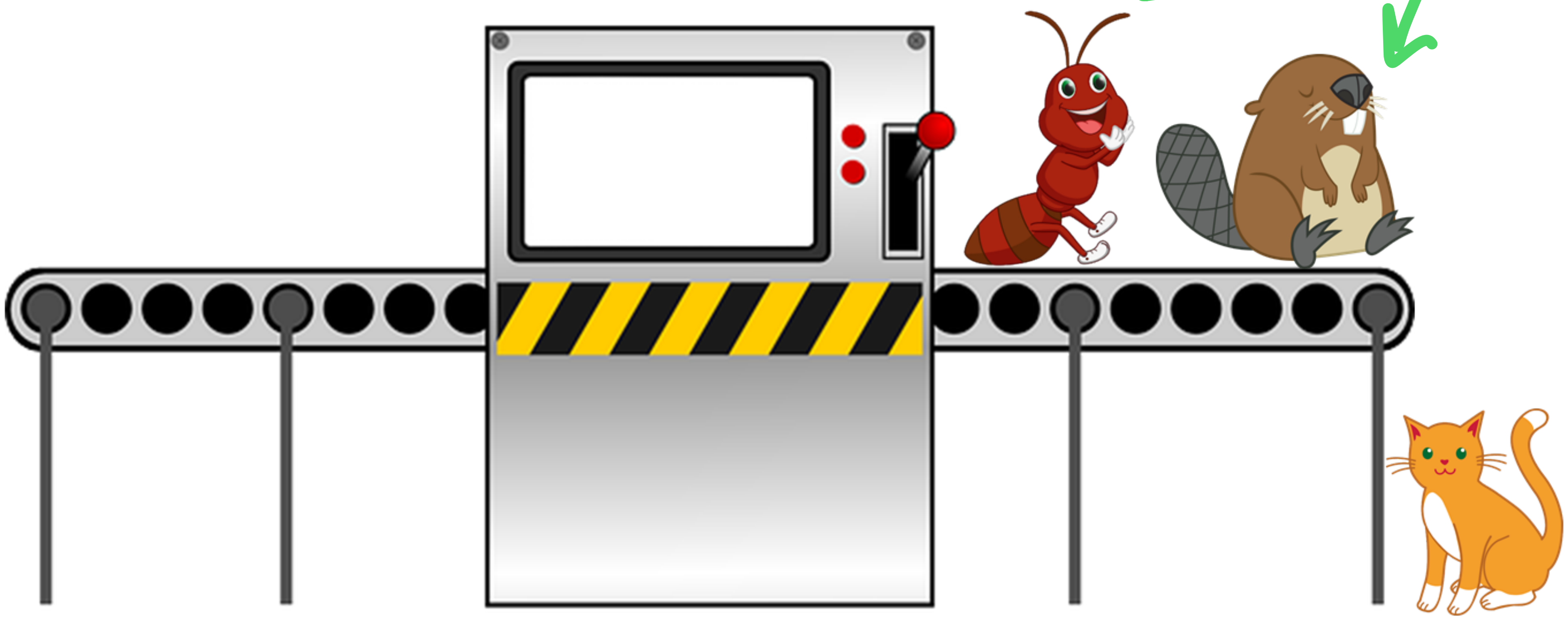
age $n+1$



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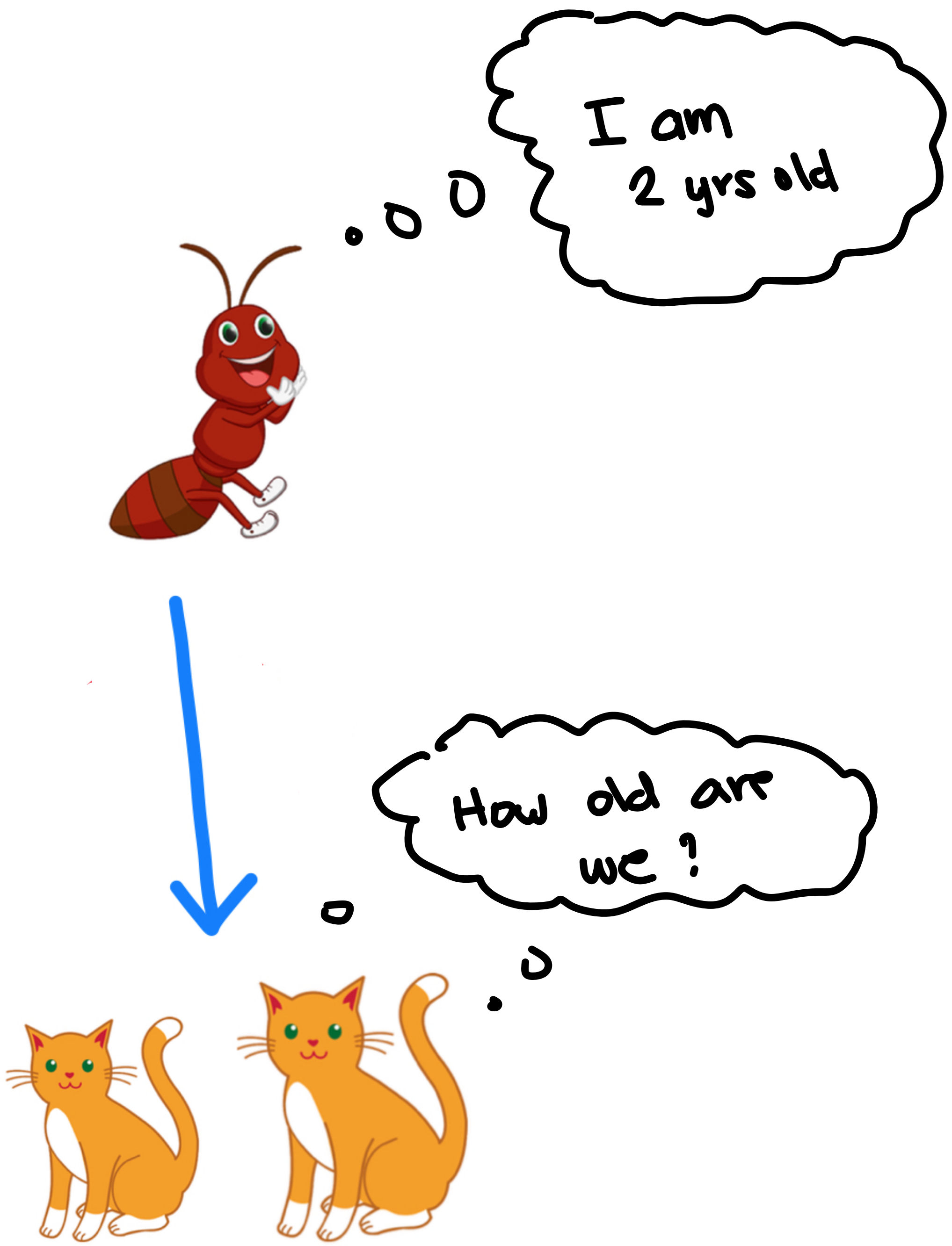

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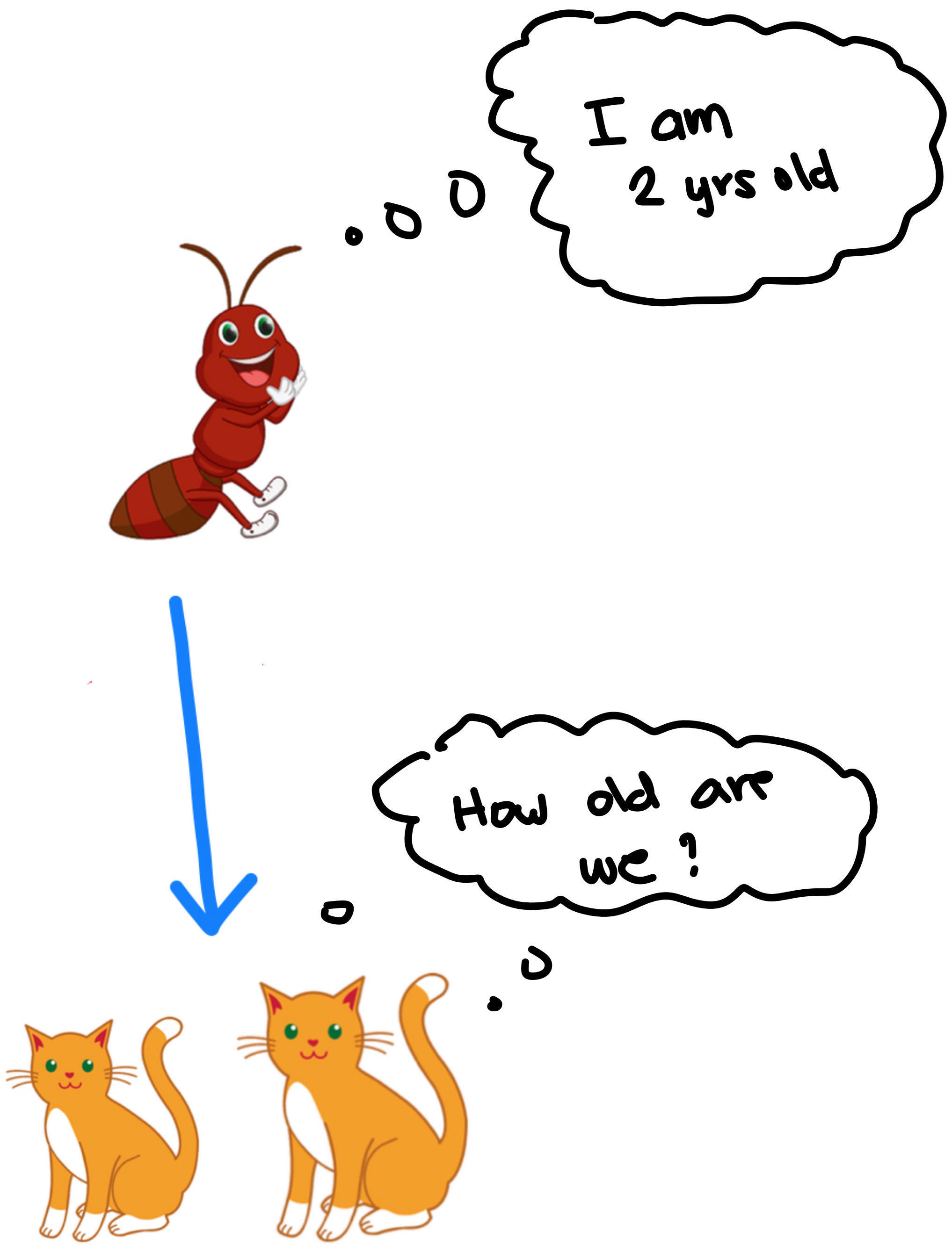
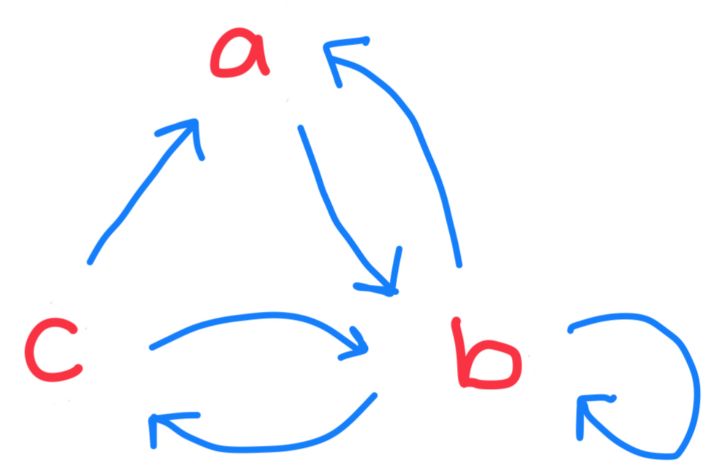


$$
\begin{aligned}
a(2) &= b(3) \\
&= a(4) + b(4) + c(4) \\
&= c(3) + c(4)
\end{aligned}
$$

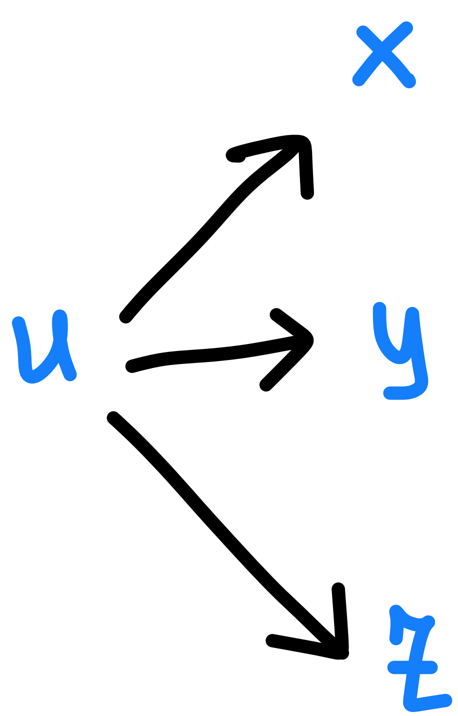


Talented Monoid

Let E be a row-finite graph. The *talented monoid* of E, denoted by T_E, is the abelian monoid generated by $\{v(i) : v \in E^0, i \in \mathbb{Z}\}$, subject to

$$v(i) = \sum_{e \in s^{-1}(v)} r(e)(i+1)$$

for every $i \in \mathbb{Z}$ and every $v \in E^0$ that is not a sink.



$$u(i) = x(i+1) + y(i+1) + z(i+1)$$

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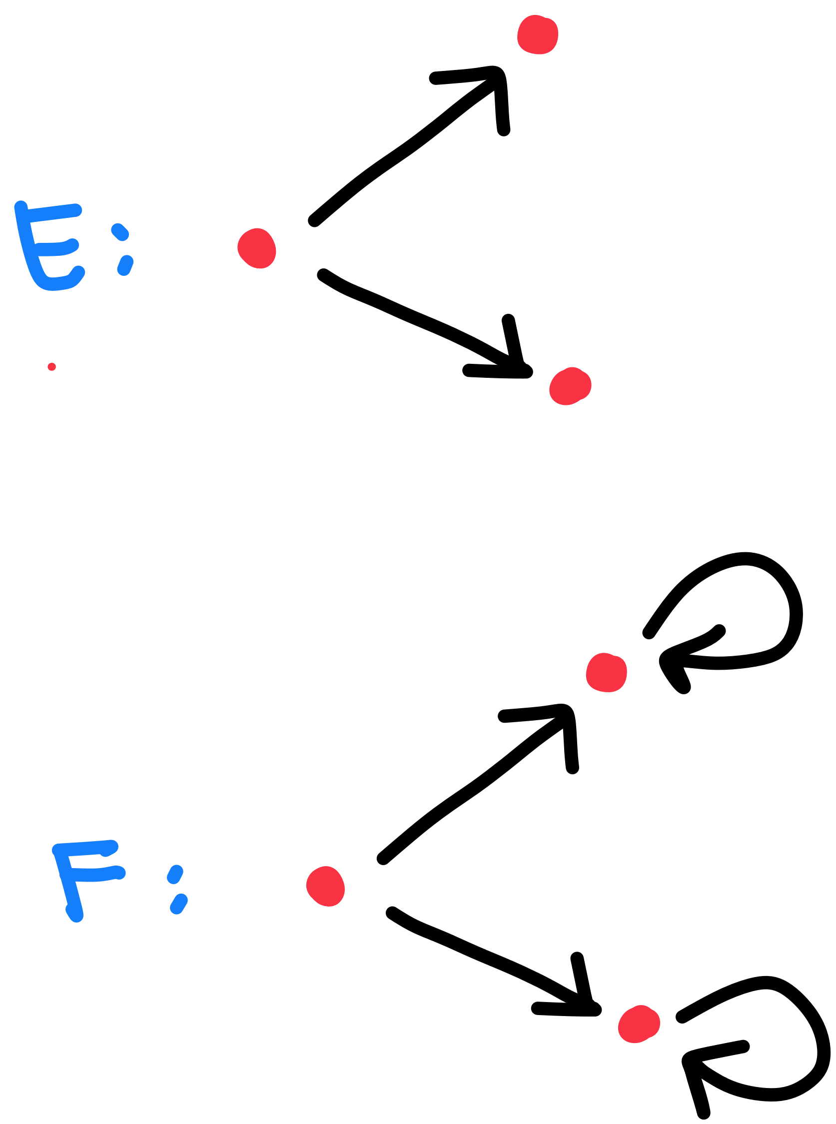
$$\begin{aligned} T_E &= \mathcal{V}^{gr}(L_K(E)) \\ &- \text{ monoid of graded finitely generated} \\ &\quad \text{ projective module of } L_K(E) \end{aligned}$$

[Ara, Hazrat, Li, 2018]

M_E VS T_E



$$M_E \cong M_F$$

$$T_E \ncong T_F$$

Leavitt path algebra

$L_K(E)$

Directed graph

E

Talented monoid

T_E

Adjacency Matrix

$\mathrm{Adj}(E)$

Leavitt path algebra

$L_K(E)$

Directed graph

E

Hereditary Saturated Set

H

Talented monoid

T_E

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Leavitt path algebra

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Graded Ideal

$I(H)$

[Abrams, Ara, Molina (2017)]

Directed graph

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Directed graph

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Hereditary Saturated Set

H

Talented monoid

T_E

$\mathbb{Z}$-order-ideal

$\langle H \rangle$

[Ara, Hazrat, Li, Sims (2018)]

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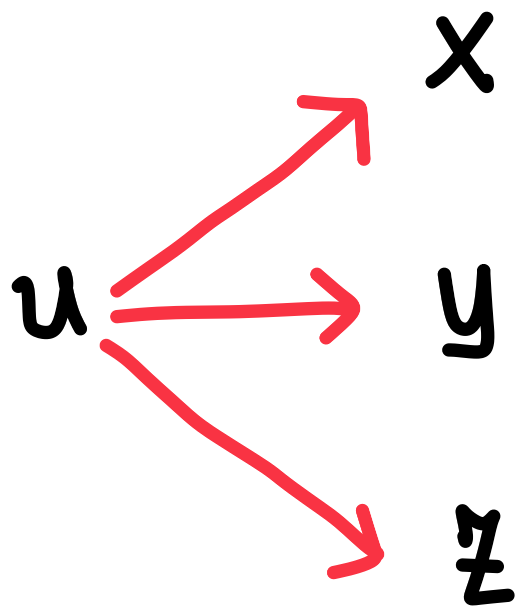
[Bock, Sebandal (2022)]



Hereditary

$$u \in H \implies x, y, z \in H$$

Saturated

$$x, y, z \in H \implies u \in H$$

Directed graph

E

Hereditary Saturated Set

H

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[Abrams, Ara, Molina (2017)]

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$\langle H \rangle$

[Ara, Hazrat, Li, Sims (2018)]

Adjacency Matrix

$\mathrm{Adj}(E)$

Hereditary Saturated submatrix

[Bock, Sebandal (2022)]

$\mathbb{Z}$ acts on T_E:

$${}^n v(i) = v(i+n) \qquad \Rightarrow \qquad T_E \text{ is a } \mathbb{Z}\text{-monoid}$$

Example

A *$\mathbb{Z}$-order ideal* of T_E is a subset I of T_E such that for $\alpha, \beta \in \mathbb{Z}$,

$${}^{\alpha}a + {}^{\beta}b \in I \quad \Longleftrightarrow \quad a, b \in I.$$

Directed graph

E

Hereditary Saturated Set

H

Leavitt path algebra

$L_K(E)$

Graded Ideal

$I(H)$

[Abrams, Ara, Molina (2017)]

Talented monoid

T_E

$\mathbb{Z}$-order-ideal

$\langle H \rangle$

[Ara, Hazrat, Li, Sims (2018)]

Adjacency Matrix

$\mathrm{Adj}(E)$

Hereditary Saturated submatrix

[Bock, Sebandal (2022)]

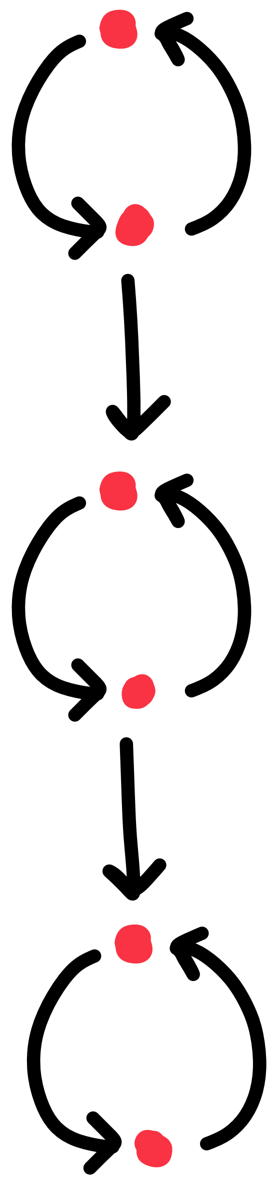
Theorem

Let E be an arbitrary graph with countably many vertices and $\varnothing \neq H \subset E^0$. Then H is a hereditary saturated subset of E^0 if and only if up to a permutation on E^0, the adjacency matrix of E could be written of the form

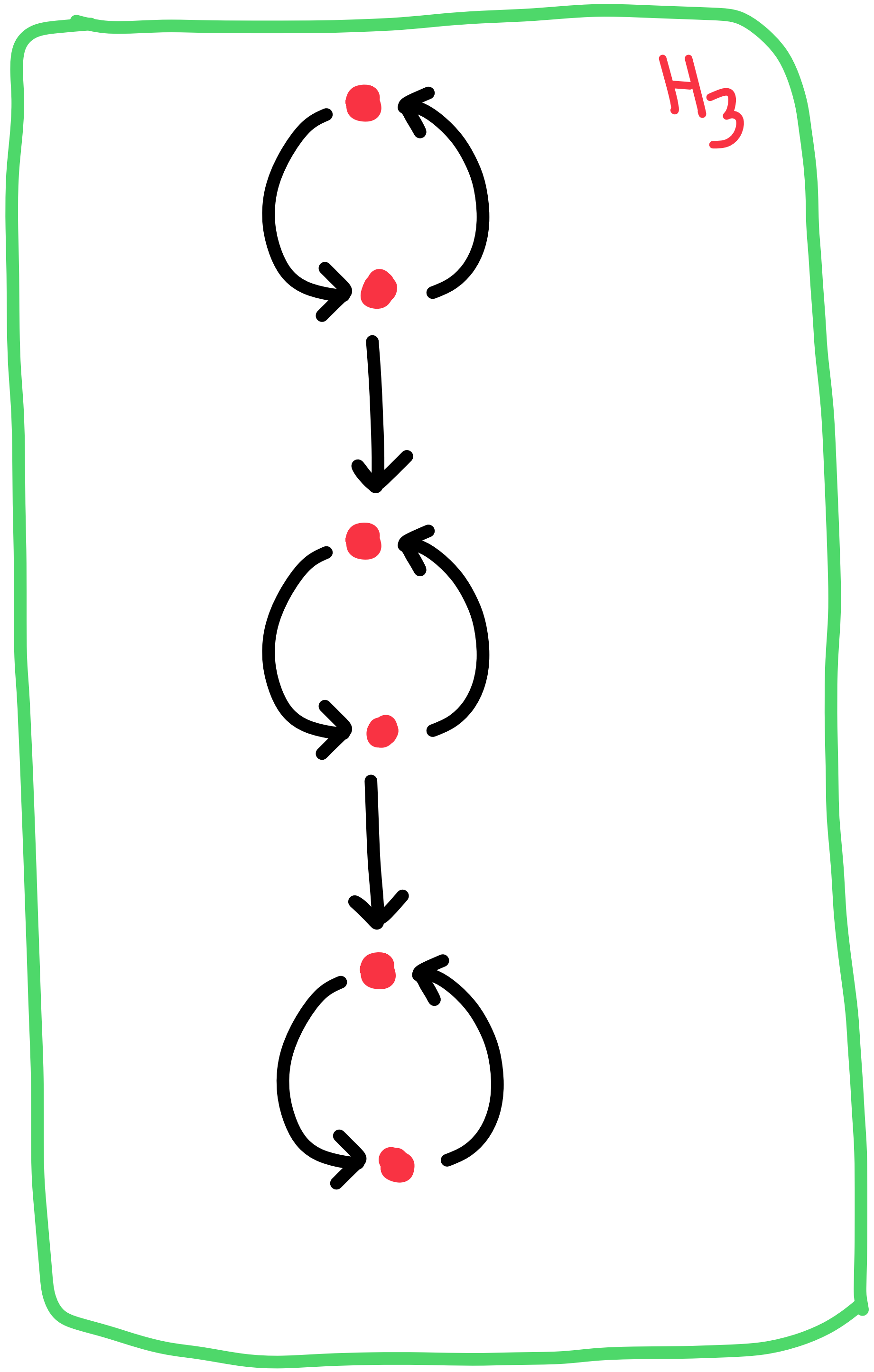
$$\mathrm{Adj}(E) = \begin{pmatrix} \mathrm{Adj}(H) & 0 \\ A & B \end{pmatrix},$$

where for each i, $A_{i,s} = 0$ for all s if $B_{i,t} = 0$ for all t.

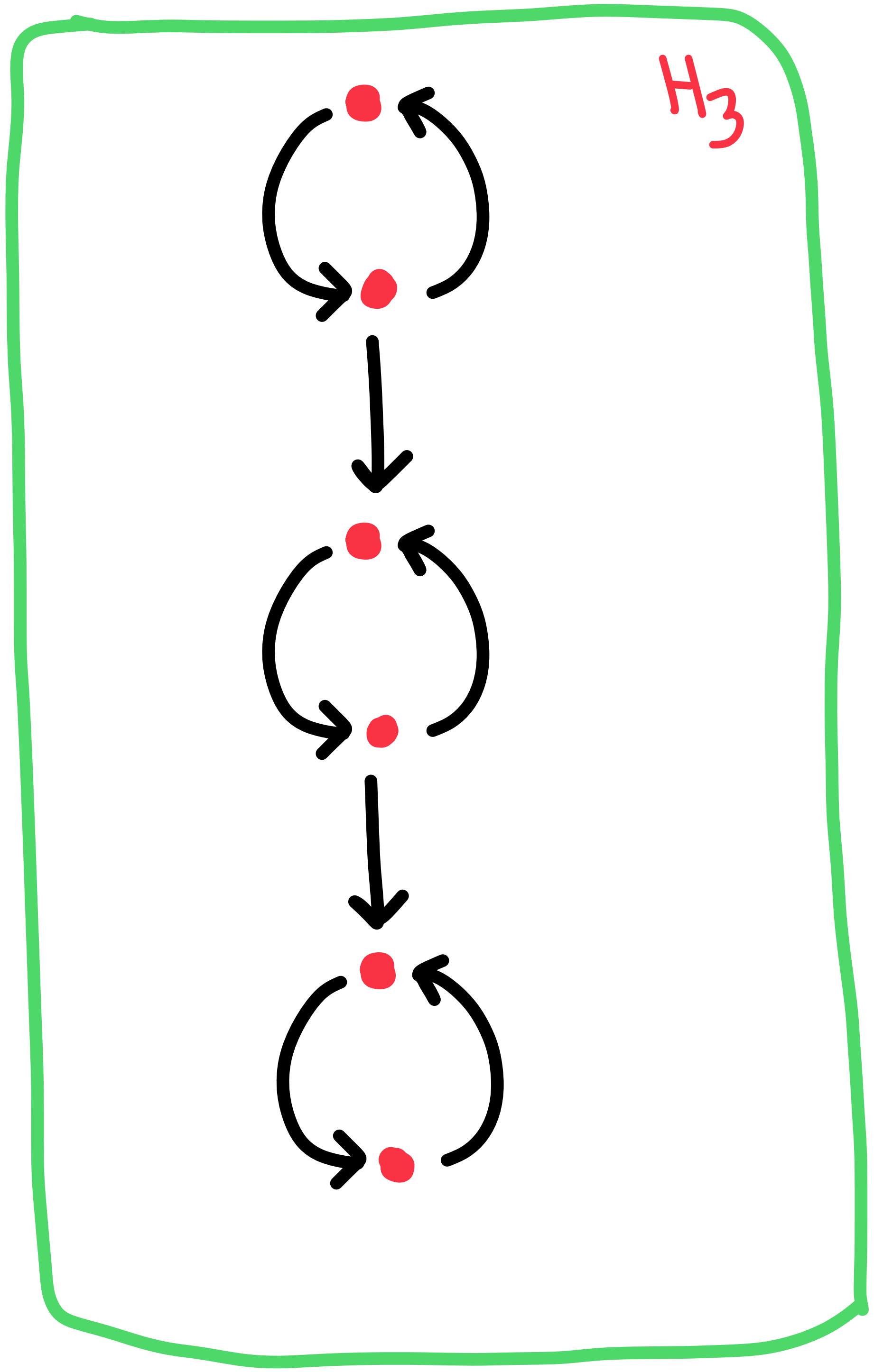
E:



E :



E:

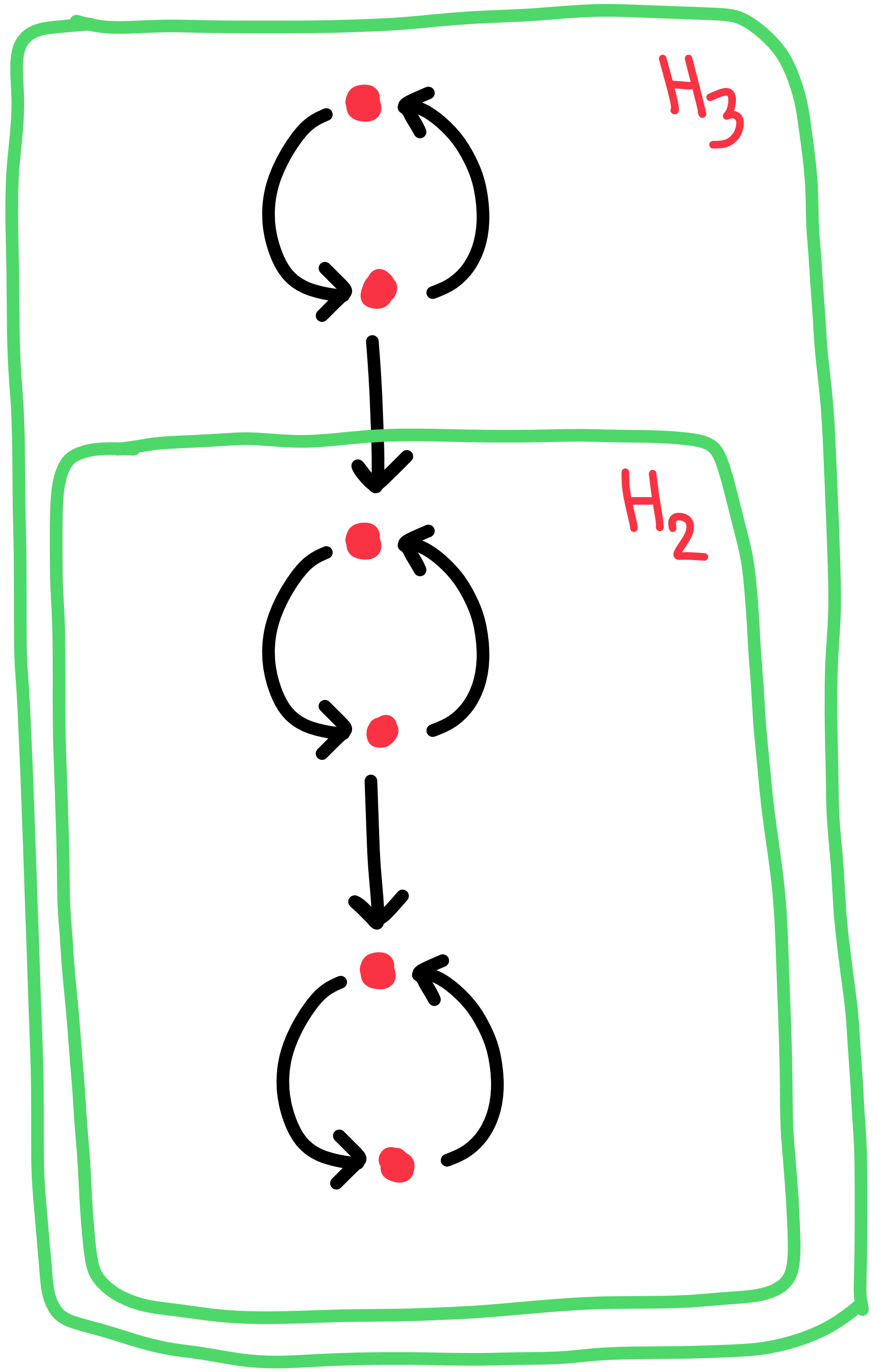


$$E^0$$

$$T_E$$

$$\mathrm{Adj}(E) = \begin{pmatrix} \quad \end{pmatrix}$$

E :

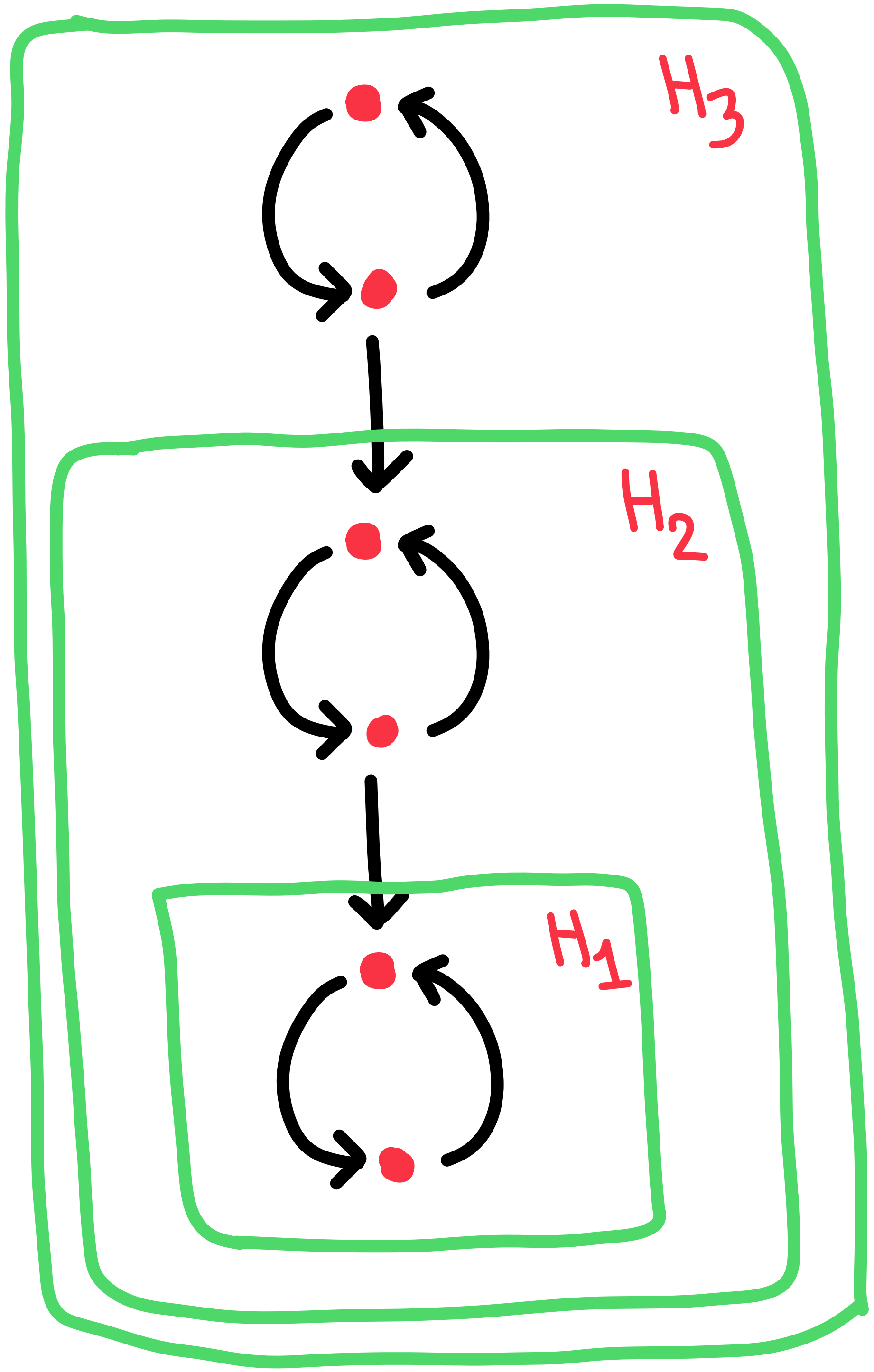


$$H_2 \subset E^0$$

$$\langle H_2 \rangle \subset T_E$$

$$\mathrm{Adj}(E) = \begin{pmatrix} \mathrm{Adj}(H_2) & 0 \\ A_3 & B_3 \end{pmatrix}$$

E:



$$H_1 \subset H_2 \subset E^0$$

$$\langle H_1 \rangle \subset \langle H_2 \rangle \subset T_E$$

$$\mathrm{Adj}(E) = \begin{pmatrix} \begin{pmatrix} \mathrm{Adj}(H_1) & 0 \\ A_2 & B_2 \end{pmatrix} & 0 \\ A_3 & B_3 \end{pmatrix}$$

sequence of hereditary saturated subsets of E

$$\varnothing \neq H_1 \subset H_2 \subset \cdots \subset H_n \subset E^0$$

chain of $\mathbb{Z}$-order ideals of T_E

$$0 \subset \langle H_1 \rangle \subset \langle H_2 \rangle \subset \langle H_3 \rangle \subset \cdots \subset \langle H_n \rangle \subset T_E.$$

chain of hereditary saturated submatrices of $\mathrm{Adj}(E)$

$$\mathrm{Adj}(E) = \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} \mathrm{Adj}(H_1) & 0 \\ A_2 & B_2 \end{pmatrix} & 0 \\ A_3 & B_3 \end{pmatrix} & 0 \\ A_4 & B_4 \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \\ A_n & & B_n \end{pmatrix} & 0 \\ A_{n+1} & B_{n+1} \end{pmatrix}$$

$$0 \subset_{fp} \mathrm{Adj}(H_1) \subset_{fp} \mathrm{Adj}(H_2) \subset_{fp} \mathrm{Adj}(H_3) \subset_{fp} \cdots \subset_{fp} \mathrm{Adj}(H_n) \subset_{fp} \mathrm{Adj}(E)$$

A *$\mathbb{Z}$-series* for T_E is a sequence of $\mathbb{Z}$-order-ideals

$$0 = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = T. \qquad (*)$$

Furthermore, we say $(*)$ is a *$\mathbb{Z}$-composition series* if for each $i = 0, 1 \cdots, n-1$, $I_i \subsetneq I_{i+1}$ and each of quotients I_{i+1}/I_i are simple $\mathbb{Z}$-monoids.

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Jordan-Hölder Theorem

Two Γ-series of a refinement Γ-monoid T have equivalent refinement. Thus, any Γ-composition series are equivalent and a Γ-monoid having a composition series determines a unique list of simple Γ-monoids.

A *$\mathbb{Z}$-series* for T_E is a sequence of $\mathbb{Z}$-order-ideals

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chain of $\mathbb{Z}$-order ideals of T_E

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chain of hereditary saturated submatrices of $\mathrm{Adj}(E)$

$$\mathrm{Adj}(E) = \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} \mathrm{Adj}(H_1) & 0 \\ A_2 & B_2 \end{pmatrix} & 0 \\ A_3 & B_3 \end{pmatrix} & 0 \\ A_4 & B_4 \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \\ A_n & & B_n \end{pmatrix} & 0 \\ A_{n+1} & B_{n+1} \end{pmatrix}$$

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Where else can we use the notion of adjacency matrices?

Proposition

If E is a finite graph with $E^0 = \{v_1, v_2, \ldots, v_n\}$ and where the vertices $v_m, v_{m+1}, \ldots, v_n$ are the sinks in E, and $\mathrm{Adj}(E)$ its adjacency matrix. Then for all $k \in \mathbb{N}$,

$$\begin{pmatrix} v_1(0) \\ \vdots \\ v_n(0) \end{pmatrix} = \mathrm{Adj}(E)^k \begin{pmatrix} v_1(k) \\ \vdots \\ v_n(k) \end{pmatrix} + \sum_{l=1}^{k-1} \mathrm{Adj}(E)^l B_l$$

where B_l is an $n \times 1$ matrix with $(B_l)_{i,1} = v_i(l)$ for all $i = m, \cdots, n$ and $(B_m)_{i,1} = 0$ for $i < m$. Notice that under these computations, $v_i(0) = 0$ if and only if v_i is a sink.

Where else can we use the notion of adjacency matrices?

Theorem

Let E be a finite graph A its adjacency matrix. Then for each $k > 1$, the number linearly independent elements of $L_K(E)$ of the form $\alpha\beta^*$ where $l(\alpha) + l(\beta) = k$ is

$$p_k(E) = \sum_{s+t=k} \left(\sum_{j=1}^{n} \left(\|A^s\|_j^c \right) \left(\|A^t\|_j^c \right) \right) - \sum_{\substack{s+t=k \\ s,t>0}} \left(\sum_{\|A\|_j^r=1} \left(\|A^{s-1}\|_j^c \right) \left(\|A^{t-1}\|_j^c \right) \right).$$

Where else can we use the notion of adjacency matrices?

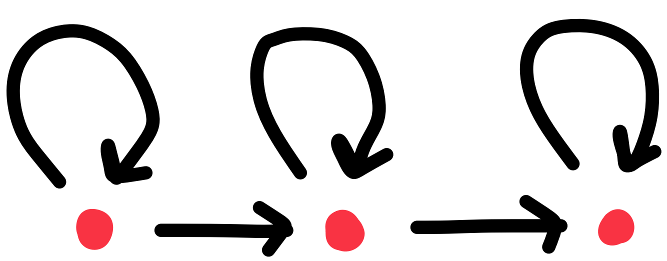
Let A be an algebra (not necessarily unital), which is generated by a finite dimensional subspace V. Let V^n denote the span of all products $v_1 v_2 \cdots v_n$, $v_i \in V$, $k \leq n$. Then $V = V^1 \subseteq V^2 \subseteq \cdots$,

$$A = \bigcup_{n \geq 1} V^n \quad \text{and} \quad g_{V(n)} = \dim V^n < \infty.$$

If $g_{(V(n))}$ is polynomially bounded, then the *Gelfand-Kirillov dimension* of A is defined as

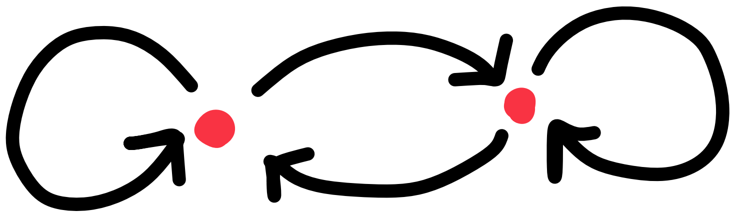
$$\mathrm{GKdim} A = \limsup_{n \to \infty} \frac{\ln g_{V(n)}}{\ln n}.$$

E:



$$\mathrm{GKdim} L_K(E) = 5$$

F :



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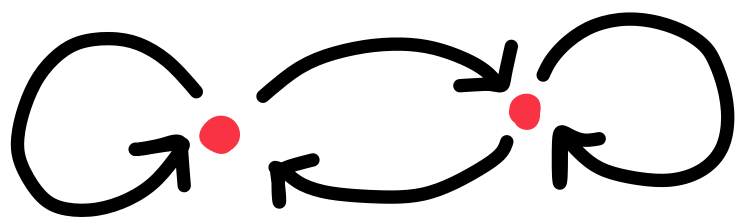
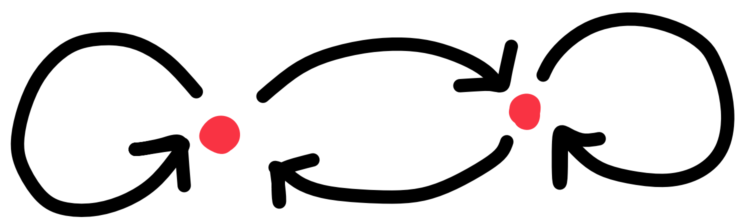


We define the *algebraic entropy of a filtered algebra* $(A, \mathcal{F})$ where $\mathcal{F} = \{V_n\}$, by

$$h(A,\mathcal{F}) := \begin{cases} 0 & \text{if A is finite dimensional,} \\ \limsup\limits_{n\to\infty} \dfrac{\log \dim(V_n/V_{n-1})}{n} & \text{otherwise.} \end{cases}$$

For $L_K(E)$ we define its *standard filtration* $\{W_i\}_{i\in\mathbb{N}}$ so that W_0 is the linear span of the set of vertices of E, being W_1 the sum of W_0 plus the linear span of the set $E^1 \cup (E^1)^*$. For W_k we take the linear span of the set of elements: $\lambda\mu^*$ with $l(\lambda) + l(\mu) \leq k$.

F : 

$$h(L_K(E)) = \log(2)$$

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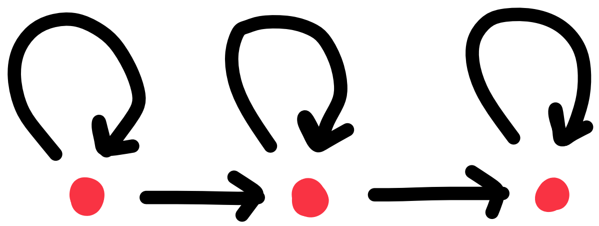
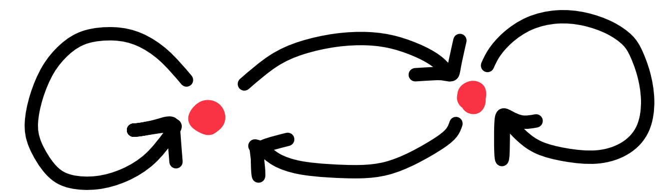
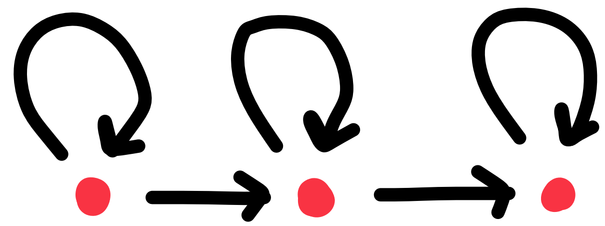
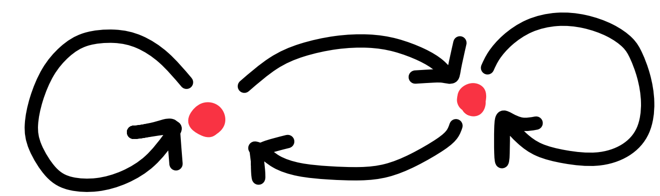
E: 

F: 

GKdim	5	∞
Entropy	0	$\log(2)$

E: 

F: 

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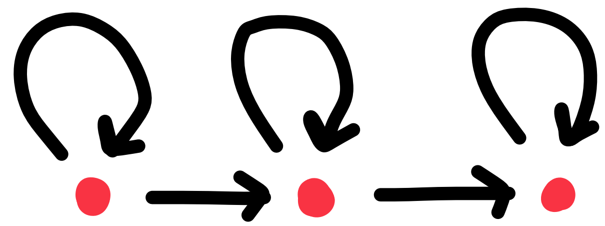
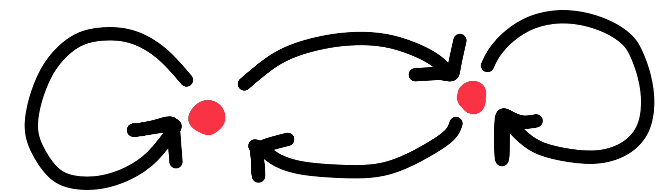
$$T_E \cong T_F \Rightarrow \mathrm{GKdim}(L_k(E)) = \mathrm{GKdim}(L_k(F))$$

E: 

F: 

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$$T_E \cong T_F \Rightarrow \mathrm{GKdim}(L_k(E)) = \mathrm{GKdim}(L_k(F))$$

$$T_E \cong T_F \overset{?}{\Rightarrow} h(L_k(E)) = h(L_k(F))$$

Where else can we use the notion of adjacency matrices?

Theorem

Let E be a finite graph, A its corresponding adjacency matrix and fix $k \in \mathbb{N}$. In $L_K(E)$, $\dim(V_k/V_{k-1})$ is equal to:

$$p'_k(E) = \sum_{s+t=k} \left(\sum_{j=1}^{n} \left(\|A^s\|_j^c \right) \left(\|A^t\|_j^c \right) \right) - \sum_{\substack{s+t=k \\ s,t>0}} \left(\sum_{j=1}^{n} \left(\|A^{s-1}\|_j^c \right) \left(\|A^{t-1}\|_j^c \right) \right).$$

Therefore,

$$h(L_K(E)) = \limsup_{n\to\infty} \frac{\log p'_n(E)}{n}$$

Leavitt path algebra

$L_K(E)$

Lie Bracket algebra

$[L_K(E), L_K(E)]$

Directed graph

E

Talented monoid

T_E

Adjacency Matrix

$\mathrm{Adj}(E)$

Balloon

We call a vertex v in a connected graph E a *balloon* over a nonempty set $W \subseteq E^0$ if

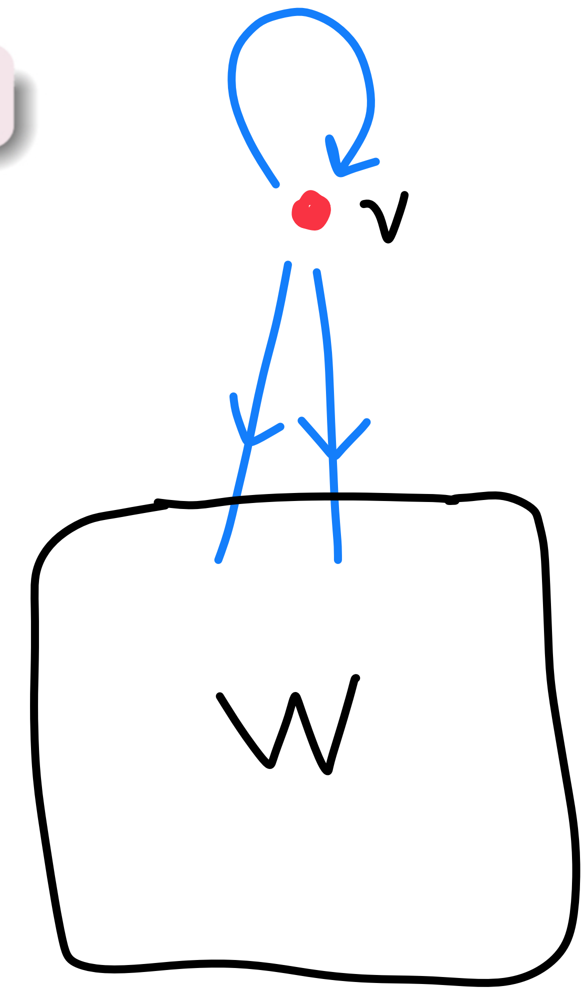
(i) $v \notin W$

(ii) there is a loop $C \in E(v, v)$

(iii) $E(v, W) \neq \emptyset$

(iv) $E(v, E^0) = \{C\} \cup E(v, W)$

(v) $E(E^0, v) = \{C\}$.



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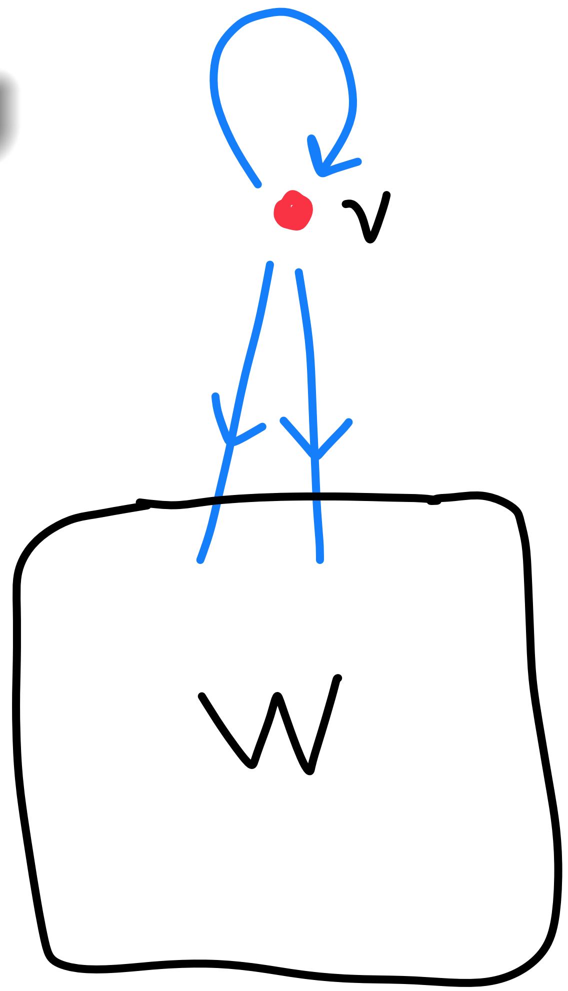

Theorem

Let E be a connected graph and $W \subseteq E^0$. A vertex $v \notin W$ is a balloon over W if and only if

(i) $\langle E \setminus \{v\} \rangle$ is the maximal $\mathbb{Z}$-order-ideal of T_E which does not contain v;

(ii) $r(s^{-1}(v)) \setminus W = \{v\}$;

(iii) $T_{E/H}$ is simple cyclic.

Theorem

Let E be connected row-finite graph with $L_K(E)$ not simple.

$$[L_K(E), L_K(E)] \text{ is simple}$$

$$\Updownarrow$$

for every vertex $v \notin I$ for some $\mathbb{Z}$-order-ideal I,
Theorem ♥ (i)-(iii) are satisfied and

$$\sum_{w \in r(E(v,W))} w \in [L_K(W), L_K(W)]$$

where $W = E^o \cap J$, J the minimal non-cyclic $\mathbb{Z}$-order-ideal of T_E.

Theorem

Let E be a finite graph and T_E its talented monoid. Then the following are equivalent:

(i) $[L_K(E), L_K(E)]$ is simple and T_E is simple.

(ii) $L_K(E)$ is simple and

$$1_{L_K(E)} = \sum_{v \in E^0} v \notin [L_K(E), L_K(E)].$$

What's on and poppin'?

Graded Classification Conjecture

For finite graphs E and F:

$$T_E \cong T_F \quad \Longleftrightarrow \quad Gr\text{-}L_K(E) \approx_{gr} Gr\text{-}L_K(F)$$

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☑ Sinkless graphs:
$\dim(M_i) < \infty$ for each i

Tenchu !

A *$\mathbb{Z}$-series* for T_E is a sequence of $\mathbb{Z}$-order-ideals

$$0 = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = T. \qquad (*)$$

Furthermore, we say $(*)$ is a *$\mathbb{Z}$-composition series* if for each $i = 0, 1 \cdots, n-1$, $I_i \subsetneq I_{i+1}$ and each of quotients I_{i+1}/I_i are simple $\mathbb{Z}$-monoids.

Jordan-Hölder Theorem

Two Γ-series of a refinement Γ-monoid T have equivalent refinement. Thus, any Γ-composition series are equivalent and a Γ-monoid having a composition series determines a unique list of simple Γ-monoids.

[Sebandal, Vilela (2021)]

$T = \{0, 1, x, y, z, s, b\}$ and an operation $(+)$ on given by

$+$	0	1	x	y	z	s	b
0	0	1	x	y	z	s	b
1	1	1	1	s	s	s	b
x	x	1	1	s	s	s	b
y	y	s	s	y	y	s	b
z	z	s	s	y	y	s	b
s	s	s	s	s	s	s	b
b	b	b	b	b	b	b	s

Non-refinement Γ-monoid where the action is trivial from a trivial group Γ

What's on and poppin'?

Graded Classification Conjecture

For finite graphs E and F:

$$T_E \cong T_F \iff Gr\text{-}L_K(E) \approx_{gr} Gr\text{-}L_K(F)$$

The Graded Classification is true for finite-dimensional case!

- $\dim\left(\bigoplus_{i \in \mathbb{Z}} M_i\right) < \infty$ ✓

- Sinkless graphs:

  $\dim(M_i) < \infty$ for each i ✓

Examples

$$E: \quad \begin{tikzcd} \bullet \arrow[loop above, "e"] \end{tikzcd}$$

$$F: \quad v_1 \xrightarrow{f_1} v_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} v_n$$

$$L_K(E) \cong K[x, x^{-1}]$$

$$v \longmapsto 1$$

$$e \longmapsto x$$

$$e^* \longmapsto x^{-1}$$

$$L_K(F) \cong M_n(K)$$

$$v_i \longmapsto e_{ii}$$

$$f_i \longmapsto e_{i,i+1}$$

$$f_i^* \longmapsto e_{i+1,i}$$

Definition 2.9. Let T be a Γ-monoid. The *upper cyclic series* of T is a chain of Γ-order-ideals

$$0 = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n,$$

where I_{i+1}/I_i is the largest cyclic ideal of T/I_i, $0 \leq i \leq n-1$. We call I_n the *leading ideal* of the series and denote n by $l_c(T)$.