

J -ideals of matrices over PIDs

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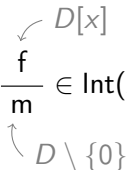
Motivation: Integer-valued polynomials on a matrix

D domain with qu. field K

$A \in M_n(D)$... $n \times n$ -matrix with entries in D

$$\text{Int}(A, M_n(D)) = \{g \in K[x] \mid g(A) \in M_n(D)\}$$

$$g = \frac{f}{m} \in \text{Int}(A, M_n(D)) \iff f(A) \in mM_n(D)$$



$$N_m(A) = \{f \in D[x] \mid f(A) \in mM_n(D)\} = ??$$

J -ideals of a matrix

D domain, J ideal of D

$A \in M_n(D)$... $n \times n$ -matrix with entries in D

$$N_J^D(A) = N_J(A) = \{f \in D[X] \mid f(A) \in M_n(J)\}$$

Classical linear algebra:

$$D \text{ field, } J = 0 \rightsquigarrow N_0(A) = \mu_A D[X]$$

null ideal of A   minimal polynomial of A

Applications

1. $\text{Int}(A, M_n(D))$
2. null ideals of matrices over D/J
3. module structure of $(D/J)[\bar{A}]$

Some facts

characteristic polynomial of A

Cayley-Hamilton:

$$\chi_A(A) = 0 \implies \chi_A \in N_J(A)$$

K quotient field, $\mu_A \in K[X]$

minimal polynomial of A over K

$$\forall n \forall A \in M_n(D) : \mu_A \in D[X] \iff D \text{ integrally closed}$$

Definition. $f \in D[X]$ J -minimal polynomial of A if

- ▶ $f(A) \in M_n(J)$
- ▶ f monic
- ▶ $\deg(f)$ minimal

Generators of J -ideals over PIDs

D ... principal ideal domain, K qu. field

$$\blacktriangleright J = \mathbf{0}: \mu_A \in D[X] \implies N_{\mathbf{0}}^D(A) = \mu_A D[X]$$

$\rightsquigarrow \mu_A$ is $\mathbf{0}$ -minimal polynomial

$$\blacktriangleright J = D: N_D^D(A) = D[X]$$

$\rightsquigarrow 1$ is D -minimal polynomial

$$\blacktriangleright J = (a) \ (a \neq 0 \text{ non-unit}): \text{ if } a = bc \text{ and } \gcd(b, c) = 1$$

$$N_{(a)}(A) = cN_{(b)}(A) + bN_{(c)}(A)$$

\rightsquigarrow investigate case $J = (p^t)$

p^t -ideal of A

Theorem (R., 2016) D PID

1. For all but finitely many prime elements p ,

$$N_{p^t}(A) = \mu_A D[X] + p^t D[X]$$

2. For $p \in \mathbb{P}$, \exists finite $S_p \subseteq \mathbb{N}$ and polynomials $\nu_{(p,s)}$, $s \in S_p$:

$$N_{p^t}(A) = \mu_A D[X] + p^t D[X] + \sum_{s \in S_p} p^{\max\{t-s, 0\}} \nu_{(p,s)} D[X]$$

Algorithm (Heuberger, R., 2017) \rightsquigarrow SageMath

Questions:

- ▶ Do these assertions hold for non-PIDs?
- ▶ Characterize these primes?

Description of $\text{Int}(A, M_n(D))$

D PID with qu. field K

$A \in M_n(D) \dots n \times n$ -matrix with entries in D

$$\begin{aligned}\text{Int}(A, M_n(D)) &= \{g \in K[x] \mid g(A) \in M_n(D)\} \\ &= \sum_{m \neq 0} \frac{1}{m} N_m(A) = \sum_{p \in \mathbb{P}} \sum_{\ell \geq 0} \frac{1}{p^\ell} N_{p^\ell}(A) \\ &= \mu_A K[x] + D[x] + \sum_p \sum_{s \in S_p} \frac{\nu_{(p,s)}}{p^s} D[X]\end{aligned}$$

$$g = \frac{f}{m} \in \text{Int}(A, M_n(D)) \iff f(A) \in mM_n(D)$$

$\swarrow D[x]$
 $\searrow D \setminus \{0\}$

\mathbb{Z} -similarity classes of matrices

$$A, B \in M_n(\mathbb{Z})$$

$$A \sim_{\mathbb{Z}} B \text{ are } \underline{\mathbb{Z}\text{-similar}} \iff \exists U \in GL_n(\mathbb{Z}): A = UBU^{-1}$$

$$A \sim_{\mathbb{Z}} B \stackrel{\text{red X}}{\implies} \forall p, t: N_{p^t}(A) = N_{p^t}(B) \\ \iff \text{Int}(A, M_n(\mathbb{Z})) = \text{Int}(B, M_n(\mathbb{Z}))$$

Theorem (Latimer, MacDuffee, 1933)

Let $f \in \mathbb{Z}[x]$ be monic, irreducible and $\lambda \in \mathbb{C}$ a root of f .

$$\{A \mid \chi_A = f\} / \sim_{\mathbb{Z}} \longleftrightarrow \text{ideal classes of } \mathbb{Z}[\lambda]$$

Questions.

1. Are (p^t) -ideals part of a characterization of \mathbb{Z} -similarity classes?
2. Representatives of \mathbb{Z} -similarity classes?