J-ideals of matrices over PIDs

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Content

1. Motivation: Integer-valued polynomials
2. Generators of J-ideals over a PID
3. Integer-valued polynomials revisited
4. $\mathbb{Z}$-similarity classes of matrices

Motivation: Integer-valued polynomials on a matrix

D domain with qu. field K

$A \in M_n(D) \ldots n \times n$-matrix with entries in D

$$\mathrm{Int}(A, M_n(D)) = \{g \in K[x] \mid g(A) \in M_n(D)\}$$

$$g = \frac{\overset{D[x]}{\mathrm{f}}}{\underset{D \setminus \{0\}}{\mathrm{m}}} \in \mathrm{Int}(A, M_n(D)) \quad \Longleftrightarrow \quad f(A) \in mM_n(D)$$

$$\mathrm{N}_m(A) = \{f \in D[x] \mid f(A) \in mM_n(D)\} = ??$$

J-ideals of a matrix

D domain, J ideal of D
$A \in M_n(D) \ldots n \times n$-matrix with entries in D

$$\mathrm{N}_J^D(A) = \mathrm{N}_J(A) = \{f \in D[X] \mid f(A) \in M_n(J)\}$$

Classical linear algebra:

D field, $J = \mathbf{0} \quad \rightsquigarrow \quad \mathrm{N}_{\mathbf{0}}(A) \; = \; \mu_A \; D[X]$

null ideal of A — minimal polynomial of A

Applications

1. $\mathrm{Int}(A, M_n(D))$
2. null ideals of matrices over D/J
3. module structure of $(D/J)[\overline{A}]$

Some facts

characteristic polynomial of A

Cayley-Hamilton: $\chi_A(A) = 0 \quad \Longrightarrow \chi_A \in \mathrm{N}_J(A)$

K quotient field, $\mu_A \in K[X]$

minimal polynomial of A over K

$$\forall n \, \forall A \in M_n(D) : \mu_A \in D[x] \iff D \text{ integrally closed}$$

Definition. $f \in D[X]$ J-minimal polynomial of A if

- $f(A) \in M_n(J)$
- f monic
- $\deg(f)$ minimal

Generators of J-ideals over **PIDs**

D ... principal ideal domain, K qu. field

- $J = \mathbf{0}$: $\mu_A \in D[X] \quad \Longrightarrow \quad \mathrm{N}_{\mathbf{0}}^D(A) = \mu_A D[X]$

  $\rightsquigarrow$ μ_A is $\mathbf{0}$-minimal polynomial

- $J = D$: $\mathrm{N}_D^D(A) = D[X]$

  $\rightsquigarrow$ 1 is D-minimal polynomial

- $J = (a)$ ($a \neq 0$ non-unit): if $a = bc$ and $\gcd(b, c) = 1$

$$\mathrm{N}_{(a)}(A) = c\mathrm{N}_{(b)}(A) + b\mathrm{N}_{(c)}(A)$$

  $\rightsquigarrow$ investigate case $J = (p^t)$

p^t-ideal of A

Theorem (R., 2016) D PID

1. For all but finitely many prime elements p,

$$\mathrm{N}_{p^t}(A) = \mu_A D[X] + p^t D[X]$$

2. For $p \in \mathbb{P}$, $\exists$ finite $S_p \subseteq \mathbb{N}$ and polynomials $\nu_{(p,s)}$, $s \in S_p$:

$$\mathrm{N}_{p^t}(A) = \mu_A D[X] + p^t D[X] + \sum_{s \in S_p} p^{\max\{t-s,0\}} \, \nu_{(p,s)} \, D[X]$$

Algorithm (Heuberger, R., 2017) $\rightsquigarrow$ SageMath

Questions:

- Do these assertions hold for non-PIDs?
- Characterize these primes?

Description of $\mathrm{Int}(A, M_n(D))$

D PID with qu. field K

$A \in M_n(D)$ ... $n \times n$-matrix with entries in D

$$\begin{aligned}
\mathrm{Int}(A, M_n(D)) &= \{g \in K[x] \mid g(A) \in M_n(D)\} \\
&= \sum_{m \neq 0} \frac{1}{m} \mathrm{N}_m(A) = \sum_{p \in \mathbb{P}} \sum_{\ell \geq 0} \frac{1}{p^\ell} \mathrm{N}_{p^\ell}(A) \\
&= \mu_A K[x] + D[x] + \sum_{p} \sum_{s \in S_p} \frac{\nu_{(p,s)}}{p^s} D[X]
\end{aligned}$$

$$g = \frac{\overset{D[x]}{\mathrm{f}}}{\underset{D \setminus \{0\}}{\mathrm{m}}} \in \mathrm{Int}(A, M_n(D)) \iff f(A) \in mM_n(D)$$

$\mathbb{Z}$-similarity classes of matrices

$A,\ B \in M_n(\mathbb{Z})$
$A \sim_{\mathbb{Z}} B$ are $\underline{\mathbb{Z}\text{-similar}}$ $\quad \Longleftrightarrow \quad \exists\ U \in GL_n(\mathbb{Z}):\ A = UBU^{-1}$

$$\begin{aligned} A \sim_{\mathbb{Z}} B \ \overset{\cancel{\Longleftarrow}}{\Longrightarrow}\ & \forall\ p,\ t:\ N_{p^t}(A) = N_{p^t}(B) \\ & \Longleftrightarrow \operatorname{Int}(A, M_n(\mathbb{Z})) = \operatorname{Int}(B, M_n(\mathbb{Z})) \end{aligned}$$

Theorem (Latimer, MacDuffee, 1933)
Let $f \in \mathbb{Z}[x]$ be monic, irreducible and $\lambda \in \mathbb{C}$ a root of f.

$$\{A \mid \chi_A = f\}/\sim_{\mathbb{Z}} \quad \longleftrightarrow \quad \text{ideal classes of } \mathbb{Z}[\lambda]$$

Questions.

1. Are (p^t)-ideals part of a characterization of $\mathbb{Z}$-similarity classes?
2. Representatives of $\mathbb{Z}$-similarity classes?