Unions of Sets of Lengths

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Context

Factorization Theory (FT) investigates various phenomena arising from the non-uniqueness of factorizations in atomic monoids, and classify them by a wide assortment of invariants of different nature (algebraic, arithmetic, combinatorial, etc.).

It had its origins in algebraic number theory, and was later extended to rings and monoids, with an emphasis on the case of integral domains and cancellative, commutative monoids.

FT has greatly profited from the interaction with Arithmetic Combinatorics, and the work this talk is based on has further contributed to strengthen the ties between the two fields.

Notations and Definitions

$H = (H, \cdot)$: monoid (a set with associated operation and identity).

$H^\times$: the set of units (or invertible elements) of H.

atom: If $a \in H \setminus H^\times$ and $a = uv$ for some $u, v \in H$ implies $u \in H^\times$ or $v \in H^\times$, then a is called an atom.

$\mathscr{A}(H)$: the set of atoms of H.

H is said to be *atomic* if every $x \in H \setminus H^\times$ is a product of (finitely many) atoms of H.

Notations and Definitions

If $x \in H \setminus \{1_H\}$ and $x = a_1 \cdots a_n$ with $a_1, \ldots, a_n \in \mathscr{A}(H)$, then n is called a factorization length of x, and the set

$$\mathsf{L}(x) := \{n \in \mathbb{N} : n \text{ is a factorization length of } x\}$$

is called the *set of lengths of x*. Also, we take $\mathsf{L}(1_H) := \{0\} \subseteq \mathbb{N}_0$.

We say that H is *BF* iff it is atomic and $|\mathsf{L}(x)| < \infty$ for all $x \in H$; and *half-factorial* (HF) iff $|\mathsf{L}(x)| = 1$ for every $x \in H \setminus H^\times$.

We take

$$\mathscr{L}(H) = \{L(x) : x \in H\}$$

and

$$\mathscr{U}_k(H) = \bigcup \{L \in \mathscr{L}(H) : k \in L\}, \quad \forall k \in \mathbb{N}_0.$$

$|\mathscr{U}_k(H)| \leq 1$ for all k iff H is HF. Or else, $|\mathscr{U}_k(H)| \nearrow \infty$ as $k \to \infty$. The set $\mathscr{U}_k(H)$ is called the *union of sets of lengths containing k*, and understanding their structure is a fundamental problem in FT.

Let G be an abelian group, G_0 a subset of G, and $\sigma : \mathscr{F}(G) \to G$ the unique (monoid) homomorphism s.t. $\sigma(x) = x$ for every $x \in G$, where $\mathscr{F}(G)$ denotes the free abelian monoid with basis G.

Then $\mathcal{B}(G_0) := \sigma^{-1}(0_G) \cap \mathscr{F}(G_0)$ is a submonoid of $\mathscr{F}(G)$, called the *monoid of zero-sum sequences over G with support in G_0*.

Theorem 1 (*Kainrath*, 1999)

If G is an infinite abelian group, then the system of sets of lengths of $\mathcal{B}(G)$ is as large as possible (for a BF*-monoid), i.e.,*

$$\mathscr{L}(\mathcal{B}(G)) = \{\{0\}, \{1\}\} \cup \mathcal{P}_{fin}(\mathbb{N}_{\geq 2}).$$

Some Literature

1. On a related note, **Freeze** & **Geroldinger (2008)** showed that, if G is a finite abelian group, then $\mathscr{U}_k(\mathscr{B}(G))$ is a finite (discrete) interval for every $k \in \mathbb{N}$.

2. Blanco, García-Sánchez & **Geroldinger (2011)** established that, under mild assumptions, unions of sets of lengths in numerical monoids are arithmetic progressions.

3. Smertning (2013) proved that, if H is the multiplicative monoid of certain maximal orders in a simple central algebra over a number field, then $\mathscr{U}_k(H)$ is $\mathbb{N}_{\geq 2}$ or $\mathbb{N}_{\geq 3}$ for every $k \geq 2$.

4. Geroldinger, Kainrath & **Reinhart (2015)** showed that the conclusions of Freeze & Geroldinger's theorem carry over to seminormal orders in number fields with finite elasticity, and more generally to certain seminormal, commutative, v-noetherian, weakly Krull monoid with finite class group.

Some Literature

5. More recently, **JunSeok Oh (2017)** has generalized Freeze & Geroldinger's result to the non-abelian case, where $\mathscr{B}(G)$ is replaced by the submonoid of the free monoid with basis G whose non-unit elements are the non-empty words $x_1 * \cdots * x_n$ such that $x_{\tau(1)} \cdots x_{\tau(n)} = 1_G$ for some permutation τ of $[1, n]$.

6. **Fan & Tringali (2017)** have shown that $\mathscr{U}_k(\mathcal{P}_{fin,0}(\mathbb{N}_0)) = \mathbb{N}_{\geq 2}$ for every $k \geq 2$, where $\mathcal{P}_{fin,0}(\mathbb{N}_0)$ is the BF-monoid of non-empty finite subsets of $\mathbb{N}_0$ endowed with the operation of set addition. In fact, it is conjectured that $\mathscr{L}(\mathcal{P}_{fin,0}(\mathbb{N}_0))$ is the same as in Kainrath's theorem, but the conjecture is still widely open.

7. Last but not least, **Frisch, Nakato** & **Rissner (2017)** have proved that the same conclusions of Kainrath's paper hold for $\mathrm{Int}(\mathbb{Z}_K)$, where K is a number field and $\mathrm{Int}(\mathbb{Z}_K)$ is the subring of $K[x]$ consisting of all polynimials $f \in K[x]$ such that $f(\mathbb{Z}_K) \subseteq \mathbb{Z}_K$. Frisch (2013) had previously settled the case $K = \mathbb{Q}$.

We call a set $L \subseteq \mathbb{Z}$ is:

- an *arithmetic progression* (AP) if there are $z \in \mathbb{Z}$ and $d \in \mathbb{N}$ s.t. $L = [\inf L, \sup L] \cap (z + d \cdot \mathbb{Z})$;

- an *almost arithmetic progression* (AAP) with difference d and bound M, for some $d \in \mathbb{N}$ and $M \in \mathbb{N}_0$, if there is $y \in \mathbb{Z}$ s.t.

$$(y + d \cdot \mathbb{Z}) \cap [\inf L + M, \sup L - M] \subseteq L \subseteq y + d \cdot \mathbb{Z}.$$

We say that the monoid H satisfies the *Structure Theorem for Unions* if there are $d \in \mathbb{N}$ and $M \in \mathbb{N}_0$ such that $\mathscr{U}_k(H)$ is an AAP with difference d and bound M for all sufficiently large $k \in \mathbb{N}$.

We let $\Delta(L)$ be the *set of distances* of L, that is, the set of all $d \in \mathbb{N}$ s.t. $L \cap [x, x + d] = \{x, x + d\}$ for some $x \in \mathbb{Z}$.

The Structure Theorem for Unions

The following result has been a cornerstone for any subsequent work aimed to show that various classes of monoids (and unital rings) satisfy the Structure Theorem for Unions:

Theorem 2 (Gao & Geroldinger, 2009)

Let H be a commutative, cancellative, atomic monoid, and suppose that $\bigcup_{x \in H} \Delta(L(x))$ is finite and there is $K \in \mathbb{N}$ s.t.

$$\sup(\mathscr{U}_{k+1}(H)) \leq \sup(\mathscr{U}_k(H)) + K$$

for all large k. Then H satisfies the Structure Theorem for Unions.

In particular, Gao & Geroldinger used this to prove that all orders in number fields and all commutative Krull monoids with finite class group satisfy the Structure Theorem for Unions.

The Structure Theorem for Unions

1. Geroldinger, Grynkiewicz, Schaeffer & **Schmid (2010)** proved that the Structure Theorem for Unions holds for $\mathcal{B}(G_0)$ when $G_0 \subseteq \mathbb{Z}$ and at least one of G_0^- and G_0^+ is finite.

2. Geroldinger (2016) extended Theorem 2 to the non-commutative, but cancellative setting and concluded that the Structure Theorem for Unions holds for all transfer Krull monoids of finite type, that is, monoids H for which there are an abelian group G, a finite set $G_0 \subseteq G$, and a weak transfer homomorphism $\theta : H \to \mathcal{B}(G_0)$.

Remark: All the monoids considered in the above are cancellative. But many monoids arising from applications are not, two major examples being **monoids of ideals** and **monoids of modules**.

Monoids of Modules

Let R be a possibly non-commutative ring and $\mathcal{C}$ a class of finitely generated (left) R-modules which is closed under finite direct sums, direct summands, and isomorphisms, namely:

$$M \cong M_1 \oplus M_2 \;\Rightarrow\; (M \in \mathcal{C} \text{ iff } M_1, M_2 \in \mathcal{C}).$$

Let $\mathcal{V}(\mathcal{C})$ denote a set of representatives of isomorphism classes in $\mathcal{C}$, and for $M \in \mathcal{C}$ let $[M]$ denote the class of all modules in $\mathcal{C}$ that are isomorphic to M.

Then $\mathcal{V}(\mathcal{C})$ is a (additively written) reduced commutative monoid, with the operation given by $[M] + [N] = [M \oplus N]$, the identity by the zero-module, and the atoms by the isomorphism classes of irreducible modules.

Monoids of Modules

If $\mathrm{End}_R(M)$ is semilocal for each $M \in \mathcal{C}$, then $\mathcal{V}(\mathcal{C})$ is a Krull monoid by [Facchini, 2002], so can be studied through the theory of Krull monoids.

This was the starting point for a new approach to the study of direct-sum decompositions of modules in terms of the algebraic and arithmetic properties of $\mathcal{V}(\mathcal{C})$ [Baeth & Wiegand, 2013]

So far, the focus has been on classes $\mathcal{C}$ for which $\mathcal{V}(\mathcal{C})$ is cancellative: We can now extend to the non-cancellative case.

For this, note that $\mathcal{V}(\mathcal{C})$ being finitely generated means that, up to isomorphism, there are only finitely many irreducible R-modules in $\mathcal{C}$.

Corollary 3 (F., Geroldinger, Kainrath & Tringali, 2016)

Let R be a ring and $\mathcal{C}$ a class of finitely generated R-modules which is closed under finite direct sums and direct summands. Assume also that there are, up to isomorphism, only finitely many irreducible modules in $\mathcal{C}$, and that, for all $M, N \in \mathcal{C}$, $M \oplus N \cong M$ implies $N = 0$. Then $\mathcal{V}(\mathcal{C})$ satisfies the Structure Theorem for Unions.

Directed Families

Let $\mathscr{L} \subseteq \mathscr{P}(\mathbb{N}_0)$. For every $k \in \mathbb{N}_0$, we set

$$\mathscr{U}_k(\mathscr{L}) := \bigcup_{L \in \mathscr{L} : k \in L} L \quad \text{and} \quad \rho_k(\mathscr{L}) := \sup \mathscr{U}_k(\mathscr{L}),$$

Moreover, we let

$$\Delta(\mathscr{L}) := \bigcup_{L \in \mathscr{L}} \Delta(L).$$

We call $\Delta(\mathscr{L})$ the set of distances (or delta set) of $\mathscr{L}$.

We say that the family $\mathscr{L}$ is directed if $1 \in L$ for some $L \in \mathscr{L}$ and, for all $L_1, L_2 \in \mathscr{L}$, there is $L' \in \mathscr{L}$ with $L_1 + L_2 \subseteq L'$.

Remark: Directed families are the additive model we use to investigate the structure of unions. Note that the system of sets of lengths of a monoid with non-empty set of atoms is a directed family.

Main Results

Theorem 4 (F., Geroldinger, Kainrath & Tringali, 2016)

Let $\mathscr{L} \subseteq \mathcal{P}(\mathbb{N}_0)$ be a directed family such that $\Delta(\mathscr{L})$ is finite and non-empty, and set $\delta = \min \Delta(\mathscr{L})$. Moreover, let $l \in \mathbb{N}_0$ such that $\{l, l+\delta\} \subseteq L$ for some $L \in \mathscr{L}$. Then $q = \frac{1}{\delta} \max \Delta(\mathscr{L})$ is a positive integer and the following are equivalent:

(a) $\mathscr{L}$ satisfies the Structure Theorem for Unions.

(b) There exists $M \in \mathbb{N}_0$ such that

$$\mathscr{U}_k(\mathscr{L}) \cap [\rho_{k-lq}(\mathscr{L}) + lq, \rho_k(\mathscr{L}) - M]$$

is an AP with difference δ for all sufficiently large k.

Main Results

Corollary 5 (F., Geroldinger, Kainrath & Tringali, 2016)

Let $\mathscr{L}$ be a directed family, and suppose that $\Delta(\mathscr{L})$ is finite and there exists $K \in \mathbb{N}$ s.t.

$$\rho_{k+1}(\mathscr{L}) \le \rho_k(\mathscr{L}) + K$$

for all large k. Then $\mathscr{L}$ satisfies the Structure Theorem for Unions.

Corollary 6 (F., Geroldinger, Kainrath & Tringali, 2016)

Let $\mathscr{L} \subseteq \mathcal{P}(\mathbb{N}_0)$ be a directed family with finite non-empty delta set satisfying the Structure Theorem for Unions. Then

$$\lim_{k\to\infty} \frac{|\mathscr{U}_k(\mathscr{L})|}{k} = \sup_{k\ge 1} \frac{|\mathscr{U}_k| - 1}{k} = \frac{1}{\min \Delta(\mathscr{L})}\left(\rho(\mathscr{L}) - \frac{1}{\rho(\mathscr{L})}\right).$$

A Counterexample

Finally, we have provided the very first example of a "nice" monoid for which the Structure Theorem for Unions does not hold.

Theorem 7 (F., Geroldinger, Kainrath & Tringali, 2016)

There exists an atomic, commutative, locally tame Krull monoid with finite set of distances which does **not** *satisfy the Structure Theorem for Unions*

Comment: We can combine the counter-example with Claborn's Realization Theorem to obtain a Dedekind domain that does not satisfy the Structure Theorem for Unions.

Closing Remarks

We say that a family $\mathscr{L} \subseteq \mathcal{P}(\mathbb{N}_0)$ satisfies the *Strong Structure Theorem (for Unions)* if there exist $d, \mu \in \mathbb{N}$, $M \in \mathbb{N}_0$, and $\mathscr{U}_0', \mathscr{U}_0'', \ldots, \mathscr{U}_{\mu-1}', \mathscr{U}_{\mu-1}'' \subseteq [0, M]$ such that, for all large $k \in \mathbb{N}$,

$$\mathscr{U}_k = \left(\lambda_k + \mathscr{U}'_{k \bmod \mu}\right) \uplus \mathcal{P}_k \uplus \left(\rho_k - \mathscr{U}''_{k \bmod \mu}\right) \subseteq k + d \cdot \mathbb{Z},$$

where $\lambda_k := \inf \mathscr{U}_k$ and $\mathcal{P}_k := (k + d \cdot \mathbb{Z}) \cap [\lambda_k + M, \rho_k - M]$.

Theorem 8 (Tringali, 2017)

Let $\mathscr{L} \subseteq \mathcal{P}(\mathbb{N}_0)$ be a directed family with accepted elasticity. Then $\mathscr{L}$ satisfies the Strong Structure Theorem for Unions.

Here, $\mathscr{L}$ has *accepted elasticity* if there exists $\bar{L} \in \mathscr{L}$ such that

$$\sup L / \inf L^+ \le \sup \bar{L} / \inf \bar{L}^+, \quad \forall L \in \mathscr{L},$$

where, for every $X \subseteq \mathbb{N}_0$, $X^+ = X \setminus \{0\}$.

Remark: The systems of sets of lengths of a large variety of monoids have accepted elasticity.

Thanks for your attention.