Metric dimension of zero divisor graphs

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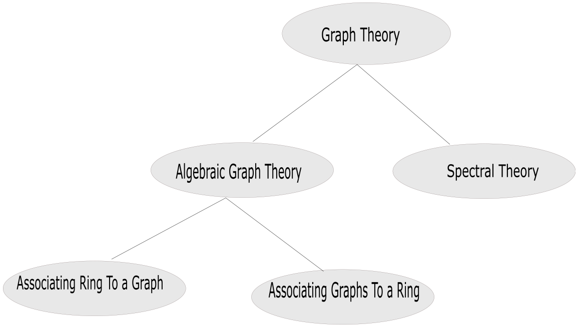

Figure: Model

Overview

Introduction

Metric dimension

Relation between Diameter, Girth and Metric Dimension

Metric dimension of Compressed Zero-divisor Graphs

A graph G is a pair (V, E) where V is a non-empty set of vertices of G and E is the edge set, each joined by a pair of distinct vertices u and v of G.

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The *distance* between two vertices u and v in G, denoted by $d(u, v)$ is the length of the shortest $u - v$ path in G. If such a path does not exist, we define $d(u, v)$ to be infinite.

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A cycle passing through all the vertices of a graph G is called a *Hamiltonian cycle* and a graph containing a Hamiltonian cycle is called a *Hamiltonian graph*.

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Zero divisor graph

A zero divisor graph $\Gamma(R)$ is the undirected graph with vertex set $Z^*(R) = Z(R) \setminus \{0\}$ the set of non-zero zero divisors of a commutative ring R with $1 \neq 0$ and the two vertices x and y are adjacent if and only if $xy = 0$.

Example. Consider $R = \mathbb{Z}_{12}$.
Here $Z^*(R) = \{2, 3, 4, 6, 8, 9, 10\}$ is the vertex set of $\Gamma(R)$

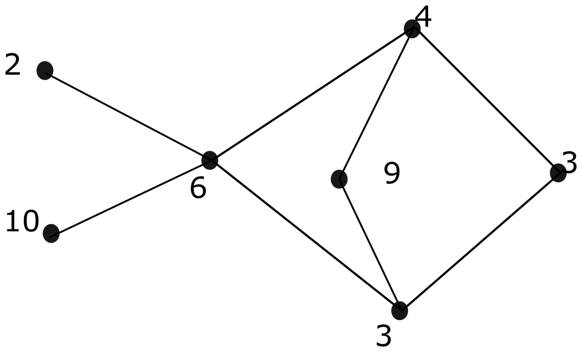

Figure: $\Gamma(\mathbb{Z}_{12})$

Beck [5] introduced the notion of zero divisor graphs of a commutative ring R and he was mainly interested in colorings. Even more, the concept has been extended to the ideal based zero divisor graphs [5], unit graphs [3], zero-divisor graphs of non-commutative rings [1], lattices and several others.

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$\Gamma(R)$ is complete graph or complete bipartite graph if it is regular.

Metric Dimension

Let G be a connected graph with $n \geq 2$ vertices.
For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of $V(G)$, we refer to the k-vector as the metric representation of v with respect to W as

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$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

The set W is a resolving set of G if distinct vertices have distinct metric representations.

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In fact, for every connected graph G of order $n \geq 2$,

$$1 \leq dim(G) \leq n - 1$$

.

Example. Consider the graph (G) given in Figure 2. Take $W_1 = \{v_1, v_3\}$. So, $r(v_1|W_1) = (0, 1)$, $r(v_2|W_1) = (1, 1)$, $r(v_3|W_1) = (1, 0)$, $r(v_4|W_1) = (1, 1)$, $r(v_5|W_1) = (2, 1)$. Notice, $r(v_2|W_1) = (1, 1) = r(v_4|W_1)$, therefore W_1 is not a resolving set. However, if we take $W_2 = \{v_1, v_2\}$, then $r(v_1|W_2) = (0, 1)$, $r(v_2|W_2) = (1, 0)$, $r(v_3|W_2) = (1, 1)$, $r(v_4|W_2) = (1, 2)$, $r(v_5|W_2) = (2, 1)$. Since distinct vertices have distinct metric representations, W_2 is a minimum resolving set and thus this graph has metric dimension 2.

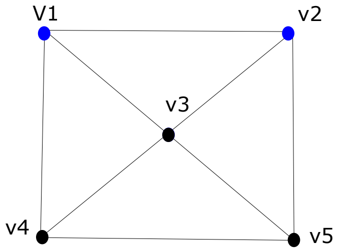

Figure: $dim(G) = 2$

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For a connected graph G of order $n \geq 3$, the metric dimension of a cycle graph C_n is 2.

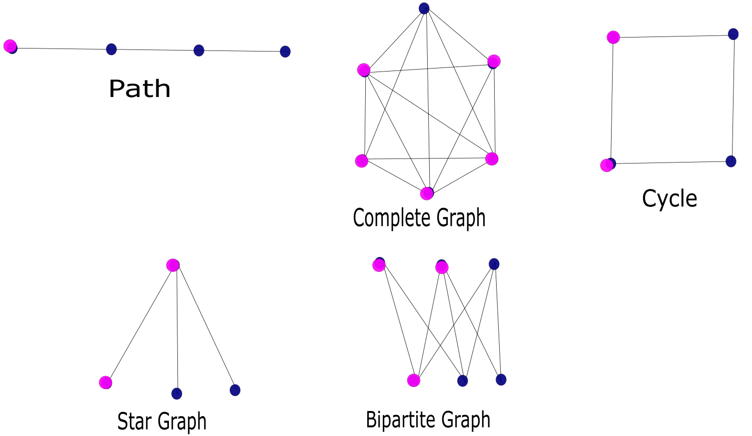

Figure: Pink Colored vertices correspond to metric basis

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Theorem. The graph $\Gamma(\mathbb{Z}_n)$ is Hamiltonian graph if and only if $dim(\Gamma(\mathbb{Z}_n)) = |Z^*(\mathbb{Z}_n)| - 1$.

If G is a connected graph of order $n \geq 2$, we say two distinct vertices u and v of G are distance similar, if $d(u,x) = d(v,x)$ for all $x \in V(G) - \{u, v\}$.

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Theorem. Let G be a connected graph partitioned into k distinct distance similar classes $V_1, V_2, \ldots, v_k$, then $dim(G) \geq |V(G)| - k$ and $|V(G)| - k \leq dim(G) \leq |V(G)| - k + m$, where m is the number of number of distance similar classes that consist of a single vertex.

Theorem. For connected graph G of order $n \geq 3$, the metric dimension of bipartite graph $K_{1,n-1}$ is $n-2$ and for for $r \geq 2, n \geq 5$, metric dimension of $K_{r,n-r} = K_{n,m}$ is $n-2$, with $n = r$ and $m = n - r$.

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Corollary. Let p be a prime number.
(i) If $n = 2p$, then $dim(\Gamma(\mathbb{Z}_n)) = p - 2$
(ii) If $n = p^2$ and $p > 2$, then $dim(\Gamma(\mathbb{Z}_n))p - 2$
(iii) If $n = p^k$ and $k \geq 3$, then
$dim(\Gamma(\mathbb{Z}_n)) = |Z^*(\mathbb{Z}_n)| - (k-1) = p^{k-1} - k$.

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Theorem. Let R be a commutative ring and $\Gamma(R)$ be the corresponding zero-divisor graph of R such that $|Z^*(R)| \geq 2$. Then

$$\lceil log_3(\triangle + 1) \rceil \leq dim(\Gamma(R)) \leq |Z^*(R)| - d$$

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Corollary. Let R be a commutative ring with unity 1 such that $dim(\Gamma(R)) = k$ where k is any non-negative integer. Then $|Z(R)| \leq 4^k + 1$.

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A ring R is called *Boolean ring* if $a^2 = a$ for every $a \in R$.

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Theorem. Let R be a finite commutative local ring with unity 1 and for prime q let $\mathbb{F}_q$ be a finite field. Then $|Z^*(R \times \mathbb{F}_q| = |U(R)| + (|Z^*(R)| + 1)q - 1.$

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Example. Let $\mathbb{F}_2[x]/(x^3, xy, y^2) = \{ax^2 + bx + cy + d \mid a, b, c, d \in \mathbb{F}_2\}$ and let $\mathbb{F}_q = \mathbb{Z}_5$.
Clearly, R is a local ring of order 16 with $Z(R) = \{0, x, y, x^2, x + y, x + x^2, y + x^2, x + x^2 + y\}$.
Therefore, $|Z^*(R \times \mathbb{F}_q)| = |U(R)| + (|Z^*(R)| + 1)q - 1 = 8 + (8)5 - 1 = 47$.

Theorem. Let $R_1, R_2, \ldots, R_n$ be n finite commutative rings (not domains) each having unity 1 with none of R_i, $1 \leq i \leq n$, being isomorphic to $\prod_{i=1}^{n} \mathbb{Z}_2$ for any positive integer n. Then for a commutative ring R with unity 1 and for any prime field $\mathbb{F}_q$,
(a) $dim(\Gamma(R_1 \times R_2 \times \cdots \times R_n)) \geq \sum_{i=1}^{n} dim(\Gamma(R_i))$
(b) $dim(R \times \mathbb{F}_q)) = |Z^*(R \times \mathbb{F}_q)| - 2^{n+1} + 2$ *or* $|Z^*(R \times \mathbb{F}_q)| - 2$ *or at least* $|U(R)| + (|Z^*(R)| + 1)q - t + 3$, where t is any positive integer.

Example. Consider the zero divisor graph shown in Fig.6 associated with the ring $\mathbb{Z}_8 \times \mathbb{Z}_2$. Partition the graph $\Gamma(\mathbb{Z}_8 \times \mathbb{Z}_2)$ into the distance similar equivalence classes which are given by sets, $\{(1,0),(3,0),(5,0),(7,0)\}$, $\{(2,1), (6,1)\}$, $\{(0,1)\}, \{(2,0),(6,0)\}$, $\{(4,0)\}$ and $\{(4,1)\}$. Therefore, by theorem, $dim(\Gamma(\mathbb{Z}_8 \times \mathbb{Z}_2)) \geq 4 + (4)2 - 4 - 3 = 5$. From Fig.6, it can be easily verified that $\{(1,0),(3,0),(5,0),(7,0),(2,0),(6,1)\}$ is the metric basis and consequently $dim(\Gamma(\mathbb{Z}_8 \times \mathbb{Z}_2)) = 6$.

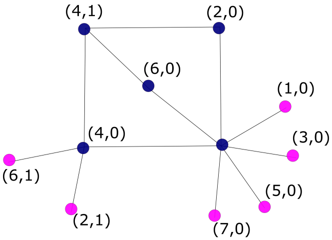

Figure: $dim(\Gamma(\mathbb{Z}_8 \times \mathbb{Z}_2)) = 6$

Theorem. Let $R_1, R_2, \ldots, R_k$ be integral domains with 1 and $|R_i| > 2$ for some i. Then $dim(\Gamma(R_1 \times R_2 \times \cdots \times R_k)) = |Z(R)| - 2^k + 2$.

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Remark. If $R \ncong \prod_{i=1}^{n} \mathbb{Z}_2$, then $|W|$ is equal to this sum. In light of above theorem, one should consider the metric dimension of $\prod_{i=1}^{n} \mathbb{Z}_2$.

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For $n > 5$, the case is still open.

The cartesian product $\Gamma(R_1) \times \Gamma(R_2)$ of two zero divisor graphs $\Gamma(R_1)$ and $\Gamma(R_2)$ with vertex set as $Z^*(R_1) \times Z^*(R_2)$ and the edge set $\{(x_1, x - 2) \mid x_1 = y_1 \text{ and } x_2y_2 \in E(\Gamma(R_2)) \text{ or } x_2 = y_2 \text{ and } x_1y_1 \in E(\Gamma(R_1))$

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Theorem. If R is any finite commutative ring with unity 1 (not a domain), then

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. The equality is possible for either bound in the above theorem is possible. For example, $dim(R \cong \mathbb{Z}_3 \times \mathbb{Z}_3) = 2$, where as the $dim(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = 3$.

Girth, diameter and metric dimension

Theorem. Let R be a finite commutative ring with $gr(\Gamma(R)) = \infty$.
$dim(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = dim(\Gamma(\mathbb{Z}_9)) = dim(\Gamma(\mathbb{Z}_3[x]/(x^3)) = 1$.
If R is reduced and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $dim(\Gamma(R)) = |Z^*(R)| - 2$.
If $R \cong \mathbb{Z}_4))$ or $\mathbb{Z}_2[x]/(x^3)$, then $dim(\Gamma(R)) = 0$.

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Theorem. If R is reduced Artinian commutative ring and $gr(\Gamma(R)) = 4$, then $dim(\Gamma(R)) = |Z^*(R)| - 2$ and $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, where each $\mathbb{F}_i$ is a field with $\mathbb{F}_i \geq 3$.

Theorem. Let R be a commutative ring. Then
(i) $diam(\Gamma(R)) = 0$ if and only if $dim(\Gamma(R)) = 0$.
(ii) $diam(\Gamma(R)) = 1$ if and only if $dim(\Gamma(R)) = |Z^*(R)| - 1$
(iii) $dim(\Gamma(R)) = 1$ or $|Z^*(R)| - 2$ if $R \cong \mathbb{F}_1 \times \mathbb{F}_2$.

Compressed zero divisor graph

Let R be a commutative ring with $1 \neq 0$.

A zero-divisor graph determined by equivalence classes or simply a compressed zero-divisor graph of a ring R is the undirected graph $\Gamma_E(R)$ with vertex set $Z(R_E) \setminus \{[0]\} = R_E \setminus \{[0], [1]\}$ defined by $R_E = \{[x] : x \in R\}$, where $[x] = \{y \in R : ann(x) = ann(y)\}$ and the two distinct vertices $[x]$ and $[y]$ of $Z(R_E)$ are adjacent if and only if $[x][y] = [xy] = [0]$, that is, if and only if $xy = 0$.

Example Consider $R = \mathbb{Z}_{12}$. For the vertex set of $\Gamma_E(R)$, we have $ann(2) = \{6\}$, $ann(3) = \{4, 8\}$, $ann(4) = \{3, 6, 9\}$, $ann(6) = \{2, 4, 6, 8, 10\}$, $ann(8) = \{3, 6, 9\}$, $ann(9) = \{4, 8\}$, $ann(10) = \{6\}$.

So, $Z(R_E) = \{[2], [3], [4], [6]\}$ is the vertex set of $\Gamma_E(R)$.

Graph of equivalence classes of zero divisors of R, $\Gamma_E(R)$.

Figure: $\Gamma_E(\mathbb{Z}_{12})$

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$\Gamma_E(R)$ may be finite when $\Gamma(R)$ is infinite.

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A ring R satisfies the a.c.c for ideals if given any sequence of ideals $I_1, I_2, \ldots$ of R with $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \ldots$ there exists an integer m such that $I_m = I_n$ for all $m \geq n$.

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Example. The ring of integers $\mathbb{Z}$, $\mathbb{Z}_n$.

The vertices of the graph $\Gamma_E(R)$ correspond to *annihilator ideals* in the ring and hence prime ideals if R is Noetherian ring.

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Theorem If R is a finite local ring with unity 1 and $\mathbb{F}_q$ is a finite prime filed, then

$$|Z^*(R \times \mathbb{F}_q)_E| = 2k \quad or \quad 2(1 + |Z^*(R_E)|)$$

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Thanks for your attention