Counting semistar operations on Prüfer domains

E. Houston, A. Mimouni, and M.H. Park

Better title:

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Attempting to count semistar operations on Prüfer domains.

- Semistar operations (on Prüfer domains)

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- Moore families and PIDs

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- The *h*-local case
- The non-*h*-local case

Definition. Let R be an integral domain with quotient field K, and let $\bar{\mathcal{F}}(R)$ denote the set of nonzero R-submodules of K. A *semistar operation* on R is a map $\star : \bar{\mathcal{F}}(R) \to \bar{\mathcal{F}}(R)$ such that

1. $A \subseteq A^\star$ and $A^\star \subseteq B^\star$ whenever $A \subseteq B$ for all $A, B \in \bar{\mathcal{F}}(R)$.
2. $A^{\star\star} = A^\star$ for all $A \in \bar{\mathcal{F}}(R)$.
3. $(uA)^\star = uA^\star$ for all $u \in K$ and $A \in \bar{\mathcal{F}}(R)$.

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Theorem (Matsuda). An integrally closed domain admits only finitely many semistar operations if and only if it is a Prüfer domain with only finitely many prime ideals.

Therefore, unless otherwise stated, R will denote a Prüfer domain with finitely many prime ideals. The simplest case is that of a valuation domain:

Theorem (Matsuda). If R is an n-dimensional valuation domain, then $|\mathrm{SStar(R)}| = \mathrm{n}+1+\mathrm{m}$, where m is the number of idempotent primes of R. In particular, if R is discrete (no nonzero idempotent prime ideals), then $|\mathrm{SStar(R)}| = \mathrm{n}+1$.

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It is a great understatement to say that things become more complicated in the nonlocal case.

Definition. For a set X, a set of subsets of 2^X is a *Moore family on X* if it is closed under arbitrary intersections. The set of all Moore families is denoted by $\mathrm{Moore(X)}$.

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"Proof": For $\mathcal{Y} \in \mathrm{Moore(Max(R))}$ and $A \in \bar{\mathcal{F}}(R)$, set

$$A^{\star_{\mathcal{Y}}} = \bigcap \{(J : (J : A)) \mid J \in \bar{F}(R) \text{ and } \{M \in \mathrm{Max(R)} \mid \mathrm{JR_M = K}\} \in \mathcal{Y}\}.$$

The map $\mathcal{Y} \mapsto \star_{\mathcal{Y}}$ works.

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	R	R_{M_1}	R_{M_2}	K	$\mathcal{Y}$
$\star_1$	K	K	K	K	$\{\{M_1, M_2\}\}$
$\star_2$	R_{M_1}	R_{M_1}	K	K	$\{\{M_2\}, \{M_1, M_2\}\}$
$\star_3$	R_{M_2}	K	R_{M_2}	K	$\{\{M_1\}, \{M_1, M_2\}\}$
$\star_4$	R	R_{M_1}	K	K	$\{\emptyset, \{M_2\}, \{M_1, M_2\}\}$
$\star_5$	R	R_{M_1}	R_{M_2}	K	$\{\emptyset, \{M_2\}, \{M_1\}, \{M_1, M_2\}\}$
$\star_6$	R	K	R_{M_2}	K	$\{\emptyset, \{M_1\}, \{M_1, M_2\}\}$
$\star_7$	R	K	K	K	$\{\emptyset, \{M_1, M_2\}\}$

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Admission. Okay, I don't have a proof. In the rest of the talk some evidence for the truth of the conjecture will emerge.

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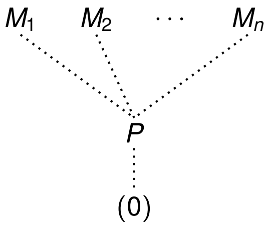

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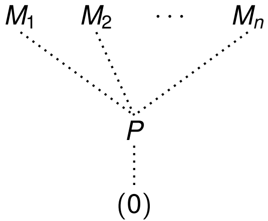

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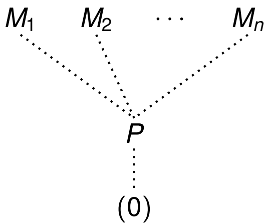

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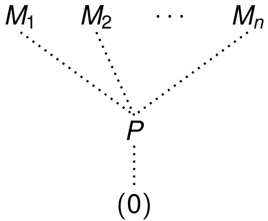

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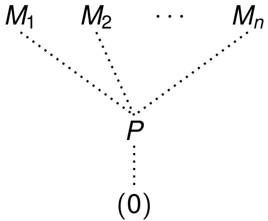

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Remark. One can prove the "in particular" case above by slightly modifying the Elliott map, essentially by replacing K by R_P.

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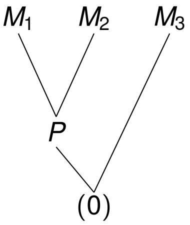

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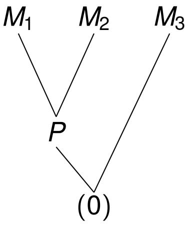

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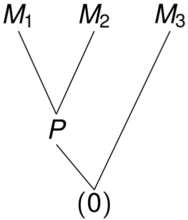
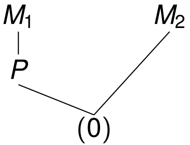


Question. For the R above, what is $|\mathrm{SStar}(R)|$?
Answer: I don't know, but it is much greater than 61.

Consider R with the following spectrum (all 3 nonzero primes non-idempotent):



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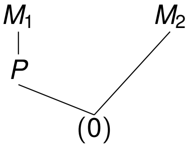

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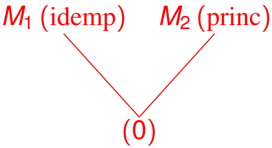

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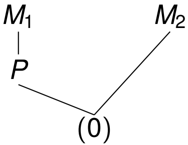

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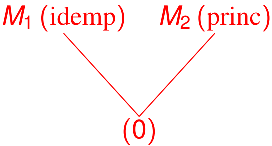

Theorem (Matsuda/HMP). We have $|\mathrm{SStar}(R)| = |\mathrm{SStar}(R)| = 14.$

In fact, here is a description of $\mathrm{SStar}(R)$ (first case):

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	R	R_{M_1}	R_{M_2}	$R_P \cap R_{M_2}$	R_P
$\star_1$	K	K	K	K	K
$\star_2$	R_{M_2}	K	R_{M_2}	R_{M_2}	K
$\star_3$	R_P	R_P	K	R_P	R_P
$\star_4$	$R_P \cap R_{M_2}$	K	K	$R_P \cap R_{M_2}$	K
$\star_5$	$R_P \cap R_{M_2}$	K	R_{M_2}	$R_P \cap R_{M_2}$	K
$\star_6$	$R_P \cap R_{M_2}$	R_P	K	$R_P \cap R_{M_2}$	R_P
$\star_7$	$R_P \cap R_{M_2}$	R_P	R_{M_2}	$R_P \cap R_{M_2}$	R_P
$\star_8$	R	K	K	$R_P \cap R_{M_2}$	K
$\star_9$	R	K	R_{M_2}	$R_P \cap R_{M_2}$	K
$\star_{10}$	R	R_P	K	$R_P \cap R_{M_2}$	R_P
$\star_{11}$	R	R_P	R_{M_2}	$R_P \cap R_{M_2}$	R_P
$\star_{12}$	R_{M_1}	R_{M_1}	K	R_P	R_P
$\star_{13}$	R	R_{M_1}	K	$R_P \cap R_{M_2}$	R_P
$\star_{14}$	R	R_{M_1}	R_{M_2}	$R_P \cap R_{M_2}$	R_P

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	M_1	$M_1 R_{M_1}$	R_{M_2}	R	R_{M_1}
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$\star_2$	R_{M_2}	K	R_{M_2}	R_{M_2}	K
$\star_3$	R_{M_1}	R_{M_1}	K	R_{M_1}	R_{M_1}
$\star_4$	R	K	K	R	K
$\star_5$	R	K	R_{M_2}	R	K
$\star_6$	R	R_{M_1}	K	R	R_{M_1}
$\star_7$	R	R_{M_1}	R_{M_2}	R	R_{M_1}
$\star_8$	M_1	K	K	R	K
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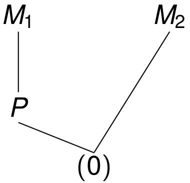
$\star_{11}$	M_1	R_{M_1}	R_{M_2}	R	R_{M_1}
$\star_{12}$	$M_1 R_{M_1}$	$M_1 R_{M_1}$	K	R_{M_1}	R_{M_1}
$\star_{13}$	M_1	$M_1 R_{M_1}$	K	R	R_{M_1}
$\star_{14}$	M_1	$M_1 R_{M_1}$	R_{M_2}	R	R_{M_1}

We repeat: Consider R with the following spectrum (all 3 nonzero primes non-idempotent):



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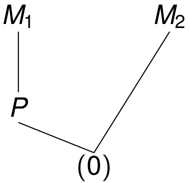

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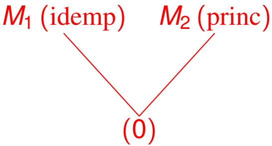

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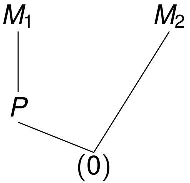

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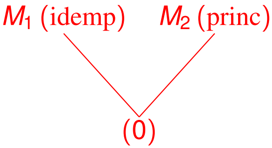

Theorem (Matsuda/HMP). We have $|\mathrm{SStar(R)}| = |\mathrm{SStar(R)} = 14.$

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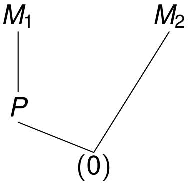

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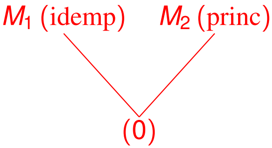

Theorem (Matsuda/HMP). We have $|\mathrm{SStar}(R)| = |\mathrm{SStar}(R) = 14$. Moreover, the lattices $\mathrm{SStar}(R)$ and $\mathrm{SStar}(R)$ are (extremely) isomorphic.

Proof:

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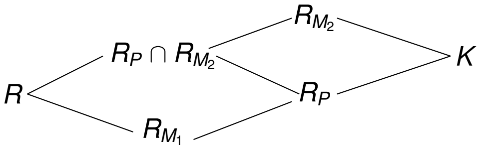
Lattice for $\mathcal{F}(\bar{R})$:



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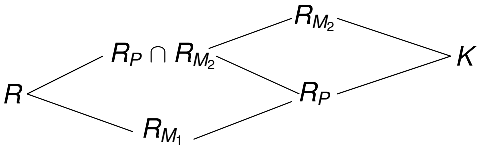

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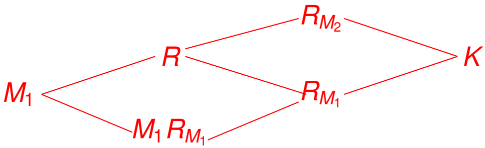

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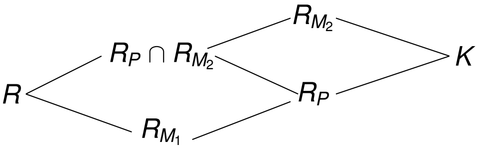

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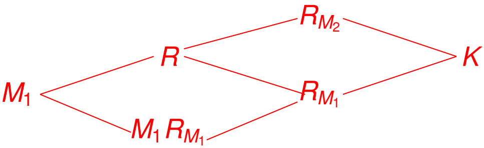

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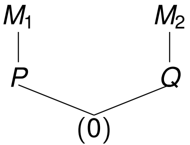
R with the following spectrum (all 4 nonzero primes non-idempotent):



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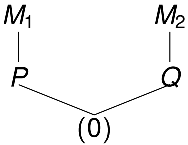

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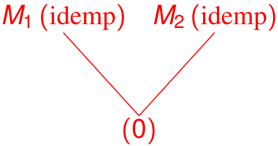

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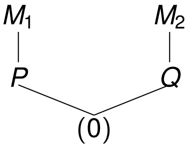

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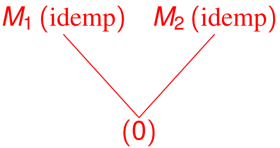

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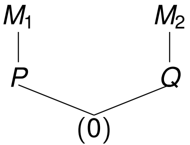

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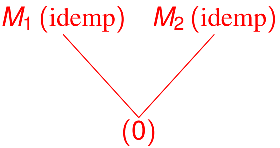

Theorem. $|\mathrm{SStar}(R)| = |\mathrm{SStar}(R)$, and the semistar lattices are the same.

Proof:

	R	$R_Q \cap R_{M_1}$	R_{M_1}	$R_P \cap R_{M_2}$	R_{M_2}	R_P	$R_P \cap R_Q$	R_Q
$\star_1$	K	K	K	K	K	K	K	K
$\star_2$	R_P	R_P	R_P	R_P	K	R_P	R_P	K
$\star_3$	R_{M_1}	R_{M_1}	R_{M_1}	R_P	K	R_P	R_P	K
$\star_4$	R_Q	R_Q	K	R_Q	R_Q	K	R_Q	R_Q
$\star_5$	$R_P \cap R_Q$	$R_P \cap R_Q$	K	$R_P \cap R_Q$	K	K	$R_P \cap R_Q$	K
$\star_6$	$R_Q \cap R_{M_1}$	$R_Q \cap R_{M_1}$	K	$R_P \cap R_Q$	K	K	$R_P \cap R_Q$	K
$\star_7$	$R_P \cap R_Q$	$R_P \cap R_Q$	R_P	$R_P \cap R_Q$	K	R_P	$R_P \cap R_Q$	K
$\star_8$	$R_Q \cap R_{M_1}$	$R_Q \cap R_{M_1}$	R_P	$R_P \cap R_Q$	K	R_P	$R_P \cap R_Q$	K
$\star_9$	$R_Q \cap R_{M_1}$	$R_Q \cap R_{M_1}$	R_{M_1}	$R_P \cap R_Q$	K	R_P	$R_P \cap R_Q$	K
$\star_{10}$	$R_P \cap R_Q$	$R_P \cap R_Q$	K	$R_P \cap R_Q$	R_Q	K	$R_P \cap R_Q$	R_Q
$\star_{11}$	$R_Q \cap R_{M_1}$	$R_Q \cap R_{M_1}$	K	$R_P \cap R_Q$	R_Q	K	$R_P \cap R_Q$	R_Q
$\star_{12}$	$R_P \cap R_Q$	$R_P \cap R_Q$	R_P	$R_P \cap R_Q$	R_Q	R_P	$R_P \cap R_Q$	R_Q
$\star_{13}$	$R_Q \cap R_{M_1}$	$R_Q \cap R_{M_1}$	R_P	$R_P \cap R_Q$	R_Q	R_P	$R_P \cap R_Q$	R_Q
$\star_{14}$	$R_Q \cap R_{M_1}$	$R_Q \cap R_{M_1}$	R_{M_1}	$R_P \cap R_Q$	R_Q	R_P	$R_P \cap R_Q$	R_Q
$\star_{15}$	$R_P \cap R_{M_2}$	$R_P \cap R_Q$	K	$R_P \cap R_{M_2}$	K	K	$R_P \cap R_Q$	K
$\star_{16}$	R	$R_Q \cap R_{M_1}$	K	$R_P \cap R_{M_2}$	K	K	$R_P \cap R_Q$	K
$\star_{17}$	$R_P \cap R_{M_2}$	$R_P \cap R_Q$	R_P	$R_P \cap R_{M_2}$	K	R_P	$R_P \cap R_Q$	K
$\star_{18}$	R	$R_Q \cap R_{M_1}$	R_P	$R_P \cap R_{M_2}$	K	R_P	$R_P \cap R_Q$	K
$\star_{19}$	R	$R_Q \cap R_{M_1}$	R_{M_1}	$R_P \cap R_{M_2}$	K	R_P	$R_P \cap R_Q$	K
$\star_{20}$	$R_P \cap R_{M_2}$	$R_P \cap R_Q$	K	$R_P \cap R_{M_2}$	R_Q	K	$R_P \cap R_Q$	R_Q
$\star_{21}$	R	$R_Q \cap R_{M_1}$	K	$R_P \cap R_{M_2}$	R_Q	K	$R_P \cap R_Q$	R_Q
$\star_{22}$	$R_P \cap R_{M_2}$	$R_P \cap R_Q$	R_P	$R_P \cap R_{M_2}$	R_Q	R_P	$R_P \cap R_Q$	R_Q
$\star_{23}$	R	$R_Q \cap R_{M_1}$	R_P	$R_P \cap R_{M_2}$	R_Q	R_P	$R_P \cap R_Q$	R_Q
$\star_{24}$	R	$R_Q \cap R_{M_1}$	R_{M_1}	$R_P \cap R_{M_2}$	R_Q	R_P	$R_P \cap R_Q$	R_Q
$\star_{25}$	R_{M_2}	R_Q	K	R_{M_2}	R_{M_2}	K	R_Q	R_Q
$\star_{26}$	$R_P \cap R_{M_2}$	$R_P \cap R_Q$	K	$R_P \cap R_{M_2}$	R_{M_2}	K	$R_P \cap R_Q$	R_Q
$\star_{27}$	R	$R_Q \cap R_{M_1}$	K	$R_P \cap R_{M_2}$	R_{M_2}	K	$R_P \cap R_Q$	R_Q
$\star_{28}$	$R_P \cap R_{M_2}$	$R_P \cap R_Q$	R_P	$R_P \cap R_{M_2}$	R_{M_2}	R_P	$R_P \cap R_Q$	R_Q
$\star_{29}$	R	$R_Q \cap R_{M_1}$	R_P	$R_P \cap R_{M_2}$	R_{M_2}	R_P	$R_P \cap R_Q$	R_Q
$\star_{30}$	R	$R_Q \cap R_{M_1}$	R_{M_1}	$R_P \cap R_{M_2}$	R_{M_2}	R_P	$R_P \cap R_Q$	R_Q

	$M_1 M_2$	M_1	$M_1 R_{M_1}$	M_2	$M_2 R_{M_2}$	R_{M_1}	R	R_{M_2}
$\star_1$	K	K	K	K	K	K	K	K
$\star_2$	R_P	R_P	R_P	R_{M_1}	K	R_P	R_{M_1}	K
$\star_3$	$M_1 R_{M_1}$	$M_1 R_{M_1}$	$M_1 R_{M_1}$	R_{M_1}	K	R_P	R_{M_1}	K
$\star_4$	R_{M_2}	R_{M_2}	K	R_{M_2}	R_{M_2}	K	R_{M_2}	R_{M_2}
$\star_5$	R	R	K	R	K	K	R	K
$\star_6$	M_1	M_1	K	R	K	K	R	K
$\star_7$	R	R	R_P	R	K	R_{M_1}	R	K
$\star_8$	M_1	M_1	R_P	R	K	R_{M_1}	R	K
$\star_9$	M_1	M_1	$M_1 R_{M_1}$	R	K	R_{M_1}	R	K
$\star_{10}$	R	R	K	R	R_{M_2}	K	R	R_{M_2}
$\star_{11}$	M_1	M_1	K	R	R_{M_2}	K	R	R_{M_2}
$\star_{12}$	R	R	R_P	R	R_{M_2}	R_{M_1}	R	R_{M_2}
$\star_{13}$	M_1	M_1	R_P	R	R_{M_2}	R_{M_1}	R	R_{M_2}
$\star_{14}$	M_1	M_1	$M_1 R_{M_1}$	R	R_{M_2}	R_{M_1}	R	R_{M_2}
$\star_{15}$	M_2	R	K	M_2	K	K	R	K
$\star_{16}$	$M_1 M_2$	M_1	K	M_2	K	K	R	K
$\star_{17}$	M_2	R	R_P	M_2	K	R_P	R	K
$\star_{18}$	$M_1 M_2$	M_1	R_P	M_2	K	R_P	R	K
$\star_{19}$	$M_1 M_2$	M_1	$M_1 R_{M_1}$	M_2	K	R_P	R	K
$\star_{20}$	M_2	R	K	M_2	R_{M_2}	K	R	R_{M_2}
$\star_{21}$	$M_1 M_2$	M_1	K	M_2	R_{M_2}	K	R	R_{M_2}
$\star_{22}$	M_2	R	R_P	M_2	R_{M_2}	R_P	R	R_{M_2}
$\star_{23}$	$M_1 M_2$	M_1	R_P	M_2	R_{M_2}	R_P	R	R_{M_2}
$\star_{24}$	$M_1 M_2$	M_1	$M_1 R_{M_1}$	M_2	R_{M_2}	R_P	R	R_{M_2}
$\star_{25}$	$M_2 R_{M_2}$	R_{M_2}	K	$M_2 R_{M_2}$	$M_2 R_{M_2}$	K	R_{M_2}	R_{M_2}
$\star_{26}$	M_2	R	K	M_2	$M_2 R_{M_2}$	K	R	R_{M_2}
$\star_{27}$	$M_1 M_2$	M_1	K	M_2	$M_2 R_{M_2}$	K	R	R_{M_2}
$\star_{28}$	M_2	R	R_P	M_2	$M_2 R_{M_2}$	R_{M_1}	R	R_{M_2}
$\star_{29}$	$M_1 M_2$	M_1	R_P	M_2	$M_2 R_{M_2}$	R_{M_1}	R	R_{M_2}
$\star_{30}$	$M_1 M_2$	M_1	$M_1 R_{M_1}$	M_2	$M_2 R_{M_2}$	R_1	R	R_2

Question. Let n be a positive integer and $0 \leq i \leq n$.

- Let R be a discrete Prüfer domain with spectrum:

$$\begin{array}{ccccc} M_1 & \cdots & M_i & \cdots & M_n \\ | & & | & & \\ P_1 & \cdots & P_i & & \end{array}$$

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THEN:

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- If e_n is the number of semistar operations on a PID with n maximal ideals, can we count $\mathrm{SStar}(R)$ and $\mathrm{SStar}(R)$ in terms of e_n?

THANKS!