Roots in Extensions of Domains

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Monadic methods

Let H be a monoid.

- A monoid M is called a *monad of H*, if $M \simeq ([[a]]_H)_{red}$ for some $a \in H$.
- $\widehat{H}$ is a root extension of H if and only if $\widehat{M}$ is a root extension of M for every monad M of H.
- If every monad of H is finitely generated, then $\widehat{H}$ is a root extension of H.
- *"monadically finitely generated"* := "every monad of H is finitely generated" = "locally finitely generated" =[*in case of domains*] "atomic and IDPF"

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Review of results of Etingof, Malcolmson and Okoh (2010)

Let $H \subseteq D$ be an extension of monoids.

- If H is monadically finitely generated and $H : D \neq \emptyset$, then $H \subseteq D$ is a root extension, $D^\times / H^\times$ is finite and D is monadically finitely generated.
- If $H \subseteq D$ is a root extension, $D^\times / H^\times$ is finite and D is monadically finitely generated, then H is monadically finitely generated.

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Review of results of Jedrzejewicz and Zielinski (2017)

Let S be a submonoid of a factorial monoid H such that $H \cap q(S) = S$ and $S^\times = H^\times$.

- If every atom of S is squarefree in H, then S is root-closed in H.
- In particular, if every squarefree element of S is squarefree in H, then S is root-closed in H.

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Extraction methods (I)

Let H be a monoid.

- The function $\lambda_H : H \times H \to [0, \infty]$ defined by $\lambda_H(x, y) := sup\{m/n | m \in \mathbb{N}_0, n \in \mathbb{N}, x^m |_H y^n\}$ is called the *extraction degree of H.*
- H is called an *extraction monoid*, if for any $x \in H \setminus H^\times$, $y \in H$ there are $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ such that $x^m |_H y^n$ and $\lambda_H(x, y) = m/n$.
- $(\mathbb{N}_0, +)$ is an extraction monoid with extraction degree λ given by $\lambda(x, y) = y/x$, if $x \neq 0$, and $\lambda(0, y) = \infty$.
- Any Krull monoid is an extraction monoid.

Let H be a monoid, $x \in q(H)$ and $c \in H$ such that $cx^k \in H$ for all $k \in \mathbb{N}$. Then the following conditions are equivalent:

- $x^m \in H$ for some $m \in \mathbb{N}$.
- If $\lambda_H(c, cx) = 1$, then $c^m |_H (cx)^m$ for some $m \in \mathbb{N}$.

Let D be an extension of an extraction monoid H. Then the following conditions are equivalent:

- $D \cap q(H)$ is a root extension of H
- $\lambda_H = \lambda_D|_{H \times H}$.

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Another application

A monoid H is called *archimedian*, if $\bigcap_{n \geq 0} x^n H = \emptyset$ for every $x \in H \setminus H^\times$.

- If H satisfies the ascending chain condition for principal ideals, then it is archimedian.
- Let H be an archimedian monoid such that $fH = rad(fH)$ for some $f \in H : \widehat{H}$. Then $H = \widehat{H}$.

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