Row-factorization matrices and type of almost Gorenstein monomial curves

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Preliminaries

Notation:

- $\mathcal{S}$ is a numerical semigroup (submonoid of $\mathbb{N}$ such that $\mathbb{N} \setminus \mathcal{S}$ is finite)
- $\mathcal{G} = \{g_1, \ldots, g_e\}$ are the minimal generators $\mathcal{S}$;
- $e = |\mathcal{G}|$ is the embedding dimension of $\mathcal{S}$;
- $F(\mathcal{S}) = \max \mathbb{Z} \setminus \mathcal{S}$ is the Frobenius number of $\mathcal{S}$;
- The set of pseudo-Frobenius numbers

$$PF(\mathcal{S}) = \{x \notin \mathcal{S} \mid x + g_i \in \mathcal{S} \text{ for all } i = 1, \ldots, e\}.$$

- $t(\mathcal{S}) = |PF(\mathcal{S})|$ is the type of $\mathcal{S}$.

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Main topic

Study the boundedness of $t(\mathcal{S})$ in terms of e.

The general case

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1. if $e = 2$, then $t(\mathcal{S}) = 1$.
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2. if $e = 3$, then $t(\mathcal{S}) \leq 2$.
3. if $e = 4$

Example (Bresinsky, 1975)

The family of numerical semigroups

$$\mathcal{S}_h = \langle (2h-1)2h, (2h-1)(2h+1), 2h(2h+1), 2h(2h+1)+2h-1 \rangle$$

is such that $t(\mathcal{S}_h) = 4h-3$.

$\Longrightarrow$ In the general case the type is not bounded.

Almost symmetric semigroups

We say that $\mathcal{S}$ is **almost symmetric** if, for every $x \notin \mathcal{S}$, we have either $F(\mathcal{S}) - x \in \mathcal{S}$ or $\{x, F(\mathcal{S}) - x\} \subseteq PF(\mathcal{S})$.

Question (Numata, 2013)

Let $\mathcal{S}$ be an almost symmetric numerical semigroup with $e(\mathcal{S}) = 4$. Is it true that $t(\mathcal{S}) \leq 3$?

Families of almost symmetric semigroups such that $t(\mathcal{S})$ is large are present in the literature [García-Sánchez and Ojeda, 2019], but for all of them we have $2e \geq t$.

Row-factorization matrices

We say that a matrix $\Lambda = (\lambda_{ij}) \in M_e(\mathbb{Z})$ is a **RF-matrix** for $f \in PF(\mathcal{S})$ if for every $i = 1, \ldots, e$, $\lambda_{ii} = -1$ $\lambda_{ij} \in \mathbb{N}$ if $i \neq j$ and $\lambda_{i1}g_1 + \ldots + \lambda_{ie}g_e = f$.

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Example

Take $\mathcal{S} = \langle 6, 7, 9, 10 \rangle$, $PF(\mathcal{S}) = \{3, 8, 11\}$. The two matrices

$$\Lambda_1 = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 3 & 0 & 0 & -1 \end{pmatrix}$$

are both RF-matrices for $8 \in PF(\mathcal{S})$.

RF-matrices: key property

Proposition

Let $\Lambda_1 = (a_{ij})$ be a RF-matrix for f and $\Lambda_2 = (b_{ij})$ a RF-matrix for $F(\mathcal{S}) - f$. Then for every $i \neq j$ we have $a_{ij} b_{ji} = 0$.

Example

Take $\mathcal{S} = \langle 6, 7, 9, 10 \rangle$. We have $PF(\mathcal{S}) = \{3, 8, 11\}$. The matrix Λ_1 is a RF-matrix for 3, while Λ_2 is a RF-matrix for $8 = F(\mathcal{S}) - 3$.

$$\Lambda_1 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 2 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 \end{pmatrix}.$$

Simple case: $e = 4$

Let $i, j \in \{1, \ldots, e\}$, $i \neq j$ and

$$m_{ij} = max\{K \in \mathbb{N} \mid Kg_j - g_i \notin \mathcal{S}\}, \quad M_{ij} = m_{ij}g_j - g_i \notin \mathcal{S},$$

and $\mathcal{M} = \{M_{ij} | i \neq j\}$, $|\mathcal{M}| \leq e(e-1)$.

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Let $f, F(\mathcal{S}) - f \in PF(\mathcal{S})$.

- If $f = kg_j - g_i \in PF(\mathcal{S})$, then $f \in \mathcal{M}$.
- If there is a row with at least $e - 2$ zeroes, then the element associated to that RF-matrix belongs to $\mathcal{M}$.
- In every couple of RF-matrices there are at least $e(e-1)$ zeroes distributed over $2e$ rows.

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Theorem

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Advanced case: $e = 5$

Computational evidence [Garcìa-Sànchez]

If $\mathcal{S}$ is almost symmetric, $e = 5$ and $g_5 \leq 200$, then $t(\mathcal{S}) \leq 5$.

If $e = 5$ then $e(e-1) = 20 = 2e(e-3)$, so the previous argument leaves out one case:

1. Every row and column of Λ_1, Λ_2 has exactly 2 zeroes.

Example

For instance we could have

$$\Lambda_1 = \begin{pmatrix} -1 & 0 & 0 & * & * \\ 0 & -1 & 0 & * & * \\ * & * & -1 & 0 & 0 \\ 0 & * & * & -1 & 0 \\ * & 0 & * & 0 & -1 \end{pmatrix} \Lambda_2 = \begin{pmatrix} -1 & * & 0 & * & 0 \\ * & -1 & 0 & 0 & * \\ * & * & -1 & 0 & 0 \\ 0 & 0 & * & -1 & * \\ 0 & 0 & * & * & -1 \end{pmatrix}$$

Advanced case: $e = 5$

Lemma

For every possible distribution of zeroes of the form described before, there are at most two elements of $PF(\mathcal{S})$ having a RF-matrix of that shape.

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If $\mathcal{S}$ is almost symmetric and $e = 5$ then $t(\mathcal{S})$ is bounded. (Actually, $t(\mathcal{S}) \leq 473$)

Unknown case: $e = 6$ and beyond

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- If $e = 4$ we could easily determine the shape of the factorization of all elements in $PF(\mathcal{S})$, showing that $f = kg_i - g_j$.
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If $e \geq 6$ all of this does not work anymore.

- There are a lot of elements not of the form $f = kg_i - g_j$.
- It seems that there are more elements associated to RF-matrices with the same shape.

It is not clear whether $t(\mathcal{S})$ is bounded if $e \geq 6$.

Unknown case: $e = 6$

Example

Take $\mathcal{S} = \langle 455, 497, 574, 589, 631, 708 \rangle$. Then $t(\mathcal{S}) = 14$ and

$$PF(\mathcal{S}) = \{3079, 3289, 3521, 3655, 3674, 3789, 3923, 4057,$$

$$4172, 4191, 4325, 4557, 4767, 7846\}.$$

The elements $\{3521, 3655, 3789, 3923, 4057, 4191\} \subset PF(\mathcal{S})$ all have RF-matrices sharing the same shape.

Some remarks:

- The example above is the first known example of almost symmetric numerical semigroup such that $t > 2e$.
- "Bad" examples occur for very high values of g_i - very hard to find by computer.

Unknown case: $e = 6$ - potential route

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- Fix the shape of a matrix.
- Play with the non-zero entries, and create a set of potential RF-matrices in this way.

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- [Easy part] Solve the linear system given by the rows of the matrices, finding the generators g_i and the elements f_j associated to those matrices (make sure that $f_j \notin \mathcal{S}$).

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- [Hard part] Check that the numerical semigroup generated by the g_is is almost symmetric.

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- Fix the shape of a matrix.
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- [Easy part] Solve the linear system given by the rows of the matrices, finding the generators g_i and the elements f_j associated to those matrices (make sure that $f_j \notin \mathcal{S}$).
- [Hard part] Check that the numerical semigroup generated by the g_is is almost symmetric.

The previous example was built using this construction. It is not known if this can be done for arbitrarily large sets of RF-matrices.

Thank you for your attention.