Dilatations of numerical semigroups

Valentina Barucci

Department of Mathematics
Sapienza Università di Roma 1

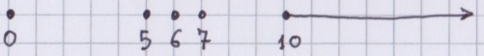
Conference on Rings and Factorizations

The results of this talk are contained in a joint paper with Francesco Strazzanti, accepted for publication on Semigroup Forum.

A numerical semigroup S is a submonoid of $(\mathbb{N}, +)$ for which $\mathbb{N} \setminus S$ is finite.
We always assume $S \neq \mathbb{N}$.

We recall some invariants of a numerical semigroup on an example.

S

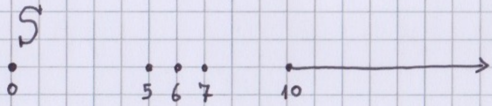


Frobenius number (the biggest gap), $F(S) = 9$

genus (number of gaps), $g(S) = 6$

multiplicity (smallest positive element), $e(S) = 5$

number of elements in S before F, $n(S) = 4$



maximal ideal, $M = S \setminus \{0\}$

$M - M = \{ z \in \mathbb{Z} \mid z + M \subseteq M \} = \{5, 6, \rightarrow\}$

Pseudo-Frobenius numbers : $(M-M) \setminus S = \{8, 9\}$

type : $|(M-M) \setminus S|$, $t(S) = 2$

embedding dimension $\nu(S) = |M \setminus 2M| = 3$

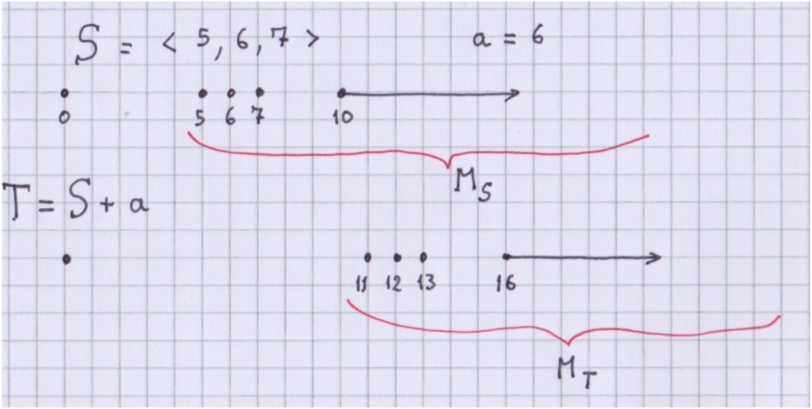
(in fact $M \setminus 2M = \{5, 6, 7\}$)

given $a \in S$, we study the numerical semigroup

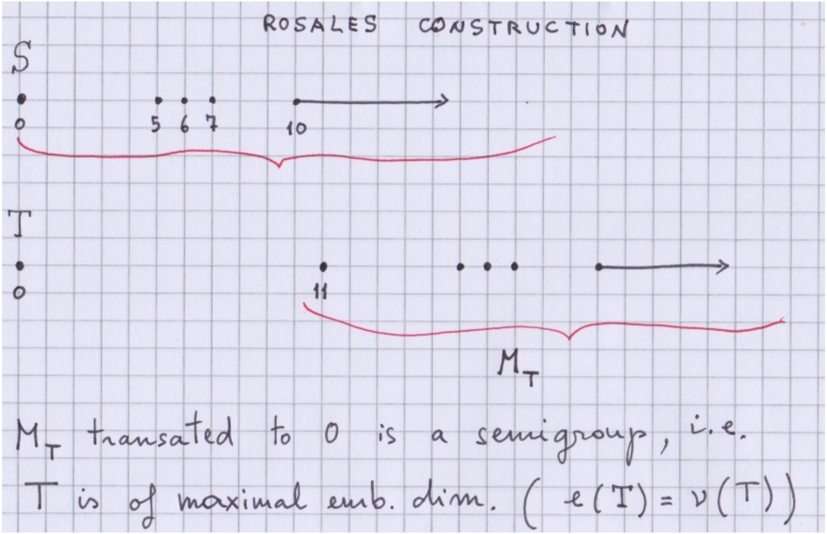
$$S + a = \{s + a; s \in M\} \cup \{0\}$$

that we call a *dilatation* of S.

In the example:

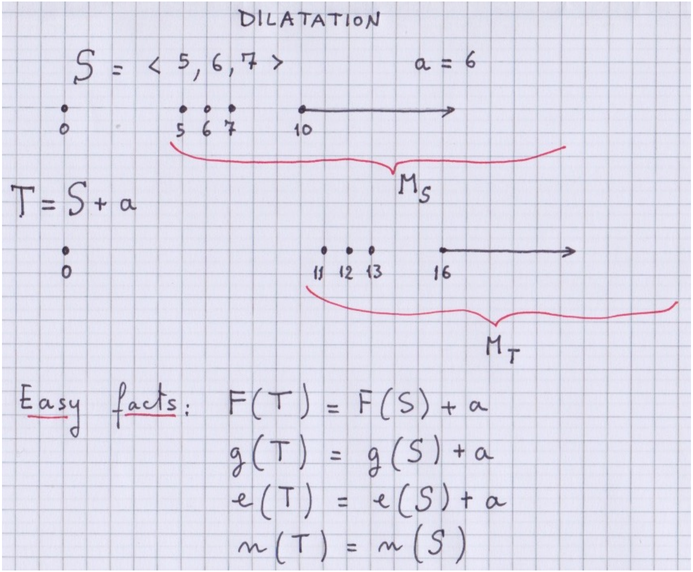

S = < 5, 6, 7 >
a = 6
0
5 6 7
10
M_S
T = S + a
11 12 13
16
M_T

In literature there are two constructions that may appear similar to the dilatation, but actually the properties of the obtained semigroups are very different.


ROSALES CONSTRUCTION
S
0
5 6 7
10
T
0
11
M_T
M_T transated to 0 is a semigroup, i.e.
T is of maximal emb. dim. (e(T) = ν(T))

Given a semigroup $S = \langle s_1, \dots, s_\nu \rangle$ Herzog, Srinivasan, Vu and others considered the semigroup generated by
$s_1 + a, \dots, s_\nu + a$, where $a \in \mathbb{N}$.
Also this construction is completely different respect to our dilatation.

Go back to the dilatation.


DILATATION
S = < 5, 6, 7 >
a = 6
0
5 6 7
10
M_S
T = S + a
0
11 12 13
16
M_T
Easy facts:
F(T) = F(S) + a
g(T) = g(S) + a
e(T) = e(S) + a
m(T) = m(S)

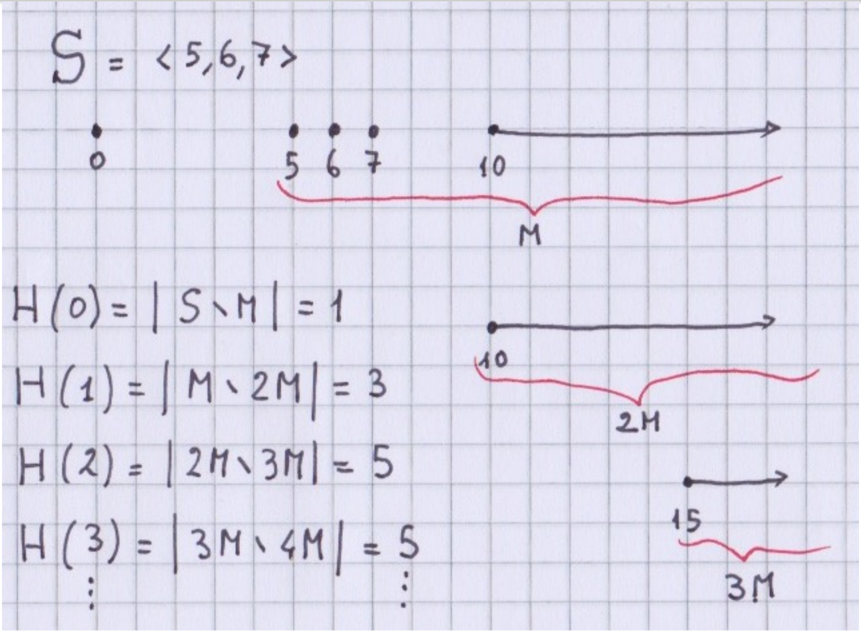
Let S be a numerical semigroup of maximal ideal M. The n-th value of the Hilbert function of S is

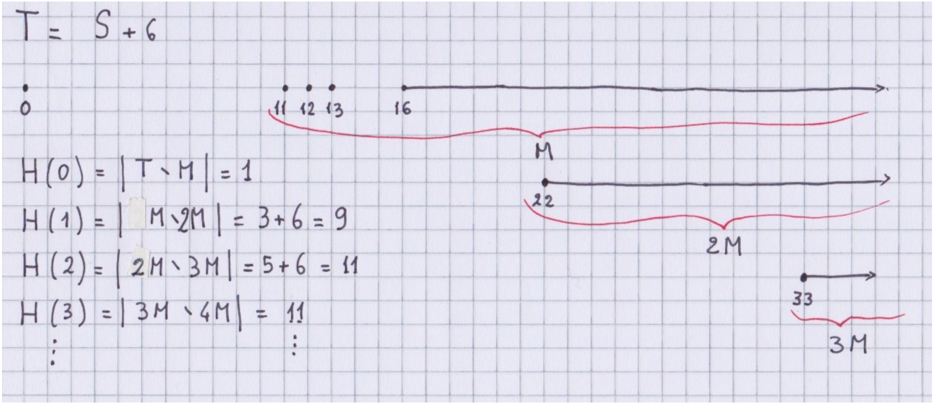
$$H(n) = |nM \setminus (n+1)M|$$

which is the number of generators of the n-multiple of M.

$H(n)$ is also the Hilbert function of the associated graded ring of $k[[S]]$.

Look at our simple example...


S = <5,6,7>
0
5 6 7
10
M
H(0) = | S \ M | = 1
H(1) = | M \ 2M | = 3
H(2) = | 2M \ 3M | = 5
H(3) = | 3M \ 4M | = 5
10
2M
15
3M


T = S + 6
0
11 12 13
16
M
H(0) = |T \ M| = 1
H(1) = |M \ 2M| = 3 + 6 = 9
H(2) = |2M \ 3M| = 5 + 6 = 11
H(3) = |3M \ 4M| = 11
22
2M
33
3M

Proposition

Let $T = S + a$ be a dilatation of S. Then:

1. $t(T) = t(S) + a$;
2. $H_T(n) = H_S(n) + a$ *for each* $n \geq 1$;
3. $\nu(T) = \nu(S) + a$.

Proof.

1. Since $(M_T - M_T) = (M_S - M_S)$, we have $t(T) = |(M_T - M_T) \setminus T| = |(M_S - M_S) \setminus T| = |(M_S - M_S) \setminus S| + a = t(S) + a$.
2. Sketch. Translating to zero the maximal ideals, $M_s - e(S) = M_T - e(T)$, and so the “shapes”of the multiples of the two maximal ideals change in the same way.
3. In particular $\nu(T) = H_T(1) = H_S(1) + a = \nu(S) + a$. $\square$

Proposition

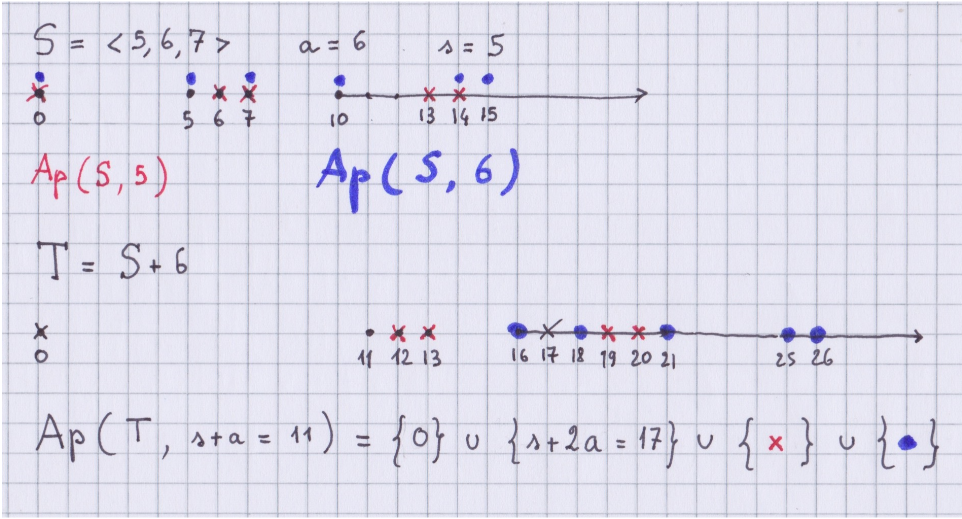
Let $T = S + a$ be a dilatation of S. Then $Ap(T, s+a)$ is given by

$\{0, s+2a\} \cup$

$\{\alpha + a \mid \alpha \in Ap(S,s) \setminus \{0\}\} \cup$

$\{\beta + s + a \mid \beta \in Ap(S,a) \setminus \{0\}\}$

Look at our example...


S = <5, 6, 7>
a = 6
s = 5
0
5 6 7
10
13 14 15
Ap(S, 5)
Ap(S, 6)
T = S + 6
0
11 12 13
16 17 18 19 20 21
25 26
Ap(T, s+a = 11) = {0} ∪ {s+2a = 17} ∪ {×} ∪ {•}

Denoting by $\Gamma(-)$ the set of minimal generators of a semigroup, there is a one to one correspondence between

$$Ap(S, e(S)) \setminus \Gamma(S) \quad \text{and} \quad Ap(T, e(T)) \setminus \Gamma(T)$$

Thus the generators of a dilatation T of S can be given in terms of the generators of S. We have an explicit formula, if S is two generated.

Wilf's conjecture is a long-standing conjecture about some invariants of a numerical semigroup:

$$F(S) + 1 \leq n(S) \cdot \nu(S)?$$

Proposition

If Wilf's conjecture holds for S, it holds for all the dilatations of S

Proof. Let $T = S + a$ and suppose that $F(S) + 1 \leq n(S) \cdot \nu(S)$. We get $F(T) + 1 = F(S) + a + 1 \leq n(S) \cdot \nu(S) + a \leq n(S) \cdot \nu(S) + n(S)a = n(S) \cdot (\nu(S) + a) = n(T) \cdot \nu(T)$. □

Fromentin - Hivert and Sammartano proved that Wilf's conjecture holds, provided that $g(S) \leq 60$ or $e(S) \leq 8$ respectively. Clearly, if S satisfies one of these properties and a is large enough, $S + a$ does not satisfy it.

Corollary

If either $g(S) \leq 60$ or $e(S) \leq 8$, then Wilf's conjecture holds for all the dilatations of S.

A numerical semigroup S is said to be *symmetric* if

$$x \in \mathbb{Z} \setminus S \implies F(S) - x \in S$$

Symmetric numerical semigroups arise naturally in numerical semigroup theory, since, if we consider all the numerical semigroups with a fixed odd Frobenius number, they are the maximal ones with respect to the inclusion or, equivalently, the ones with minimal genus.

On the other hand, their importance is due to the fact that $k[[S]]$ is Gorenstein if and only if S is symmetric.

A related notion is that of canonical ideal of S, i.e. the relative ideal

$$\Omega_S = \{x \in \mathbb{N}; F(S) - x \notin S\}$$

S is symmetric if and only if $S = \Omega_S$ or, equivalently, S has type one.

Thus, if a is positive, $S + a$ is never symmetric.

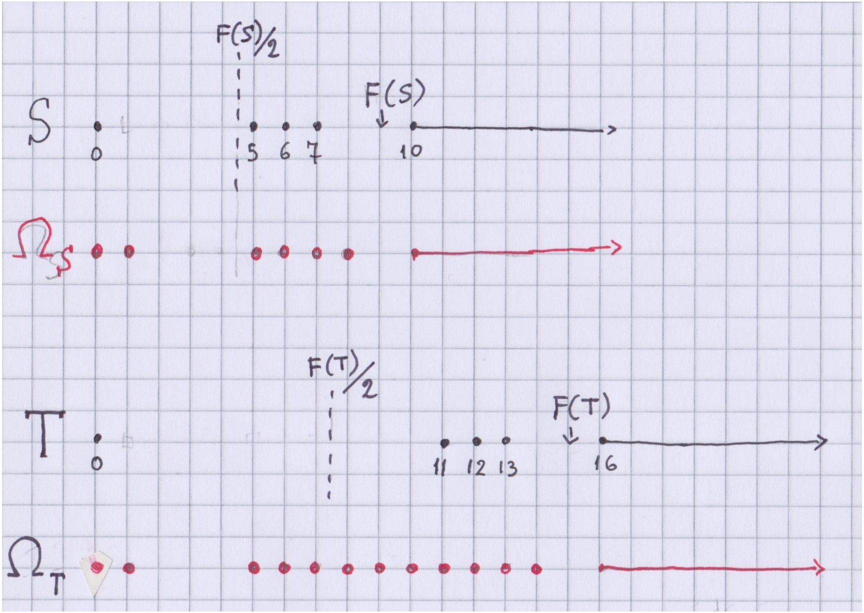
On the other hand, it is possible to use the dilatation to find numerical semigroups that are, in some sense, *near* to be symmetric. In particular, we consider the following properties: almost symmetric, nearly Gorenstein and 2-almost Gorenstein.

Lemma

Let $T = S + a$. Then:

1. $\Omega_T = (\Omega_S \cup \{F(S)\}) \setminus \{F(T)\}$;
2. $\Omega_S = (\Omega_T \cup \{F(T)\}) \setminus \{F(S)\}$.

Proof. Suppose that $x \in \mathbb{Z}$, $x \neq F(S), F(T)$. We have that $F(S) - x = F(T) - a - x \notin S$ if and only if $(F(T) - a - x) + a = F(T) - x \notin T$; then, $x \in \Omega_S$ if and only if $x \in \Omega_T$.
Moreover, since $F(T) - F(S) = a \notin T$, we get that $F(S) \in \Omega_T$ and, obviously, $F(T) \in S \subseteq \Omega_S$; hence, the conclusion follows.
□


F(S)/2
F(S)
S
0
5 6 7
10
Ω_S
F(T)/2
F(T)
T
0
11 12 13
16
Ω_T

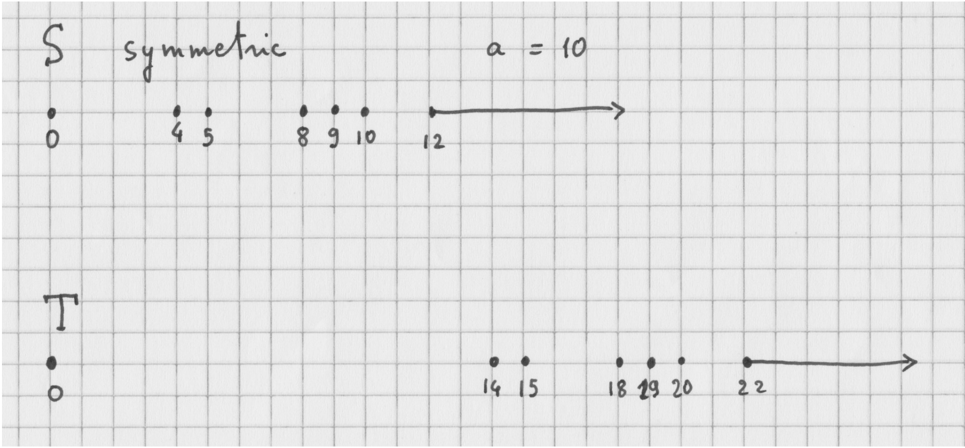
A numerical semigroup S is *almost symmetric* if $\Omega_S + M_S \subseteq M_S$ or, equivalently, if

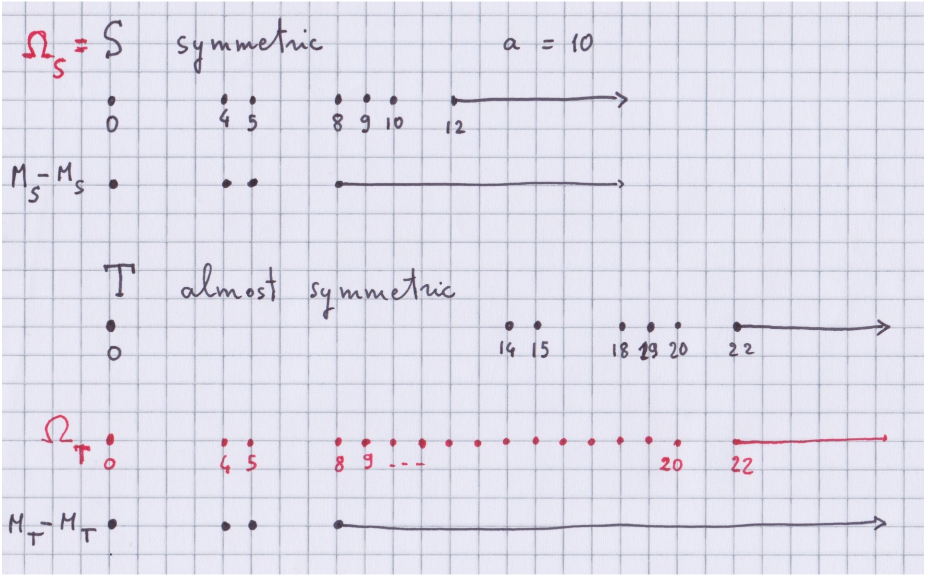
$$\Omega_S \subseteq M_S - M_S$$

Proposition

Let $T = S + a$. Then, S is almost symmetric if and only if T is almost symmetric.

Proof. S is almost symmetric if and only if $\Omega_S \subset M_S - M_S$ and T is almost symmetric if and only if $\Omega_T \subset M_T - M_T = M_S - M_S$. Since $F(S)$ and $F(T)$ are always in $M_S - M_S$, we conclude by the previous lemma. $\square$


S symmetric
a = 10
0
4 5
8 9 10
12
T
0
14 15
18 19 20
22


Ω_S = S symmetric
a = 10
0
4 5
8 9 10
12
M_S − M_S
T almost symmetric
0
14 15
18 19 20
22
Ω_T
0
4 5
8 9 - - -
20
22
M_T − M_T

Nearly Gorenstein rings (introduced by Herzog, Hibi, and Stamate) generalize in the one-dimensional case almost Gorenstein rings. In particular, the authors define *nearly Gorenstein numerical semigroups* that generalize almost symmetric semigroups.

The trace ideal of S is defined as

$$\mathrm{tr}(S) = \Omega_S + (S - \Omega_S)$$

Then, S is said to be nearly Gorenstein if $M_S \subseteq \mathrm{tr}(S)$.

The semigroup S is symmetric if and only if $\mathrm{tr}(S) = S$, otherwise S is nearly Gorenstein exactly when $\mathrm{tr}(S) = M_S$, since $\mathrm{tr}(S)$ is an ideal contained in S.

Proposition

Each almost symmetric semigroup is nearly Gorenstein.

Proof. If S is symmetric, $\mathrm{tr}(S) = S$ and, then, it is nearly Gorenstein. If S is a non-symmetric almost symmetric semigroup, we have $S - \Omega_S = M_S$, since $\Omega_S \subseteq M_S - M_S$. It follows that $\mathrm{tr}(S) = \Omega_S + (S - \Omega_S) = \Omega_S + M_S = M_S$. $\square$

Lemma

If S is not symmetric, then $\mathrm{tr}(S + a) = \mathrm{tr}(S) + a$.

Recall that, if S is symmetric, $S + a$ is always almost symmetric and, then, it is nearly Gorenstein. Thus, we get the following:

Corollary

S is nearly Gorenstein if and only if $S + a$ is nearly Gorenstein for all $a \in S$.

Let R be a one-dimensional Cohen-Macaulay local ring with canonical ideal I. Let $\ell_R(-)$ denote the length of an R-module and $e_i(I)$ denote the Hilbert coefficients of R with respect to I. It is known that $s = e_1(I) - e_0(I) + \ell_R(R/I)$ is positive and independent of the choice of I. In fact, s is the rank of Sally modules of I. Since R is almost Gorenstein, but not Gorenstein, if and only if $s = 1$, Chau, Goto, Kumashiro, and Matsuoka study the rings for which $s = 2$, that they call 2-almost Gorenstein local rings or, briefly, 2-AGL rings. If ω is a canonical module of R such that $R \subseteq \omega \subseteq \overline{R}$, where $\overline{R}$ denotes the integral closure of R, they prove that R is 2-AGL if and only if $\omega^2 = \omega^3$ and $\ell_R(\omega^2/\omega) = 2$.
Similarly, we say that a numerical semigroup S is 2-AGL if the reduction number of Ω_S is 2 and $|2\Omega_S \setminus \Omega_S| = 2$. Clearly, S is 2-AGL if and only if $k[[S]]$ is 2-AGL.

Although the definition of 2-AGL rings come from Gorenstein rings, Nearly Gorenstein and 2-AGL numerical semigroups are two disjoint classes. However we also prove that

Proposition

$S + a$ is 2-AGL, for all $a \in S$, if and only if S is 2-AGL.

Some references

V. Barucci, F. Strazzanti. *Dilatations of numerical semigroups*. In print on Semigroup Forum. Available on line (http://link.springer.com/article/10.1007/s00233-018-9922-9).

J. Fromentin, F. Hivert, *Exploring the tree of numerical semigroups*, Math. Comput. **85** (2016), no. 301, 2553–2568.

J.C. Rosales, *Principal ideals of numerical semigroups*, Bull. Belg. Math. Soc. **10** (2003), 329-343.

A. Sammartano, *Numerical semigroups with large embedding dimension satisfy Wilf's conjecture*, Semigroup Forum **85** (2012), 439–447.

H.S. Wilf, *A circle-of-lights algorithm for the money changing problem*, Amer. Math Monthly **85** (1978), 562-565.