Cosilting Modules

Flaviu Pop
(joint work with Simion Breaz)

Babeș - Bolyai University, Cluj-Napoca

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Introduction

- Keller and Vossieck: silting objects in triangulated categories (bounded derived categories);
  - (co)t-structures;
  - simply-minded collections of objects;

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- Keller and Vossieck: silting objects in triangulated categories (bounded derived categories);
  - (co)t-structures;
  - simply-minded collections of objects;
- Angeleri-Hugel, Marks and Vitoria: (partial) silting modules;
  - study the class of kernels of those homomorphisms between projective modules which represent silting objects in the derived categories.

Notations

- R - unital associative ring;
- $\mathrm{Mod}\text{-}R$ - the category of all right R-modules;
- T - right R-module;

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- R - unital associative ring;
- $\mathrm{Mod}\text{-}R$ - the category of all right R-modules;
- T - right R-module;
- Consider the following orthogonal classes:
  - ${}^{\circ}T = \{X \in \mathrm{Mod}\text{-}R \mid \mathrm{Hom}_R(X, T) = 0\}$.
  - ${}^{\perp}T = \{X \in \mathrm{Mod}\text{-}R \mid \mathrm{Ext}^1_R(X, T) = 0\}$.

Notations

- An R-module X is called T-cogenerated if it can be embedded into a direct product of copies of T, i.e. there is a monomorphism
$$0 \to X \longrightarrow T^I.$$
We denote by $\mathrm{Cogen}(T)$ the class of all T-cogenerated R-modules.

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- An R-module X is called T-copresented if it is the kernel of a homomorphism between direct products of copies of T, i.e. there is an exact sequence
$$0 \to X \longrightarrow T^I \longrightarrow T^J.$$
We denote by $\mathrm{Copres}(T)$ the class of all T-copresented R-modules.

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We denote by $\mathrm{Copres}(T)$ the class of all T-copresented R-modules.
- We denote by $\mathrm{Prod}(T)$ the class of all R-modules which are isomorphic to direct summands of direct products of copies of T.

Cotilting modules

Recall that a right R-module T is

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   (a) $\mathrm{Cogen}(T) \subseteq {}^{\perp}T$ and the class ${}^{\perp}T$ is a torsion-free class.
   (b) $\mathrm{Cogen}(T) \subseteq {}^{\perp}T$ and $\mathrm{id}(T) \leq 1$.

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  (a) $\mathrm{Cogen}(T) \subseteq {}^{\perp}T$ and the class ${}^{\perp}T$ is a torsion-free class.
  (b) $\mathrm{Cogen}(T) \subseteq {}^{\perp}T$ and $\mathrm{id}(T) \leq 1$.

(2) *cotilting* if and only if $\mathrm{Cogen}(T) = {}^{\perp}T$.

Cotilting Modules

Theorem

An R-module T is a cotilting module if and only if

(i) $id(T) \leq 1$;

(ii) $\mathrm{Ext}^1_R(T^I, T) = 0$, *for all sets* I;

(iii) *an injective cogenerator* C *admits an exact sequence*

$$0 \to T_1 \longrightarrow T_0 \longrightarrow C \to 0$$

with $T_0, T_1 \in \mathrm{Prod}(T)$.

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(iii) $\mathrm{Ker}\mathrm{Hom}_R(-, T) \cap {}^{\perp}T = 0$.

The class $\mathcal{D}_\sigma$

If $\sigma : P_1 \to P_0$ is an R-homomorphism of R-modules, then

$$\mathcal{D}_\sigma = \{X \in \mathrm{Mod}\text{-}R \mid \mathrm{Hom}_R(\sigma, X) \text{ is an epimorphism}\}.$$

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Lemma

Let $\sigma : P_1 \to P_0$ be an R-homomorphism of projective R-modules with cokernel T.

(a) *The class $\mathcal{D}_\sigma$ is closed under epimorphic images, extensions and direct products.*

(b) *The class $\mathcal{D}_\sigma$ is contained in $T^\perp$.*

(Partial) Silting Modules

We say that an R-module T is

(1) *partial silting (with respect to σ)* if there is a projective presentation

$$P_1 \xrightarrow{\sigma} P_0 \xrightarrow{g} T \to 0$$

of T such that

  (a) $\mathcal{D}_\sigma$ is a torsion class.

  (b) T lies in the class $\mathcal{D}_\sigma$.

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of T such that $\mathrm{Gen}(T) = \mathcal{D}_\sigma$.

The class B_ζ

If $\zeta : Q_0 \to Q_1$ is an R-homomorphism of R-modules, then

$$\mathcal{B}_\zeta = \{X \in \mathrm{Mod}\text{-}R \mid \mathrm{Hom}_R(X, \zeta) \text{ is an epimorphism}\}.$$

The Codefect Functor

We mention that the class $\mathcal{B}_\zeta$ is in fact the kernel of CoDef_ζ.

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Definition

Let $\zeta : Q_0 \to Q_1$ be an R-homomorphism.
For an object $X \in \mathrm{Mod}\text{-}R$, we set

$$\mathrm{CoDef}_\zeta(X) = \mathrm{CokerHom}_R(X, \zeta).$$

For a morphism $f : X \to Y$ in $\mathrm{Mod}\text{-}R$, we define

$$\mathrm{CoDef}_\zeta(f) : \mathrm{CoDef}_\zeta(Y) \to \mathrm{CoDef}_\zeta(X)$$

by

$$\mathrm{CoDef}_\zeta(f) = \phi,$$

where ϕ is given by the universal property of the cokernel.

The Codefect Functor

$$
\begin{array}{ccccccc}
\mathrm{Hom}_R(Y, Q_0) & \xrightarrow{\mathrm{Hom}_R(Y,\zeta)} & \mathrm{Hom}_R(Y, Q_1) & \xrightarrow{\pi_Y} & \mathrm{CokerHom}_R(Y, \zeta) & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \mathrm{Hom}_R(f,Q_0)} & & \Big\downarrow{\scriptstyle \mathrm{Hom}_R(f,Q_1)} & & \Big\downarrow{\scriptstyle \phi} & & \\
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Lemma

Let $\zeta : Q_0 \to Q_1$ be an R-homomorphism and assume that $\zeta = \tau_\zeta \circ \pi_\zeta$ is the canonical decomposition of ζ. The following are equivalent for an R-module X:

(1) $X \in \mathcal{B}_\zeta$;

(2) $\mathrm{Hom}_R(X, \tau_\zeta)$ *is an isomorphism and* $\mathrm{Hom}_R(X, \pi_\zeta)$ *is an epimorphism.*

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Corollary

Let $\zeta : Q_0 \to Q_1$ be an R-homomorphism with $T = \mathrm{Ker}(\zeta)$ and let $\zeta = \tau_\zeta \circ \pi_\zeta$ be the canonical decomposition. The following statements are equivalent for an R-module X which belongs to ${}^\perp T$:

(1) $X \in \mathcal{B}_\zeta$;

(2) $\mathrm{Hom}_R(X, \tau_\zeta)$ *is an isomorphism.*

Closure properties of the class B_ζ

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(4) *If Q_0 is injective and $T = \mathrm{Ker}(\zeta)$ then $\mathcal{B}_\zeta \subseteq {}^\perp T$.*

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(4) *If Q_0 is injective and $T = \mathrm{Ker}(\zeta)$ then $\mathcal{B}_\zeta \subseteq {}^\perp T$.*

(5) *Assume that Q_1 is injective. If $T = \mathrm{Ker}(\zeta)$ and*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} X \to 0$$

is an exact sequence such that A and B belong to the class $\mathcal{B}_\zeta$ and $X \in {}^\perp T$ then $X \in \mathcal{B}_\zeta$.

Definition of (partial) cosilting module

We say that an R-module T is:

(1) *partial cosilting* (*with respect to* ζ), if there exists an injective copresentation of T

$$0 \to T \xrightarrow{f} Q_0 \xrightarrow{\zeta} Q_1$$

such that:

(a) $T \in \mathcal{B}_\zeta$, and

(b) the class $\mathcal{B}_\zeta$ is closed under direct products;

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of T such that $\mathrm{Cogen}(T) = \mathcal{B}_\zeta$.

(Partial) Cosilting Modules vs.(Partial) Cotilting Modules

Lemma

If the R-module T is partial cosilting with respect to the injective copresentation $\zeta : Q_0 \to Q_1$, then

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Corollary

Let T be an R-module. Then T is (partial) cotilting if and only if T is (partial) cosilting with respect to an epimorphic injective copresentation of T.

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Let T be an R-module. Then T is (partial) cotilting if and only if T is (partial) cosilting with respect to an epimorphic injective copresentation of T.

Corollary

Let T be a cosilting R-module. Then $\mathrm{Cogen}(T) = \mathrm{Copres}(T)$.

Torsion pairs

In the following we will denote by

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Proposition

Let $0 \to T \to Q_0 \xrightarrow{\zeta} Q_1$ *be an injective copresentation for* T*.*

(1) *If* T *is partial cosilting then the pair* $({}^{\circ}T, \mathrm{Cogen}(T))$ *is a torsion pair.*

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(2) T *is cosilting with respect to* ζ *if and only if* $({}^{\circ}T, \mathcal{B}_{\zeta})$ *is a torsion pair.*

Partial Silting - Partial Cosilting

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If E is an injective cogenerator for the category $\mathrm{Mod}\text{-}S$, we denote by

$$(-)^d = \mathrm{Hom}_S(-, E) : R\text{-}\mathrm{Mod} \rightleftarrows \mathrm{Mod}\text{-}R : \mathrm{Hom}_S(-, E) = (-)^d$$

the Hom-contravariant functors induced by E.

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Suppose that $P_2 \to P_1 \stackrel{\zeta}{\to} P_0 \to M \to 0$ is a projective presentation of the left R-module M. The following are statements are true:

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(1) *If $X \in \mathcal{B}_{\zeta^d}$ then $X^d \in \mathcal{D}_\zeta$.*

(2) *If all projective modules P_i are finitely presented, and $Y \in \mathcal{D}_\zeta$ then $Y^d \in \mathcal{B}_{\zeta^d}$.*

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(2) *If all projective modules P_i are finitely presented, and $Y \in \mathcal{D}_\zeta$ then $Y^d \in \mathcal{B}_{\zeta^d}$.*

(3) *Suppose that all projective modules P_i are finitely presented. Then the left R-module M is partial silting with respect to ζ if and only if the dual M^d is a partial cosilting right R-module with respect to ζ^d.*

Characterization of Cosilting Modules

Theorem

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(1) *T is cosilting with respect to* ζ*;*

(2) *T has the following properties:*

  (a) *T is partial cosilting with respect to* ζ*, and*

  (b) *if E is an injective cogenerator in* $\mathrm{Mod}\text{-}R$ *there exists an exact sequence*

$$0 \to T_1 \to T_0 \xrightarrow{\gamma} E$$

  such that $T_0, T_1 \in \mathrm{Prod}(T)$ *and for every* $T' \in \mathcal{B}_\zeta$ *the homomorphism* $\mathrm{Hom}(T', \gamma)$ *is epic.*

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Proposition

An R-module T is a cosilting module with respect to $\zeta : Q_0 \to Q_1$ if and only if the following statements hold:

(i) $T^I \in \mathrm{KerCoDef}_\zeta$, *for all sets I;*

(ii) $\mathrm{KerHom}_R(-, T) \cap \mathrm{KerCoDef}_\zeta = 0$.

Silting - Cosilting

Corollary

Let S be a commutative ring, let R be an S-algebra and let E be an injective cogenerator for the category $\mathrm{Mod}\text{-}S$. Assume that $P_2 \to P_1 \xrightarrow{\zeta} P_0 \to M \to 0$ is a projective presentation of the left R-module M with all projective modules P_i finitely presented.

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Corollary

Let S be a commutative ring, let R be an S-algebra and let E be an injective cogenerator for the category Mod-S. Assume that $P_2 \to P_1 \xrightarrow{\zeta} P_0 \to M \to 0$ is a projective presentation of the left R-module M with all projective modules P_i finitely presented.

(1) *If M is a silting module with respect to ζ then M^d is a cosilting module with respect to ζ^d.*

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Let S be a commutative ring, let R be an S-algebra and let E be an injective cogenerator for the category $\mathrm{Mod}\text{-}S$. Assume that $P_2 \to P_1 \xrightarrow{\zeta} P_0 \to M \to 0$ is a projective presentation of the left R-module M with all projective modules P_i finitely presented.

(1) *If M is a silting module with respect to ζ then M^d is a cosilting module with respect to ζ^d.*

(2) *Suppose that R is an Artin algebra, S is the center of R and $(-)^d$ is the standard duality between finitely presented left and right modules induced by the injective envelope of $S/J(S)$. Then M is a silting module with respect to ζ if and only if M^d is a cosilting module with respect to ζ^d.*

Direct Summand

Theorem

Let T be a partial cosilting R-module with respect to an injective copresentation

$$0 \to T \to Q_0 \xrightarrow{\zeta} Q_1$$

Then there exists

- *an R-module M and*
- *an injective copresentation*

$$0 \to T \oplus M \to Q_0' \xrightarrow{\zeta'} Q_1'$$

such that

- *$T \oplus M$ is cosilting with respect to ζ' and*
- *$\mathcal{B}_\zeta = \mathcal{B}_{\zeta'}$.*

Well known result

Corollary

Every partial cotilting module is a direct summand of a cotilting module.

Pure-Injective Modules

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(2) An R-module U is *pure-injective* if the contravariant functor $\mathrm{Hom}_R(-, U)$ preserves the exactness of every pure-exact sequence.

Proposition

An R-module U is pure-injective if and only if the contravariant functor $\mathrm{Hom}_R(-, U)$ preserves the exactness of the canonical short exact sequence

$$0 \to U^{(\lambda)} \longrightarrow U^{\lambda} \longrightarrow U^{\lambda}/U^{(\lambda)} \to 0$$

for every cardinal λ.

The construction given by Bazzoni

Proposition

Let T be an R-module and let λ be a cardinal. Then there is a submodule $T^{(\lambda)} \leq V \leq T^{\lambda}$ such that $V/T^{(\lambda)} \cong X^{(\lambda^{\aleph_0})}$, where $X = T^{\aleph_0}/T^{(\aleph_0)}$.

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Proposition

Let T be a partial cosilting R-module with respect to $\zeta : Q_0 \to Q_1$. Then $T^{\aleph_0}/T^{(\aleph_0)} \in \mathcal{B}_{\zeta}$.

All cosilting modules are pure-injective

Proposition

Let X and U be two R-modules. If $X^{\aleph_0}/X^{(\aleph_0)} \in \mathrm{Cogen}(U)$ then $X^{\lambda}/X^{(\lambda)} \in \mathrm{Cogen}(U)$, for every cardinal λ.

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Theorem

Let T be an R-module. If T is cosilting then T is pure-injective.

The class $\mathcal{B}_\zeta$ is definable

A full subcategory (or a subclass) of $\mathrm{Mod}\text{-}R$ is a *definable subcategory* if it is closed in $\mathrm{Mod}\text{-}R$ under

- direct products;
- direct limits;
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- pure submodules.

Corollary

If T is a cosilting R-module with respect to ζ, then the class $\mathrm{Cogen}(T) = \mathcal{B}_\zeta$ is definable.

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If R is a ring then we denote by:

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- $\mathrm{mod}\text{-}R$ (respectively, by $R\text{-}\mathrm{mod}$) the subcategory of $\mathrm{Mod}\text{-}R$ (respectively, of $R\text{-}\mathrm{Mod}$) consisting of finitely generated R-modules.

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- $\mathrm{pres}\text{-}R$ (respectively, by $R\text{-}\mathrm{pres}$) the subcategory of $\mathrm{Mod}\text{-}R$ (respectively, of $R\text{-}\mathrm{Mod}$) consisting of finitely presented R-modules.

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- we consider the following classes
  - $\mathcal{Y}_R = \mathrm{Cogen}(T_R) \cap \mathrm{mod}\text{-}R$;
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Notations

For a right R-module T_R:

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  - $\mathcal{Y}'_R = \mathrm{Cogen}(T_R) \cap \mathrm{pres}\text{-}R$;
  - $\mathcal{X}'_R = \mathrm{Ker}\Delta_R \cap \mathrm{pres}\text{-}R$.

Finitely cosilting module

Definition

Let T be a right R-module with $S = \mathrm{End}_R(T)$ and let $T = \mathrm{Ker}(\zeta)$, where $\zeta : Q_0 \to Q_1$ is an R-homomorphism between injective R-modules. We say that T is *finitely cosilting with respect to* ζ if the following conditions are satisfied:

(1) $T \in \mathrm{KerCoDef}_\zeta$;

(2) $\mathrm{KerHom}_R(-, T) \cap \mathrm{KerCoDef}_\zeta = 0$;

(3) T is finitely generated;

(4) $\mathrm{Hom}_R(-, T) : \mathrm{Mod}\text{-}R \to S\text{-}\mathrm{Mod}$ carries finitely generated modules to finitely generated modules.

Finitely cosilting bimodule

Definition

Let ${}_S T_R$ be an (S, R)-bimodule. Then T is called *finitely cosilting bimodule* if:

(1) ${}_S T_R$ is a faithfully balanced bimodule;

(2) T_R and ${}_S T$ are finitely cosilting modules (with respect to the injective copresentations $\zeta : Q_0 \to Q_1$ and $\tau : S_0 \to S_1$, respectively).

Cosilting Theorem

Theorem

Let ${}_S T_R$ be a finitely cosilting bimodule. Then the following assertions hold.

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Let ${}_ST_R$ be a finitely cosilting bimodule. Then the following assertions hold.

(1) *The pairs* $(\mathcal{X}_R, \mathcal{Y}_R)$ *and* $({}_S\mathcal{X}, {}_S\mathcal{Y})$ *are torsion pairs in* mod-R *and* S-mod*, respectively.*

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(2) *The pairs of contravariant functors*

$$\Delta_R : \mathcal{Y}'_R \rightleftarrows {}_S\mathcal{Y}' : \Delta_S$$

and

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(3) (i) $\Gamma_S\Delta_R$ *and* $\Gamma_R\Delta_S$ *carries finitely generated modules to zero.*
(ii) $\Delta_S\Gamma_R$ *and* $\Delta_R\Gamma_S$ *carries finitely presented modules to zero.*

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