Delta Set of Affine Semigroups

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This is a work collecting some ideas published in the next three papers:

- García-Sánchez P.A., Ll. D., Moscariello A. *Delta sets for nonsymmetric numerical semigroups with embedding dimension three.* **Forum Math. 30** pp. 15-30 (2018).
- García-Sánchez P.A., Ll. D., Moscariello A. *Delta sets for symmetric numerical semigroups with embedding dimension three.* **Aequat. Math. 91** pp. 579-600 (2017).
- García-Sánchez P.A., O'Neill Ch., Webb G. *On the computation of factorization invariants for affine semigroups* **J. Algebra & Appl.** to appear.

Congruences and minimal presentations

$S = \langle s_1, \ldots, s_n \rangle \subset \mathbb{N}^d$

is an *affine semigroup* generated by $s_1, \ldots, s_n$ if

$$S = \{z_1 s_1 + \cdots + z_n s_n \mid z_1, \ldots, z_n \in \mathbb{N} \cup \{0\}\}.$$

When $d = 1$ and $gcd(s_1, \ldots, s_n) = 1$ we say a numerical semigroup.

$\varphi_S : \mathbb{N}^n \to S\,;\, \varphi(z_1, \ldots, z_n) = z_1 s_1 + \cdots + z_n s_n$

is surjective and

- $S \cong \mathbb{N}^n / \ker \varphi_S$, where
  $\ker \varphi_S = \{(\mathbf{z}, \mathbf{z}') \in \mathbb{N}^n \times \mathbb{N}^n : \varphi_S(\mathbf{z}) = \varphi_S(\mathbf{z}')\}$.
- $\ker \varphi_S$ is a *congruence* on $\mathbb{N}^n$.

Congruences and minimal presentations

$\ker \varphi_S$

- Is finitely generated and
- a subset $\rho \subset \ker \varphi_S$ that generates $\ker \varphi$ which is minimal w.r.t. containment is called *minimal presentation* of S.
- This minimal presentation need not be unique

Example 1: $S = \langle 3, 5, 7 \rangle$ a numerical semigroup

$\rho = \{((1,0,1),(0,2,0)),((4,0,0),(0,1,1)),((3,1,0),(0,0,2))\}$
In this case ρ is unique.

Example 2: $S = \langle 4, 6, 11 \rangle$ a numerical semigroup

$\rho = \{((1,3,0),(0,0,2)),((3,0,0),(0,2,0))\}$.
$\rho' = \{((4,1,0),(0,0,2)),((3,0,0),(0,2,0))\}$
In this case, there exist several minimal presentations.

Betti Elements

S an affine semigroup

- The *Betti elements* of S are the elements $\varphi_S(\mathbf{z})$ for $(\mathbf{z}, \mathbf{z}') \in \rho$
- This definition is independent of the minimal presentation chosen

Examples

- $S = \langle 3, 5, 7\rangle$; then $\mathrm{Betti}(S) = \{10, 12, 14\}$ from
$$\rho = \{((1,0,1),(0,2,0)),((4,0,0),(0,1,1)),((3,1,0),(0,0,2))\}.$$
- $S = \langle 4, 6, 11\rangle$; then $\mathrm{Betti}(S) = \{12, 22\}$ from
$$\rho = \{((1,3,0),(0,0,2)),((3,0,0),(0,2,0))\} \text{ or}$$
$$\rho' = \{((4,1,0),(0,0,2)),((3,0,0),(0,2,0))\}.$$

Factorizations

Factorizations of an element $s \in S = \langle s_1, \dots, s_n \rangle$

$$Z(s) = \{(z_1, \dots, z_n) \in \mathbb{N}^n, | \text{ with } s = z_1 s_1 + \cdots + z_n s_n\}$$

Length of a factorization $\mathbf{z} = (z_1, \dots, z_n)$

$$\ell(\mathbf{z}) = z_1 + \cdots + z_n$$

Set of lengths of factorizations of $s \in S$

$$\mathrm{L}(s) = \{\ell(\mathbf{z}) \mid \mathbf{z} \in Z(s)\}$$. For each $s \in S$

Betti elements

Betti elements are those whose factorizations are used to build minimal presentations

Factorizations and Delta Set

$S =< 3, 5, 7 >= \{0, 3, 5, 6, 7, 8, 9, 10, 11, \dots\}$

Elements $0, 3, 5, 6, 7, 8, 9, 11,$ have only one factorization. But:

$$\begin{array}{lll} \mathsf{Z}(10) & = & \{(1,0,1),(0,2,0)\} \quad \mathsf{L}(10) = \{2\} \\ \mathsf{Z}(12) & = & \{(4,0,0),(0,1,1)\} \quad \mathsf{L}(12) = \{2,4\} \\ \mathsf{Z}(14) & = & \{(3,1,0),(0,0,2)\} \quad \mathsf{L}(14) = \{2,4\} \end{array}$$

$\mathsf{Z}(30) = \{(10,0,0),(5,3,0),(0,6,0),(6,1,1),(1,4,1),(2,2,2),(3,0,3)\}$
$\mathsf{L}(30) = \{6, 8, 10\}$

Delta Set

We order the set $\mathrm{L}(s)$ which always is finite

$$\mathsf{L}(s) = \{l_1 < l_2 < \cdots < l_n\}$$

And define the Delta sets as:

- $\Delta(s) = \{l_i - l_{i-1} \mid i = 2, \dots n\}$.
- $\Delta(S) = \cup_{s \in S} \Delta(s)$. We going to focus in the set $\Delta(S)$.

Factorization and Delta Sets

Geroldinger 1991. Let S be a numerical semigroup, then:

$$\min \Delta(S) = \gcd \Delta(S).$$

Set $d = \gcd \Delta(S)$. There exists $k \in \mathbb{N} \setminus \{0\}$ such that

$$\Delta(S) \subseteq \{d, 2d, \ldots, kd\}.$$

Chapman,García-Sánchez,Ll,Malyshev,Steinberg 2013.

For ρ a minimal presentation of S we have:

$$\min \Delta(S) = \gcd\{\left|\ell(\mathbf{z}) - \ell(\mathbf{z}')\right| : (\mathbf{z}, \mathbf{z}') \in \rho\}$$

$$\max \Delta(S) = \max\{\max \Delta(s) : s \in \mathrm{Betti}(S)\}.$$

Numerical Semigroups embedding dimension 3

The M_S group associated to $S = \langle s_1, s_2, s_3 \rangle$

- $M_S = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1 s_1 + x_2 s_2 + x_3 s_3 = 0\}$.
- $\mathbf{v}_1 = (4, -1, -1)$ and $\mathbf{v}_2 = (3, 1, -2)$ span M_S as a group.
- $\delta_1 = \ell(\mathbf{v}_1) = 2$ and $\delta_2 = \ell(\mathbf{v}_2) = 2$. And $\Delta(S) = \{2\}$.

For δ_1 and δ_2 coprime integer, define

$\eta_1 = \max\{\delta_1, \delta_2\}$, $\eta_2 = \min\{\delta_1, \delta_2\}$ and $\eta_{i+2} = \eta_i \bmod \eta_{i+1}$. Euclid's algorithm.

The Euclid's set, $\mathrm{Euc}(\delta_1, \delta_2)$, can be constructed as

$\mathrm{D}(\eta_1, \eta_2) = \{\eta_1, \eta_1 - \eta_2, \ldots, \eta_1 \bmod \eta_2 = \eta_3\}$,
$\mathrm{D}(\eta_2, \eta_3) = \{\eta_2, \eta_2 - \eta_3, \ldots, \eta_2 \bmod \eta_3 = \eta_4\}$,
$\mathrm{D}(\eta_3, \eta_4) = \{\eta_3 - \eta_4, \ldots, \eta_3 \bmod \eta_4 = \eta_5\}$,
$\ldots$
$\mathrm{D}(\eta_i, \eta_{i+1}) = \{\eta_i - \eta_{i+1}, \ldots, \eta_i \bmod \eta_{i+1} = \eta_{i+2} = 0\}$.
Then:

$$\mathrm{Euc}(\delta_1, \delta_2) = \bigcup_{i \in I} \mathrm{D}(\eta_i, \eta_{i+1}).$$

Numerical Semigroups embedding dimension 3

Theorem

For $S = \langle n_1, n_2, n_3 \rangle$ we have:

$$\Delta(S) = \mathrm{Euc}(\delta_1, \delta_2)$$

In this case $\sharp\mathrm{Betti}(S) = 1, 2$ or 3

- For $\sharp\mathrm{Betti}(S) = 3$. ρ is unique. And all elements have 2 factorizations. Then: $\mathbf{v}_1 = (c_1, -r_{12}, -r_{13})$ and $\mathbf{v}_2 = (r_{31}, r_{32}, -c_3)$.
- For $\sharp\mathrm{Betti}(S) = 2$. ρ is not unique. But we can choose: $\mathbf{v}_1 = (m_2, -m_1, 0)$ and $\mathbf{v}_2 = (b + \lambda m_2, c - \lambda m_1, -a)$.
- For $\sharp\mathrm{Betti}(S) = 1$. ρ is not unique. But we can choose: $\mathbf{v}_1 = (s_1, -s_2, 0)$ and $\mathbf{v}_2 = (0, s_2, -s_3)$.

Numerical Semigroups embedding dimension 3

Example: $S = \langle 1407, 26962, 35413 \rangle$

$\mathbf{v}_1 = (411, -7, -11)$, $\mathbf{v}_2 = (127, 84, -69)$, and so: $\delta_1 = 393$, $\delta_2 = 142$.

$\mathbf{v}_1$=(411,−7,−11) 393	$\mathbf{v}_1-\mathbf{v}_2$=(284,−91,58) 251	$\mathbf{v}_1-2\mathbf{v}_2$=(157,−175,127)=$\mathbf{v}_3$ $109 = \eta_3$	
$\mathbf{v}_2$=(127,84,−69) 142	$\mathbf{v}_2-\mathbf{v}_3$=(−30,259,−196)=$\mathbf{v}_4$ $33 = \eta_4$		
$\mathbf{v}_3$=(157,−175,127) 109	(187,−434,323) 76	(217,−693,519) 43	(247,−952,715) $10 = \eta_5$
(−30,259,−196) 33	(−277,1211,−911) 23	(−524,2163,−1626) 13	(−771,3115,−2341) $3 = \eta_6$
(247,−952,715) 10	(1018,−4067,3056) 7	(1789,−7182,5397) 4	(2560,−10297,7738) $1 = \eta_7$
(−771,3115,−2341) 3	(−3331,13412,−10079) 2	(−5891,23709,−17817) 1	(−8451,34006,−25555) $0 = \eta_8$

$$\Delta(S) = \{1, 2, 3, 4, 7, 10, 13, 23, 33, 43, 76, 109, 142, 251, 393\}.$$

Higher dimensions in numerical semigroups

Colton and Kaplan example

They show, in this example, that we can not generalize the results to higher dimensions.

$$S = \langle 14, 29, 30, 32, 36\rangle$$

$$\Delta(S) = \{1, 4\}$$

If we try to extrapolate our results, necessarily $\{2, 3\}$ must be contained in the Delta set of $\langle 14, 29, 30, 32, 36\rangle$.

A Generalization to affine semigroups

O'Neill Ch.(2017) A first Theorem:

For $S = \langle s_1, \dots, s_n \rangle$, consider the ideals:

$I_j = \langle y^{\mathbf{z}} - y^{\mathbf{z}'} : (\mathbf{z}, \mathbf{z}') \in \mathsf{Z}(s), s \in S,$ and $|\ell(\mathbf{z}) - \ell(\mathbf{z}')| \leq j \rangle \subset \mathbb{K}[y_1, \dots, y_n] \rangle$

Then $j \in \Delta(S)$ if and only if $I_{j-1} \subset I_j$ strictly.
We denote $y^{\mathbf{z}}$ for the monomial $y_1^{z_1} \cdots y_n^{z_n}$

The trick: homogenize

The new variable stores lengths, and splits factorizations in layers of factorizations with the same length
For $S_H = \langle (1,0), (1, s_1), \dots, (1, s_n) \rangle \subset \mathbb{N}^{d+1}$ consider
$I_{S_H} = \langle t^{\ell(\mathbf{z}) - \ell(\mathbf{z}')} y^{\mathbf{z}} - y^{\mathbf{z}'} : \mathbf{z}, \mathbf{z}' \in \mathsf{Z}(s), s \in S, \ell(\mathbf{z}) \leq \ell(\mathbf{z}') \rangle \subset \mathbb{K}[t, y_1, \dots, y_n] \rangle$
For G a reduced Groebner basis of I_{S_H} with some order satisfying $t > y_i$ then

$$J_j = \langle y^{\mathbf{z}} - y^{\mathbf{z}'} : t^i y^{\mathbf{z}} - y^{\mathbf{z}'} \in G, i \leq j \rangle = I_j$$

Algorithm

function DeltaSetOfAffineSemigroup(S)

$\rho \leftarrow$ minimal presentation of S_H

G $\leftarrow$ reduced lex Groebner basis for $\langle t^i y^{\mathbf{z}} - t^j y^{\mathbf{z}'} : ((i,\mathbf{z}),(j,\mathbf{z}')) \in \rho \rangle$

return $\{j : t^j y^{\mathbf{z}} - y^{\mathbf{z}'} \in G\}$

Return to $S = \langle 1407, 26962, 35413 \rangle$

$\rho = \{((411;411,0,0),(18;0,7,11)),((211;127,84,0),(69;0,0,69)),((342;284,0,58),(91;0,91,0))\}$.

$\langle t^{411}x^{411} - t^{18}y^7z^{11}, t^{211}x^{127}y^{84} - t^{69}z^{69}, t^{342}x^{284}z^{58} - t^{91}y^{91}\rangle$, with $t > x > y > z$.

$\langle t^{393}x^{411} - y^7z^{11}, t^{142}x^{127}y^{84} - z^{69}, t^{251}x^{284}z^{58} - y^{91}\rangle = \langle f_1, f_2, f_3\rangle$

$S(f_1,f_2) = \frac{t^{393}x^{411}y^{84}}{t^{393}x^{411}} f_1 - \frac{t^{393}x^{411}y^{84}}{t^{142}x^{127}y^{84}} f_2 = t^{251}x^{284}z^{69} - y^{91}z^{11} \rightarrow t^{251}x^{284}z^{58} - y^{91} = f_3$

$S(f_3,f_2) = \frac{t^{251}x^{284}y^{84}z^{58}}{t^{251}x^{284}z^{58}} f_3 - \frac{t^{251}x^{284}y^{84}z^{58}}{t^{142}x^{127}y^{84}} f_2 = t^{109}x^{157}z^{127} - y^{175} = f_4$

$S(f_2,f_4) = \frac{t^{251}x^{284}z^{127}}{t^{109}x^{157}z^{58}} f_2 - \frac{t^{251}x^{284}y^{84}z^{58}}{t^{142}x^{127}y^{84}} f_4 = t^{109}x^{157}z^{127} - y^{175} = f_4$

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$$S(f_1,f_2) = \frac{t^{393}x^{411}y^{84}}{t^{393}x^{411}} f_1 - \frac{t^{393}x^{411}y^{84}}{t^{142}x^{127}y^{84}} f_2 = t^{251}x^{284}z^{69} - y^{91}z^{11} \to t^{251}x^{284}z^{58} - y^{91} = f_3$$

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... And, so on

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$\langle t^{411}x^{411} - t^{18}y^7z^{11}, t^{211}x^{127}y^{84} - t^{69}z^{69}, t^{342}x^{284}z^{58} - t^{91}y^{91} \rangle$, with $t > x > y > z$.
$\langle t^{393}x^{411} - y^7z^{11}, t^{142}x^{127}y^{84} - z^{69}, t^{251}x^{284}z^{58} - y^{91} \rangle = \langle f_1, f_2, f_3 \rangle$.

$$S(f_1, f_2) = \frac{t^{393}x^{411}y^{84}}{t^{393}x^{411}} f_1 - \frac{t^{393}x^{411}y^{84}}{t^{142}x^{127}y^{84}} f_2 = t^{251}x^{284}z^{69} - y^{91}z^{11} \rightarrow t^{251}x^{284}z^{58} - y^{91} = f_3$$

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Thanks for your attention!!