Metric dimension of zero divisor graphs

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Figure: Model
Overview

Introduction

Metric dimension

Relation between Diameter, Girth and Metric Dimension

Metric dimension of Compressed Zero-divisor Graphs
A graph $G$ is a pair $(V, E)$ where $V$ is a non-empty set of vertices of $G$ and $E$ is the edge set, each joined by a pair of distinct vertices $u$ and $v$ of $G$. 
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The *distance* between two vertices $u$ and $v$ in $G$, denoted by $d(u, v)$ is the length of the shortest $u - v$ path in $G$. If such a path does not exist, we define $d(u, v)$ to be infinite.
The *diameter* of $G$ is $\sup\{d(u, v)\}$, where $u$ and $v$ are distinct vertices of $G$.  

The *girth* of a graph $G$, denoted by $gr(G)$, is the length of a smallest cycle in $G$. 

A cycle passing through all the vertices of a graph $G$ is called a *Hamiltonian cycle* and a graph containing a Hamiltonian cycle is called a *Hamiltonian graph*. 
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**Zero divisor graph**

A zero divisor graph $\Gamma(R)$ is the undirected graph with vertex set $Z^*(R) = Z(R) \setminus \{0\}$ the set of non-zero zero divisors of a commutative ring $R$ with $1 \neq 0$ and the two vertices $x$ and $y$ are adjacent if and only if $xy = 0$. 
Example. Consider $R = \mathbb{Z}_{12}$.
Here $\mathbb{Z}^*(R) = \{2, 3, 4, 6, 8, 9, 10\}$ is the vertex set of $\Gamma(R)$.
Beck [5] introduced the notion of zero divisor graphs of a commutative ring $R$ and he was mainly interested in colorings. Even more, the concept has been extended to the ideal based zero divisor graphs [5], unit graphs [3], zero-divisor graphs of non-commutative rings [1], lattices and several others.

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- $\Gamma(R)$ is connected, $\text{diam}(\Gamma(R)) \leq 3$ and $\text{gr}(\Gamma(R)) \leq 4$.
- $\Gamma(R) \cong K_{1,n}$ (star graph) if and only if $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a finite field.
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\( \Gamma(R) \) is complete graph or complete bipartite graph if it is regular.
Metric Dimension
Let $G$ be a connected graph with $n \geq 2$ vertices. For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of $V(G)$, we refer to the $k$-vector as the metric representation of $v$ with respect to $W$ as

$$r(v \mid W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

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$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

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A resolving set containing the minimum number of vertices is called a *metric basis* for $G$. 
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In fact, for every connected graph $G$ of order $n \geq 2$,

$$1 \leq \text{dim}(G) \leq n - 1$$
**Example.** Consider the graph \((G)\) given in Figure 2. Take \(W_1 = \{v_1, v_3\}\). So, \(r(v_1|W_1) = (0, 1)\), \(r(v_2|W_1) = (1, 1)\), \(r(v_3|W_1) = (1, 0)\), \(r(v_4|W_1) = (1, 1)\), \(r(v_5|W_1) = (2, 1)\). Notice, \(r(v_2|W_1) = (1, 1) = r(v_4|W_1)\), therefore \(W_1\) is not a resolving set. However, if we take \(W_2 = \{v_1, v_2\}\), then \(r(v_1|W_2) = (0, 1)\), \(r(v_2|W_2) = (1, 0)\), \(r(v_3|W_2) = (1, 1)\), \(r(v_4|W_2) = (1, 2)\), \(r(v_5|W_2) = (2, 1)\). Since distinct vertices have distinct metric representations, \(W_2\) is a minimum resolving set and thus this graph has metric dimension 2.
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A connected graph $G$ of order $n \geq 2$ has metric dimension $n - 1$ if and only if $G \cong K_n$. 
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For a connected graph \( G \) of order \( n \geq 3 \), the metric dimension of a cycle graph \( C_n \) is 2.
Figure: Pink Colored vertices correspond to metric basis
Theorem. \( \dim(\Gamma(R)) \) is finite if and only if \( R \) is finite and is undefined if and only if \( R \) is an integral domain.
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Theorem. If $R$ is a finite commutative ring and $\Gamma(R)$ is a regular graph, then $\dim(\Gamma(R)) = |\mathbb{Z}^*(R)| - 1$ or $|\mathbb{Z}^*(R)| - 2$. 
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Theorem. If \( R \) is a finite commutative ring and \( \Gamma(R) \) is a regular graph, then \( \dim(\Gamma(R)) = |Z^*(R)| - 1 \) or \( |Z^*(R)| - 2 \).

Theorem. The graph \( \Gamma(\mathbb{Z}_n) \) is Hamiltonian graph if and only if \( \dim(\Gamma(\mathbb{Z}_n)) = |Z^*(\mathbb{Z}_n)| - 1 \).
If $G$ is a connected graph of order $n \geq 2$, we say two distinct vertices $u$ and $v$ of $G$ are distance similar, if $d(u, x) = d(v, x)$ for all $x \in V(G) - \{u, v\}$.
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It can be easily checked that the distance similar relation ($\sim$) is an equivalence relation on $V(G)$. 

Theorem. Let $G$ be a connected graph partitioned into $k$ distinct distance similar classes $V_1, V_2, \ldots, V_k$, then $\dim(G) \geq |V(G)| - k$ and $|V(G)| - k \leq \dim(G) \leq |V(G)| - k + m$, where $m$ is the number of distance similar classes that consist of a single vertex.
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Theorem. For connected graph $G$ of order $n \geq 3$, the metric dimension of bipartite graph $K_{1,n-1}$ is $n-2$ and for for $r \geq 2, n \geq 5$, metric dimension of $K_{r,n-r} = K_{n,m}$ is $n-2$, with $n = r$ and $m = n - r$. 
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Theorem. Let $R$ be a finite commutative ring with 1 and odd characteristics and let $\Gamma(R)$ be partitioned into $k$ distance similar classes. Then $\dim(\Gamma(R)) = |Z^*(R)| - k$. 
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**Theorem.** Let $R$ be a finite commutative ring with 1 and odd characteristics and let $\Gamma(R)$ be partitioned into $k$ distance similar classes. Then $\dim(\Gamma(R)) = |Z^*(R)| - k$.

**Corollary.** Let $p$ be a prime number.
(i) If $n = 2p$, then $\dim(\Gamma(\mathbb{Z}_n)) = p - 2$
(ii) If $n = p^2$ and $p > 2$, then $\dim(\Gamma(\mathbb{Z}_n))p - 2$
(iii) If $n = p^k$ and $k \geq 3$, then $\dim(\Gamma(\mathbb{Z}_n)) = |Z^*(\mathbb{Z}_n)| - (k - 1) = p^{k-1} - k$. 
Lemma. Let $R$ be a commutative ring and let $x, y$ be adjacent vertices of $\Gamma(R)$. Then $|d(x, z) - d(y, z)| \leq 1$ for every vertex $z \in \Gamma(R)$. 

Theorem. Let $R$ be a commutative ring and $\Gamma(R)$ be the corresponding zero-divisor graph of $R$ such that $|Z^*(R)| \geq 2$. Then $\lceil \log_3(\Delta + 1) \rceil \leq \dim(\Gamma(R)) \leq |Z^*(R)| - d$, where $\Delta$ is the maximum degree and $d$ is the diameter of $\Gamma(R)$.

Corollary. Let $R$ be a commutative ring with unity 1 such that $\dim(\Gamma(R)) = k$ where $k$ is any non-negative integer. Then $|Z(R)| \leq 4^k + 1$. 
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Corollary. Let $R$ be a commutative ring with unity 1 such that $dim(\Gamma(R)) = k$ where $k$ is any non-negative integer. Then $|Z(R)| \leq 4^k + 1$. 
An ideal $I$ is a subring of $R$ such that $ar \in I$ for all $a \in I, r \in R$. 
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A ring \( R \) is said to be a local ring if it has a unique maximal ideal. Example \( \mathbb{Z}_9 \)
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A ring $R$ is called Boolean ring if $a^2 = a$ for every $a \in R$. 
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Theorem. Let $R$ be a finite commutative local ring with unity 1 and for prime $q$ let $\mathbb{F}_q$ be a finite field. Then

$$|\mathbb{Z}^*(R \times \mathbb{F}_q)| = |U(R)| + (|\mathbb{Z}^*(R)| + 1)q - 1.$$
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Theorem. Let $R$ be a finite commutative local ring with unity 1 and for prime $q$ let $\mathbb{F}_q$ be a finite field. Then
\[ |\mathbb{Z}^*(R \times \mathbb{F}_q)| = |U(R)| + (|\mathbb{Z}^*(R)| + 1)q - 1. \]

Example. Let
\[ \mathbb{F}_2[x]/(x^3, xy, y^2) = \{ ax^2 + bx + cy + d | a, b, c, d \in \mathbb{F}_2 \} \]
and let $\mathbb{F}_q = \mathbb{Z}_5$.
Clearly, $R$ is a local ring of order 16 with
\[ Z(R) = \{ 0, x, y, x^2, x+y, x+x^2, y+x^2, x+x^2+y \}. \]
Therefore,
\[ |\mathbb{Z}^*(R \times \mathbb{F}_q)| = |U(R)| + (|\mathbb{Z}^*(R)| + 1)q - 1 = 8 + (8)5 - 1 = 47. \]
Theorem. Let $R_1, R_2, \ldots, R_n$ be $n$ finite commutative rings (not domains) each having unity 1 with none of $R_i$, $1 \leq i \leq n$, being isomorphic to $\prod_{i=1}^{n} \mathbb{Z}_2$ for any positive integer $n$. Then for a commutative ring $R$ with unity 1 and for any prime field $\mathbb{F}_q$,
(a) $\dim(\Gamma(R_1 \times R_2 \times \cdots \times R_n)) \geq \sum_{i=1}^{n} \dim(\Gamma(R_i))$
(b) $\dim(R \times \mathbb{F}_q) = |Z^*(R \times \mathbb{F}_q)| - 2^{n+1} + 2$ or $|Z^*(R \times \mathbb{F}_q)| - 2$ or at least $|U(R)| + (|Z^*(R)| + 1)q - t + 3$, where $t$ is any positive integer.
**Example.** Consider the zero divisor graph shown in Fig.6 associated with the ring $\mathbb{Z}_8 \times \mathbb{Z}_2$. Partition the graph $\Gamma(\mathbb{Z}_8 \times \mathbb{Z}_2)$ into the distance similar equivalence classes which are given by sets, $\{(1,0), (3,0), (5,0), (7,0)\}$, $\{(2,1), (6,1)\}$, $\{(0,1)\}$, $\{(2,0), (6,0)\}$, $\{(4,0)\}$ and $\{(4,1)\}$. Therefore, by theorem, $\text{dim}(\Gamma(\mathbb{Z}_8 \times \mathbb{Z}_2)) \geq 4 + (4)2 - 4 - 3 = 5$. From Fig.6, it can be easily verified that $\{(1,0), (3,0), (5,0), (7,0), (2,0), (6,1)\}$ is the metric basis and consequently $\text{dim}(\Gamma(\mathbb{Z}_8 \times \mathbb{Z}_2)) = 6$.

**Figure:** $\text{dim}(\Gamma(\mathbb{Z}_8 \times \mathbb{Z}_2)) = 6$
**Theorem.** Let $R_1, R_2, \ldots, R_k$ be integral domains with 1 and $|R_i| > 2$ for some $i$. Then

\[ \dim(\Gamma(R_1 \times R_2 \times \cdots \times R_k)) = |Z(R)| - 2^k + 2. \]
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**Remark.** If $R \not\cong \prod_{i=1}^{n} \mathbb{Z}_2$, then $|W|$ is equal to this sum. In light of above theorem, one should consider the metric dimension of $\prod_{i=1}^{n} \mathbb{Z}_2$.

**Theorem.** For any $k \geq 2$, \[\text{dim}(\Gamma(\prod_{i=1}^{k} \mathbb{Z}_2)) \leq k.\]
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**Theorem.** For any $k \geq 2$, $\dim(\Gamma(\prod_{i=1}^{k} \mathbb{Z}_2)) \leq k$.

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dim(\Gamma(\prod_{i=1}^{n} \mathbb{Z}_2 = n - 1)) \text{ for } n = 2, 3, 4
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Remark. If $R \not\cong \prod_{i=1}^{n} \mathbb{Z}_2$, then $|\mathcal{W}|$ is equal to this sum. In light of above theorem, one should consider the metric dimension of $\prod_{i=1}^{n} \mathbb{Z}_2$.

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\[ \dim(\Gamma(\prod_{i=1}^{n} \mathbb{Z}_2 = n - 1)) \text{ for } n = 2, 3, 4 \]
\[ \dim(\Gamma(\prod_{i=1}^{5} \mathbb{Z}_2 = 5)) \]

For $n > 5$, the case is still open.
The cartesian product $\Gamma(R_1) \times \Gamma(R_2)$ of two zero divisor graphs $\Gamma(R_1)$ and $\Gamma(R_2)$ with vertex set as $Z^*(R_1) \times Z^*(R_2)$ and the edge set $\{ (x_1, x - 2) \mid x_1 = y_1 \text{ and } x_2 y_2 \in E(\Gamma(R_2)) \text{ or } x_2 = y_2 \text{ and } x_1 y_1 \in E(\Gamma(R_1)) \}$
The cartesian product $\Gamma(R_1) \times \Gamma(R_2)$ of two zero divisor graphs $\Gamma(R_1)$ and $\Gamma(R_2)$ with vertex set as $\mathbb{Z}^*(R_1) \times \mathbb{Z}^*(R_2)$ and the edge set $\{(x_1, x-2) \mid x_1 = y_1 \text{ and } x_2y_2 \in E(\Gamma(R_2)) \text{ or } x_2 = y_2 \text{ and } x_1y_1 \in E(\Gamma(R_1))\}$

**Theorem.** If $R$ is any finite commutative ring with unity 1 (not a domain), then

$$\dim(\Gamma(R)) \leq \dim(\Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) \leq \dim(\Gamma(R)) + 1$$
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**Theorem.** If $R$ is any finite commutative ring with unity 1 (not a domain), then

$$dim(\Gamma(R)) \leq dim(\Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) \leq dim(\Gamma(R)) + 1$$

. The equality is possible for either bound in the above theorem is possible. For example, $dim(R \cong \mathbb{Z}_3 \times \mathbb{Z}_3) = 2$, where as the $dim(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = 3$. 
Girth, diameter and metric dimension

**Theorem.** Let $R$ be a finite commutative ring with $gr(\Gamma(R)) = \infty$.

$$dim(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = dim(\Gamma(\mathbb{Z}_9)) = dim(\Gamma(\mathbb{Z}_3[x]/(x^3))) = 1.$$  

If $R$ is reduced and $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $dim(\Gamma(R)) = |\mathbb{Z}^*(R)| - 2$.  
If $R \cong \mathbb{Z}_4)$ or $\mathbb{Z}_2[x]/(x^3)$, then $dim(\Gamma(R)) = 0$.  


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If $R$ is reduced and $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $dim(\Gamma(R)) = |\mathbb{Z}^*(R)| - 2$.

If $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^3)$, then $dim(\Gamma(R)) = 0$.

**Theorem.** If $R$ is reduced Artinian commutative ring and $gr(\Gamma(R)) = 4$, then $dim(\Gamma(R)) = |\mathbb{Z}^*(R)| - 2$ and $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, where each $\mathbb{F}_i$ is a field with $\mathbb{F}_i \geq 3$. 
Theorem. Let \( R \) be a commutative ring. Then

(i) \( \text{diam}(\Gamma(R)) = 0 \) if and only if \( \text{dim}(\Gamma(R)) = 0 \).

(ii) \( \text{diam}(\Gamma(R)) = 1 \) if and only if \( \text{dim}(\Gamma(R)) = |\mathbb{Z}^*(R)| - 1 \).

(iii) \( \text{dim}(\Gamma(R)) = 1 \) or \( |\mathbb{Z}^*(R)| - 2 \) if \( R \cong \mathbb{F}_1 \times \mathbb{F}_2 \).
**Compressed zero divisor graph**
Let $R$ be a commutative ring with $1 \neq 0$.

A zero-divisor graph determined by equivalence classes or simply a compressed zero-divisor graph of a ring $R$ is the undirected graph $\Gamma_E(R)$ with vertex set $Z(R_E) \setminus \{[0]\} = R_E \setminus \{[0], [1]\}$ defined by $R_E = \{[x] : x \in R\}$, where $[x] = \{y \in R : \text{ann}(x) = \text{ann}(y)\}$ and the two distinct vertices $[x]$ and $[y]$ of $Z(R_E)$ are adjacent if and only if $[x][y] = [xy] = [0]$, that is, if and only if $xy = 0$. 
Example Consider $R = \mathbb{Z}_{12}$. For the vertex set of $\Gamma_E(R)$, we have $\text{ann}(2) = \{6\}$, $\text{ann}(3) = \{4, 8\}$, $\text{ann}(4) = \{3, 6, 9\}$, $\text{ann}(6) = \{2, 4, 6, 8, 10\}$, $\text{ann}(8) = \{3, 6, 9\}$, $\text{ann}(9) = \{4, 8\}$, $\text{ann}(10) = \{6\}$.

So, $Z(R_E) = \{[2], [3], [4], [6]\}$ is the vertex set of $\Gamma_E(R)$.

Graph of equivalence classes of zero divisors of $R$, $\Gamma_E(R)$.

Figure: $\Gamma_E(\mathbb{Z}_{12})$
Observations: [3, 6] For realizable graphs $\Gamma_E(R)$:

- $\Gamma_E(R)$ is connected, $\text{diam}(\Gamma_E(R)) \leq 3$ and $\text{gr}(\Gamma_E(R)) \leq 3$.
- $\text{diam}(\Gamma_E(R)) \leq \text{diam}(\Gamma(R))$.
- If $|\Gamma_E(R)| \geq 3$, then $\Gamma_E(R)$ is not complete.
- If $\Gamma_E(R)$ is complete $r$-partite, then $r = 2$ and $\Gamma_E(R) = K_n^1$.
- $\Gamma_E(R)$ is not a cycle graph $C_n$.
- $\Gamma_E(R)$ may be finite when $\Gamma(R)$ is infinite.
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In contrast to the zero divisor graph, the authors [6] pose the question of whether, given a positive integer $n$, the graph $K_{1,n}$ can be realized as $\Gamma_E(R)$ for some ring $R$. 

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An *Associated Prime*, $P$ is a prime ideal of $R$ such that $P = \text{ann}(x)$ for some $x \in R$. 

The associated primes of $R$ correspond to vertices in $\Gamma_{E}(R)$. 

A ring $R$ satisfies the a.c.c for ideals if given any sequence of ideals $I_1, I_2, \ldots$ of $R$ with $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ there exists an integer $m$ such that $I_m = I_n$ for all $m \geq n$. 

A ring in which every ascending chain holds for right (left) as well as left ideals is called a *Noetherian Ring*. 

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**Theorem** If $R$ is a finite local ring with unity 1 and $\mathbb{F}_q$ is a finite prime filed, then

$$|Z^*(R \times \mathbb{F}_q)_E| = 2k \text{ or } 2(1 + |Z^*(R_E)|)$$

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References


S. Pirzada, An Introduction to Graph Theory, University Press, Orient Blackswan, India, 2012.


Thanks for your attention