Products of idempotent matrices over Prüfer domains

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based on a joint work with P. Zanardo

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Università degli Studi di Padova

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Standard form of a 2×2 non-identity idempotent matrix over R:

$$\begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}, \text{ with } a(1-a) = bc.$$

Motivations and first results

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- **Euclidean domains** satisfy property (ID_n) for every $n > 0$.

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- (ID_n) is equivalent to other properties in the class of PID's.
- The ring of integers $\mathbb{Z}$ and DVR's satisfy property (ID_n) for every $n > 0$.

(ID$_n$) and (GE$_n$) in Bézout domains

The property (GE$_n$)

An integral domain R satisfies property (GE$_n$) if any INVERTIBLE $n \times n$ matrix over R is a product of ELEMENTARY matrices.

(ID_n) and (GE_n) in Bézout domains

The property (GE_n)

An integral domain R satisfies property (GE_n) if any **INVERTIBLE** $n \times n$ matrix over R is a product of **ELEMENTARY** matrices.

NOTE: fields and Euclidean domains satisfy (GE_n) for every $n > 0$,
not every PID satisfies (GE_n) for every $n > 0$.

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Theorem (Ruitenburg - 1993)

For a Bézout domain R (every f.g. ideal of R is principal) TFAE:

(i) R satisfies (GE_n) for every integer $n > 0$;

(ii) R satisfies (ID_n) for every integer $n > 0$.

W. RUITENBURG, Products of idempotent matrices over Hermite domains, *Semigroup Forum, 46(3): 371–378*, 1993.

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Thus, in a Bézout domain

$$(\mathrm{ID}_2) \Leftrightarrow (\mathrm{ID}_n)_{\forall n} \Leftrightarrow (\mathrm{GE}_n)_{\forall n} \Leftrightarrow (\mathrm{GE}_2).$$

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NOTE: (GE$_2$)$\not\Leftrightarrow$(ID$_2$) outside Bézout domains: local non-valuation domains satisfy (GE$_2$) but not (ID$_2$).

The conjecture $(\mathrm{ID}_2) \Longrightarrow$ Bézout

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Theorem (Bhaskara Rao - 2009)

Let R be a projective-free domain (every projective R-module is free). If R satisfies property (ID_2), then R is a Bézout domain.

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This result and those by Laffey and Ruitenburg suggested the following:

Conjecture (Salce, Zanardo - 2014)

If an integral domain R satisfies property (ID_2), then it is a Bézout domain.

L. SALCE, P. ZANARDO, Products of elementary and idempotent matrices over integral domains, *Linear Algebra Appl., 452:130–152*, 2014.

The conjecture $(\mathrm{ID}_2) \Rightarrow$ Bézout

NOTE:

- In view of Laffey's lift, if this conjecture would be true, then every domain satisfying property (ID_2) would satisfy property (ID_n) for any $n > 0$.

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  Local non-valuation domains are non-Bézout domains satisfying (GE_2).

EXAMPLES:

Unique factorization domains Projective-free domains Local domains PRINC domains (introduced by Salce and Zanardo)	+ (ID_2) $\Rightarrow$ Bézout

(ID$_2$) $\Rightarrow$ Prüfer

Our first result in support of the conjecture is the following

Theorem 1

If R is an integral domain satisfying property (ID$_2$), then R is a Prüfer domain (a domain in which every finitely generated ideal is invertible).

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Our first result in support of the conjecture is the following

Theorem 1

If R is an integral domain satisfying property (ID_2), then R is a Prüfer domain (a domain in which every finitely generated ideal is invertible).

Thus, it is NOT RESTRICTIVE to study the conjecture inside the class of Prüfer domains.

Sketch of the proof of Th.1

Idea of the proof of Theorem 1:

- we first prove as a preliminary result that, given $a, b \in R$ non-zero
$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \text{ product of idempotents} \Rightarrow (a, b) \text{ invertible.}$$

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- If we assume that R has (ID_2), then every two-generated ideal of R is invertible.
- We conclude since R is a Prüfer domain iff every two-generated ideal of R is invertible.

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Proving a preliminary technical result and using a characterization of the property (GE_2) over an arbitrary domain proved by Salce and Zanardo in 2014, we get that

Theorem 2

If an integral domain R satisfies property (ID_2), then it also satisfies property (GE_2).

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Corollary 1

If R is an integral domain satisfying property (ID_2), then R is a Prüfer domain satisfying property (GE_2).

Conjecture: an equivalent formulation

Conjecture - Equivalent formulation

If R is a Prüfer non-Bézout domain, then there exist invertible 2×2 matrices over R that are not products of elementary matrices, i.e. R does not satisfy property (GE_2).

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If R is a Prüfer non-Bézout domain, then there exist invertible 2×2 matrices over R that are not products of elementary matrices, i.e. R does not satisfy property (GE_2).

We prove that the coordinate rings of a large class of plane curves and the ring of integer-valued polynomials $\mathrm{Int}(\mathbb{Z})$ satisfy the conjecture in this last formulation.

The case of the coordinate rings

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Theorem 3

Let C be a plane smooth curve over the field k, having degree ≥ 2, such that all the points at infinity are conjugate by elements of the Galois group $G_{\bar{k}/k}$. Then, the coordinate ring $R = k[C_0] = k[x, y]$ of C_0 does not satisfy property (GE_2).

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STRATEGY OF THE PROOF:
we prove that the group of units of R is k (i.e. R is a k-ring) and that $d = -\sum_{P \in C_\infty} \mathrm{ord}_P$ is a degree-function. We conclude applying a Cohn's proposition on k-rings with degree functions.

An example

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- if its coordinate ring R is not a PID, then R is a Dedekind domain that satisfies the conjecture.

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EXAMPLE:
Let $x^4 + y^4 + 1 = 0$ be the defining equation of C over $\mathbb{R}$.
Then R is a non-UFD Dedekind domain:

$$(x^2 + y^2 - 1)(x^2 + y^2 + 1) = 2(xy - 1)(xy + 1)$$

is a non-unique factorization into indecomposable factors.
R does not satisfy properties (GE_2) and (ID_2).

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If d is also different from $19, 43, 67, 163$, then the ring of integers I in $\mathbb{Q}(\sqrt{-d})$ is a Dedekind domain, non UFD, that does not satisfy (GE_2) thus satisfying the conjecture.

P.M. Cohn, On the structure of the GL_2 of a ring, *Inst. Hautes Études Sci. Publ. Math.,30: 5–53*, 1966.

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  EXAMPLES:
  The coordinate ring over $\mathbb{R}$ of the curve $x^2 + y^2 + 1 = 0$ and the coordinate ring over $\mathbb{Q}$ of the curve $x^2 - 3y^2 + 1 = 0$ are non-Euclidean PID's not satisfying (GE_2).

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  Examples:
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- If the coordinate ring R of a plane smooth curve $\mathcal{C}$ of degree ≥ 2 having conjugate points at infinity is a PID, then R is a non-Euclidean PID not satisfying (GE_2).

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- The rings of integers I in $\mathbb{Q}(\sqrt{-d})$ with $d = 19, 43, 67, 163$ are non-Euclidean PID's not satisfying (GE_2).

These classes of non-Euclidean PID's verify another conjecture proposed by Salce and Zanardo in 2014: "every non-Euclidean PID does not satisfy (GE_2)".

L.Cossu, P. Zanardo, U. Zannier , Products of elementary matrices and non-Euclidean principal ideal domains, *Journal of Algebra,501: 182 - 205*, 2018.

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$$\binom{X}{n} = \frac{X(X-1)\cdots(X-n+1)}{n!},$$

with $\binom{X}{0} = 1$ and $\binom{X}{1} = X$, form a basis of $\mathrm{Int}(\mathbb{Z})$ as a $\mathbb{Z}$-module;

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with $\binom{X}{0} = 1$ and $\binom{X}{1} = X$, form a basis of $\mathrm{Int}(\mathbb{Z})$ as a $\mathbb{Z}$-module;

- $\mathrm{Int}(\mathbb{Z})$ is a Prüfer domain but it is not a Bézout domain.

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Proposition

The ring of integer-valued polynomials $\mathrm{Int}(\mathbb{Z})$ is a discretely ordered ring.

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Proposition

The ring of integer-valued polynomials Int($\mathbb{Z}$) is a discretely ordered ring.

Strategy of the proof:

Every $f \in \mathrm{Int}(\mathbb{Z})$ is of the form $f = a_0 + a_1 X + \cdots + a_n \binom{X}{n}$, with $a_0, \ldots, a_n \in \mathbb{Z}$ for some $n \in \mathbb{N}$.
Set $f > 0$ if and only if $a_n > 0$ and $f > g$ if and only if $f - g > 0$, with $f, g \in \mathrm{Int}(\mathbb{Z})$. Then $f > 0 \Rightarrow f \geq 1$.

The case of Int($\mathbb{Z}$)

Using a collection of Cohn's results which characterize the products of elementary matrices over discretely ordered rings, we proved that the invertible matrix

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Then:

Theorem 4

The ring Int($\mathbb{Z}$) does not satisfy property (GE_2). Thus, it is a Prüfer non-Bézout domain satisfying the conjecture (ID_2)$\Rightarrow$ Bézout in its equivalent formulation.

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Essential bibliography

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