Especially Short Sequences and Full Sumsets: A Combinatorial Curiosity

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Preliminaries and Notation

Let G be a finite abelian group. A **sequence** over G is an element $S = g_1 \cdots g_\ell \in \mathcal{F}(G)$, the free abelian monoid over G.

- ℓ is the *length* of S
- $\sigma(S) = g_1 + \cdots + g_\ell$ is the *sum* of S
- $\Sigma(S) = \{\sigma(T) : T|S,\ T \neq \emptyset\}$ is the *set of subsums* of S

The **(small) Davenport constant** of G is

$$d(G) = \max\{|S| : S \in \mathcal{F}(S),\, 0 \notin \Sigma(S)\}$$

Motivation

For any finite abelian group G, if $S \in \mathcal{F}(G)$ has $|S| = d(G)$ and $0 \notin \Sigma(S)$, then we have

$$\Sigma(S) = G \setminus \{0\} \qquad (*)$$

Motivating question: What can we learn about sequences S satisfying (*)?

We will say such a sequence has *(nearly) full sumset*. We focus in particular on *shortest* sequences with nearly full sumset. We can try to determine their

- length
- structure
- rarity

Let $fs(G)$ be the length of a shortest sequence over G with nearly full sumset.

An Elementary Lower Bound

We want

$$|\Sigma(S)| = |G| - 1$$

This implies

$$2^{|S|} - 1 \geq |G| - 1$$

so

$$|S| \geq \lceil \log |G| \rceil$$

(That is, $fs(G) \geq \lceil \log |G| \rceil$)

A Construction for Cyclic Groups

The lower bound is exact in this case; if $G = \mathbb{Z}_n$, let $\ell = \lceil \log n \rceil$ and

$$S = 1 \cdot 2 \cdot 4 \cdots 2^{\ell-2} \cdot (n - 2^{\ell-1})$$

To see this, observe

$$\Sigma(S) \supseteq \Sigma(1 \cdot 2 \cdot 4 \cdots 2^{\ell-2}) = [1, 2^{\ell-1} - 1]$$

and

$$\Sigma(S) \supseteq \Sigma(1 \cdot 2 \cdot 4 \cdots 2^{\ell-2}) + (n - 2^{\ell-1} - 1) = [n - 2^{\ell-1}, n - 1]$$

so

$$[1, 2^{\ell-1}] \cup [n - 2^{\ell-1}, n - 1] \subseteq \Sigma(S) \subseteq [1, \sigma(S)]$$

Generalization of the Cyclic Construction

Proposition

Let $S \in \mathcal{F}(\mathbb{Z}_n)$ *have* $\Sigma(S) = \mathbb{Z}_n \setminus \{0\}$ *and length* $\ell = \lceil \log n \rceil$*.*
Write $S = \bar{x}_1 \bar{x}_2 \cdots \bar{x}_\ell$ *with* $x_1 \leq \cdots \leq x_\ell$*.*
If $x_1 + \cdots + x_\ell \leq n$ *then* $x_1 = 1$ *and, for all* $i < \ell$*,*

$$x_{i+1} \leq x_1 + \ldots x_i + 1$$

Rank-2 Groups

Let $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$.

- For a lower bound: use the elementary bound
- For an upper bound: construct a full-sumset sequence coordinatewise

We have

$$\lceil \log(n^2) \rceil \leq fs(\mathbb{Z}_n^2) \leq 2fs(\mathbb{Z}_n)$$

or

$$\lceil 2 \log n \rceil \leq fs(\mathbb{Z}_n^2) \leq 2\lceil \log n \rceil$$

($fs(\mathbb{Z}_n)$ falls within a gap of at most 1)

Computer Search for Sequences

Looking at values of n where $\ell = \lceil 2 \log n \rceil = 2\lceil \log n \rceil - 1$, either

(a) Find sequence(s) over $\mathbb{Z}_n^2$ of length ℓ with full sumset or

(b) Exhaustively show that no length-ℓ sequence has full sumset

The Search Procedure: Outline

- Idea: populate a list with all sequences of length ℓ, delete the "bad" ones
- Problem: this is computationally expensive (on the order of $\left(\binom{n^2+\ell-1}{n^2-1}\right)$)
- New idea: iteratively construct all "good" sequences; during construction, discard those
    - with any zero sum subsequence
    - missing any values from the sumset

Constructing/Discarding Sequences

1. Inductively/recursively construct length-ℓ sequences, starting with the empty sequence.
2. For each sequence S of length i, concatenate a group element g to get a sequence of length $i+1$.
3. Check if the sequence Sg is "bad":
   - If $g \in -\Sigma(S)$, Sg has a zero sum.
   - Check fullness of sumset; however, we cannot rule out all sequences with $\Sigma(Sg) \subsetneq G \setminus \{0\}$. Instead monitor *expansion of sumsets*.
4. Repeat until reaching length ℓ.

Expansion of Sumsets

Let $S = g_1 \cdots g_\ell$ and write $S_i = g_1 \cdots g_i$ for $1 \leq i \leq \ell$.

$$\begin{aligned} |\Sigma(S_\ell)| &\leq |\Sigma(S_{\ell-1})| + |\Sigma(S_{\ell-1}) + g_\ell| + 1 \\ &= 2\,|\Sigma(S_{\ell-1})| + 1 \\ &\ \vdots \\ &\leq 2^{\ell-i}\,|\Sigma(S_i)| + 2^{\ell-i} - 1 \end{aligned}$$

If S has (nearly) full sumset, this implies

$$|\Sigma(S_i)| \geq \frac{1}{2^{\ell-i}} |G|$$

This gives us a numerical check to perform for each sequence at each stage of construction.

Results from the Search Procedure

n	$\lceil 2 \log n \rceil$	$2 \lceil \log n \rceil$
5	5	6
9	7	8
10	7	8
11	7	8