A FIRST-ORDER SEQUENTIAL PREDICTOR-CORRECTOR REGULARIZATION METHOD FOR ILL-POSED VOLTERRA EQUATIONS

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ABSTRACT. A discrete sequential predictor-corrector regularization method based on affine continuation into the future interval for the solution of ill-posed Volterra equations is presented. An infinite dimensional analogue is derived and analyzed. Stability estimates and convergence in the case of exact and noisy data are proved. A convergence rate result is proved for finitely smoothing kernels if the exact solution is $\mathcal{C}^{1,\alpha}$-smooth. Numerical examples are presented. The results are compared to other standard regularization methods.

1. INTRODUCTION

In this work we consider a sequential predictor-corrector regularization method for the solution of first-kind Volterra equations of the form

$$\int_0^t k(t-s)\, u(s)\, ds = f(t) \tag{1.1}$$

on $[0,1]$. The solution of a Volterra equation at time t depends only on the data $f(s)$ for $s \leq t$. Because of this property numerical approaches to the solution of Volterra equations are usually *sequential*, which means that if we consider a finite number of time-steps $t_0, \ldots, t_N$, the solution at time t_i is constructed from the current data value $f(t_i)$ and (already known) values of the solution at times t_j with $j < i$. In a somewhat sloppy use of language we call every method sequential which produces the solution time-step after time-step. The first-kind Volterra operators we are considering are usually compact, which leads to discontinuous dependence of the solution on the data. Therefore, *ad-hoc* numerical methods (either sequential or not) produce highly unstable numerical results.

In this paper we consider a regularization method for Volterra equations which uses information also from future data points to obtain stable numerical results. Moreover, the method is sequential and therefore fast and suitable for online (real-time) computations. This structure preserving property of our regularization method distinguishes it from many established regularization methods; for example Tikhonov regularization would use information of the data over the whole time-interval $[0,1]$ to construct the solution at some intermediate time $0 < t < 1$. Moreover, a discretization of Tikhonov's method usually gives full matrices which cannot be solved sequentially.

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Sequential regularization methods for ill-posed Volterra equations using information from the current and from a few future data points (in contrast to using information over the whole time-domain) were introduced and investigated by J. V. Beck already in the early 1960's. A discretized version of the method we are considering can be found in Beck, Blackwell and St. Clair [1, p. 131] under the name of "sequential linear function specification method".

More recently, P. K. Lamm made considerable contributions to the development and understanding of sequential regularization methods. Specifically with [5], [6], and [4] it could be shown that Beck's sequential constant function specification method can be interpreted as discretization of a properly defined Volterra equation of the second kind. Furthermore, it can be shown that the second-kind equation is solvable in a stable way [6]. Also it has been shown that, as the data noise goes to zero, solutions of the approximating second-kind Volterra equations converge to the exact solution for finitely smoothing kernels under a certain stability condition [6]. For 1-smoothing kernels, a convergence rate of order $O(\delta^{\frac{1}{2}})$ was derived in [4]. Several generalizations were considered in [9], [8], and [3]. We also refer to the survey article [7] and the numerous references cited therein for an overview of the literature on stable numerical methods for the sequential solution of ill-posed Volterra equations. We should also mention that the class of regularization methods under consideration here occurs under various names in the literature such as "local regularization methods", "predictor-corrector regularization", "future-sequential regularization", "Beck's method", or combinations of these notions.

The idea behind Beck's method and its variants in the discrete case is to choose the value of the solution u_i on the time-interval $[t_{i-1}, t_i]$ in such a way that a constant continuation of the solution with this value fits the given data not only for the current time t_i but also, in a least-squares sense, at a few time steps t_{i+j}, $j = 1, \dots, r$ *in the future.* In this paper we investigate a variant of Beck's method where we replace constant continuation by affine continuation into the future interval. Moreover, the continuation is assumed to be continuously connected to the already determined part of the solution. We refer to our approach as a "first-order sequential regularization method" in contrast to the "zero-order method" based on constant continuation. In the infinite dimensional limit our method leads to a Volterra *integro-differential equation* of the second kind.

A similar generalization is presented in [3], where a polynomial of degree $d \geq 0$ is used for the continuation into the future interval. The structural difference between this approach and ours is that this polynomial does not connect continuously to the already determined part of the solution. Numerically our first order method produces smoother results that the polynomial method in [3].

In section 2 we derive the regularization scheme for the discrete case and generalize the discrete approach to find a second-kind Volterra integro-differential equation representing the regularization method in the infinite dimensional case. Section 3 is devoted to the convergence analysis of the (infinite dimensional) regularization method. We obtain stability estimates and a convergence result in the case of exact data. For the class of finitely smoothing kernels we can also prove a convergence rate result. As usual in inverse problems the rate of convergence is determined by smoothness properties of the exact solution (cf. Theorem 3). Here we can prove more than what is known in the literature for the zero-order method for which a convergence rate result is still lacking. Moreover, we could avoid to impose strong smoothness conditions on the exact solution to get convergence, as it is necessary in the analysis in [6]. The last section 4 contains numerical experiments and comparisons with other regularization methods. Here it is seen that the first-order method gives much smoother solutions than the zero-order method with approximately the same error in the supremum-norm (see Figure 2). It is also seen (Figure 3) that our method is less sensitive to over-regularization than the zero-order method.

2. A first-order sequential regularization method

We consider the linear Volterra equation of the first kind

$$\int_0^t k(t-s)\,u(s)\,ds = f(t) \quad \text{for } t \in [0,1] \tag{2.1}$$

with $k, f \in \mathcal{C}[0,1]$. It follows from the Arzela-Ascoli Theorem that the integral operator in (2.1) is compact on $\mathcal{C}[0,1]$. Hence, a solution to (2.1), if it exists, does not depend continuously on the right hand side f. In the following, we define a family of approximating well-posed problems which allows to solve (2.1) in a numerically stable way.

Let us develop the main idea for a discretized version of (2.1). Let $0 = t_0 < t_1 < \cdots < t_N = 1$ with $t_i = \frac{1}{i}$ for $i = 0, \cdots, N$ be a uniform partition of the interval $[0,1]$. We set $h = \frac{1}{N}$. Let $\mathcal{B}_N$ be the space of continuous, piecewise affine functions with respect to the partition $\{t_i\}_{i=0}^N$, i.e.,

$$\begin{aligned}
\mathcal{B}_N &= \{\varphi \in \mathcal{C}[0,1] \,:\, \varphi|_{[t_{i-1},t_i]} = a_i\,t + b_i;\ a_i, b_i \in \mathbb{R} \text{ for } i = 1, \cdots, N\} \\
&= \operatorname{span}\{\varphi_i^N \,:\, \varphi_i^N \text{ continuous on } [0,1], \\
&\qquad\quad \text{affine on } [t_{k-1}, t_k] \text{ for } k = 1, \ldots, N, \\
&\qquad\quad \varphi_i^N(t_j) = \delta_{i,j} \text{ for } i,j = 0, \cdots, N\}.
\end{aligned}$$

We seek a function $u^N \in \mathcal{B}_N$ for which (2.1) is exactly satisfied at the collocation points t_i. We set $u_i^N = u^N(t_i)$, $f_i^N = f(t_i)$ for $i = 0, \cdots, N$, $\mathbf{u}^N = (u_0^N, \cdots, u_N^N)^T$, and $\mathbf{f}^N = (f_0^N, \cdots, f_N^N)^T$. It is obvious from (2.1) that u_0^N can be chosen arbitrarily and $f_0^N = 0$ has to be satisfied. For $i \geq 1$ the collocation assumption yields

$$\int_0^{t_i} k(t_i - s)\Big(\sum_{j=0}^{i} u_j^N\,\varphi_j^N(s)\Big)\,ds - f_i^N =$$

$$\sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} k(t_i - s)\left(u_{j-1}^N + \frac{u_j^N - u_{j-1}^N}{h}(s - t_{j-1})\right)ds - f_i^N = 0. \tag{2.2}$$

Equation (2.2) is a linear system for the unknown vector $\mathbf{u}^N$. We write the system (2.2) in Matrix form:

$$A^N \mathbf{u}^N = \mathbf{f}^N. \tag{2.3}$$

The system matrix A^N is lower triangular and given by

$$A^N = \begin{pmatrix} 0 & & & & \\ \Delta_{1,0} & \Delta_{1,1} & & \mathbf{0} & \\ \Delta_{2,0} & \Delta_{2,1} & \Delta_{2,2} & & \\ \vdots & & & \ddots & \\ \Delta_{N,0} & & \cdots & & \Delta_{N,N} \end{pmatrix}$$

with

$$\Delta_{i,j} = \int_0^{t_i} k(t_i - s)\, \varphi_j^N(s)\, ds. \tag{2.4}$$

Note that the reduced system matrix

$$\tilde{A}^N = \begin{pmatrix} \Delta_{1,1} & & & & \\ \Delta_{2,1} & \Delta_{2,2} & & \mathbf{0} & \\ \Delta_{3,1} & \Delta_{3,2} & \Delta_{3,3} & & \\ \vdots & & & \ddots & \\ \Delta_{N,1} & & \cdots & & \Delta_{N,N} \end{pmatrix}$$

has Toeplitz-structure and we can define Δ_m via

$$\Delta_{i-j+1} = \Delta_{i,j} \text{ for } i, j \geq 0. \tag{2.5}$$

Obviously, A^N is singular. We have already seen that we have free choice for u_0^N. But once we have chosen u_0^N, equation (2.3) can be solved *sequentially* for the remaining variables $u_1^N, \ldots, u_N^N$, i.e., we have

$$u_i^N = \frac{1}{\Delta_1}\Big(f_i^N - \Delta_{i,0}\, u_0^N - \sum_{j=1}^{i-1} \Delta_{i-j+1}\, u_j^N \Big). \tag{2.6}$$

If we replace the right-hand side in (2.6) by a perturbation $f_i^N + \delta f_i^N$ we obtain

$$\delta u_i^N = \frac{\delta f_i^N}{\Delta_1}$$

for the error in the solution u_i^N. Thus, noise in the data is amplified by a factor $\frac{1}{\Delta_1}$ in the solution. The condition of the linear system therefore depends crucially on the magnitude of

$$\Delta_1 = \frac{1}{h} \int_0^h k(s)\,(h - s)\, ds.$$

We have:

$$\Delta_1 \sim k(0)\frac{h}{2!} + k'(0)\frac{h^2}{3!} + k''(0)\frac{h^3}{4!} + \cdots$$

and thus, for small h, the magnitude of Δ_1 is determined by the first non-vanishing derivative of k at $t = 0$. We say that the kernel k is *ν-smoothing* if:

$$k \in \mathcal{C}^{(\nu)}[0,1] \text{ and } k^{(l)}(0) = 0 \text{ for } l = 0, \dots, \nu - 2, \text{ and } k^{(\nu-1)}(0) \neq 0, \tag{2.7}$$

where $k^{(l)}$ denotes the lth derivative of k. We see that (2.3) becomes more and more ill-conditioned the larger ν is. **We assume for the rest of the paper that k is ν-smoothing with $\nu \geq 1$.**

We pursue the following idea to find an approximating solution to (2.3) which is stable and sequential. We do not try to match every data point f_i^N exactly. Instead, we choose the solution u^N to be an affine function on the interval $[t_{i-1}, t_i]$, for which an (affine) continuation onto the interval $[t_{i-1}, t_{i+r}]$ fits the data $(f_i^N, \dots, f_{i+r}^N)$ in a weighted least-squares sense, for some $r > 0$. Thus, we also take future information into account to choose an appropriate solution on the current interval. Let us make this idea precise. We set $u^N(t) = \sum_{i=0}^N u_i^N \varphi_i^N(t)$ and we consider the optimization problem:

$$\begin{aligned}\sum_{l=0}^{r} \omega_l \Bigg| &\int_0^{t_{i-1}} k(t_{i+l} - s) \sum_{j=0}^{i-1} u_j^N \varphi_j^N(s)\, ds + \\ &\int_{t_{i-1}}^{t_{i+l}} k(t_{i+l} - s)\Big(u_{i-1}^N + (u_{i-1}^N)'(s - t_{i-1})\Big)\, ds - f_{i+l}^N \Bigg|^2 \to \min\end{aligned} \tag{2.8}$$

for $i = 1, 2, \dots$, where $(u_{i-1}^N)' = \frac{1}{h}(u_i^N - u_{i-1}^N)$. The numbers $\omega_0, \dots, \omega_r$ are given weights. In general (2.8), is solved for $(u_{i-1}^N)'$ with known $(u_0^N, \dots, u_{i-1}^N)$. Setting $u_i^N = u_{i-1}^N + h(u_{i-1}^N)'$, we can then go to the next step $i \mapsto i + 1$. Only in the first step, $i = 1$, we have to solve for two unknowns u_0^N and $(u_0^N)'$.

For $i = 1$, the necessary optimality conditions for (2.8) are given by:

$$\begin{aligned}&\sum_{l=0}^{r} \omega_l \left(\int_0^{t_{l+1}} k(t_{l+1} - s)\Big(u_0^N + (u_0^N)' s\Big)\, ds \right) \int_0^{t_{l+1}} k(t_{l+1} - s)\, ds \\ &\qquad = \sum_{l=0}^{r} \omega_l\, f_{l+1}^N \int_0^{t_{l+1}} k(t_{l+1} - s)\, ds,\end{aligned} \tag{2.9}$$

$$\begin{aligned}&\sum_{l=0}^{r} \omega_l \left(\int_0^{t_{l+1}} k(t_{l+1} - s)\Big(u_0^N + (u_0^N)' s\Big)\, ds \right) \int_0^{t_{l+1}} k(t_{l+1} - s)\, s\, ds \\ &\qquad = \sum_{l=0}^{r} \omega_l\, f_{l+1}^N \int_0^{t_{l+1}} k(t_{l+1} - s)\, s\, ds.\end{aligned} \tag{2.10}$$

For $i > 1$, the optimality condition for (2.8) is given by:

$$
\begin{aligned}
&\sum_{l=0}^{r} \omega_l \Bigg(\int_0^{t_{i-1}} k(t_{i+l}-s) \sum_{j=0}^{i-1} u_j^N \, \varphi_j^N(s)\, ds \\
&\qquad + \int_{t_{i-1}}^{t_{i+l}} k(t_{i+l}-s) \Big(u_{i-1}^N + (u_{i-1}^N)'(s-t_{i-1}) \Big)\, ds \Bigg) \\
&\qquad\qquad \cdot \int_{t_{i-1}}^{t_{i+l}} k(t_{i+l}-s)\,(s-t_{i-1})\, ds \\
&= \sum_{l=0}^{r} \omega_l\, f_{i+l}^N \int_{t_{i-1}}^{t_{i+l}} k(t_{i+l}-s)\,(s-t_{i-1})\, ds.
\end{aligned} \tag{2.11}
$$

We write (2.9)–(2.11) in a more compact notation where we omit the discretization index N:

$$
\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix} = \begin{pmatrix} \hat{g} \\ \hat{f}_0 \end{pmatrix}, \tag{2.12a}
$$

$$
\begin{aligned}
&\sum_{j=0}^{i} \tilde{\Delta}_{i,j} u_j + \beta\, u_i + \gamma\, u_i' = \hat{f}_i \text{ for every } i \geq 1, \\
&\qquad u_{i+1} = u_i + h u_i' \text{ for every } i \geq 0,
\end{aligned} \tag{2.12b}
$$

where we set:

$$
\begin{aligned}
\kappa_l &= \int_{t_{i-1}}^{t_{i+l}} k(t_{i+l}-s)\, ds = \int_0^{(l+1)h} k(s)\, ds, \\
\lambda_l &= \int_{t_{i-1}}^{t_{i+l}} k(t_{i+l}-s)\,(s-t_{i-1})\, ds = \int_0^{(l+1)h} ((l+1)h-s)\, k(s)\, ds,
\end{aligned}
$$

for $l = 0, \dots, r$,

$$
\begin{aligned}
\alpha &= \sum_{l=0}^{r} \omega_l\, \kappa_l^2, \quad \beta = \sum_{l=0}^{r} \omega_l\, \kappa_l\, \lambda_l, \quad \gamma = \sum_{l=0}^{r} \omega_l\, \lambda_l^2, \\
\hat{g} &= \sum_{l=0}^{r} \omega_l\, f_{l+1}\, \kappa_l, \\
\hat{f}_i &= \sum_{l=0}^{r} \omega_l\, f_{i+l+1}\, \lambda_l \text{ for } i \geq 0,
\end{aligned}
$$

and

$$
\tilde{\Delta}_{i,j} = \sum_{l=0}^{r} \omega_l\, \lambda_l \int_0^{t_i} k(t_{i+l+1}-s)\, \varphi_j(s)\, ds \text{ for } 0 \leq j \leq i.
$$

If we solve the 2×2-system (2.12a) for the initial values u_0 and u_0', and if we assign $u_1 = u_0 + h u_0'$, then we can solve (2.12b) *sequentially* by forward substitution:

$$u_i' = \frac{1}{\gamma}\Big(\hat{f}_i - \beta\, u_i - \sum_{j=0}^{i} \tilde{\Delta}_{i,j}\, u_j\Big)$$
$$u_{i+1} = u_i + h\, u_i' \text{ for } i \geq 1.$$

The system (2.12) can be interpreted as a *discretization of a second-kind Volterra integro-differential equation.* This is seen as follows. We suppose that ω is a positive, regular Borel measure on $[0, \rho]$ for $\rho > 0$ and we set:

$$\kappa(\tau) = \int_0^\tau k(s)\, ds, \tag{2.13}$$

$$\lambda(\tau) = \int_0^\tau (\tau - s)\, k(s)\, ds, \tag{2.14}$$

$$a = \int_0^\rho \kappa^2(\tau)\, d\omega(\tau), \quad b = \int_0^\rho \kappa(\tau)\, \lambda(\tau)\, d\omega(\tau), \tag{2.15}$$
$$c = \int_0^\rho \lambda^2(\tau)\, d\omega(\tau),$$

$$\tilde{g} = \int_0^\rho f(\tau)\, \kappa(\tau)\, d\omega(\tau), \tag{2.16}$$

$$\tilde{f}(t) = \int_0^\rho \lambda(\tau)\, f(t+\tau)\, d\omega(\tau), \tag{2.17}$$

$$\tilde{k}(t) = \int_0^\rho \lambda(\tau)\, k(t+\tau)\, d\omega(\tau). \tag{2.18}$$

A careful examination gives $\kappa((l+1)h) = \kappa_l$, and $\lambda((l+1)h) = \lambda_l$, for $l = 0, \ldots, r$. Let $\delta_x(\tau)$ denote the Dirac point measure located at $\tau = x$ and let $\rho = (r+1)h + \varepsilon$ with $\varepsilon > 0$. Then, for the special case where ω is the sum of discrete point measures given by

$$\omega = \sum_{l=0}^{r} \omega_l \delta_{(l+1)h}(\tau) \tag{2.19}$$

we have:

$$a = \alpha, \qquad b = \beta, \qquad c = \gamma,$$
$$\tilde{g} = \hat{g}, \qquad \tilde{f}(t_i) = \hat{f}_i \text{ for } i = 0, 1, 2, \ldots$$

and

$$\sum_{j=0}^{i} \tilde{\Delta}_{i,j}\, u_j = \int_0^{t_i} \tilde{k}(t_i - s)\, u^N(s)\, ds.$$

Thus, if we consider the Volterra integro-differential equation:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} = \begin{pmatrix} \tilde{g} \\ \tilde{f}(0) \end{pmatrix}, \tag{2.20a}$$

$$\int_0^t \tilde{k}(t-s)\, u(s)\, ds + bu(t) + cu'(t) = \tilde{f}(t) \tag{2.20b}$$

we find that (2.12) is in fact a discretization of (2.20), where ω is given by (2.19), $u = u^N \in \mathcal{B}_N$, equation (2.20b) is evaluated at the collocation points t_i, and u_i' is interpreted as left-sided derivative at t_i. Note that the second component in (2.20a) is actually equation (2.20b) at time $t = 0$. We prefer, however, the redundant notation (2.20a) since it highlights the role of the initial values $(u(0), u'(0))$.

The following section is devoted to the analysis of Volterra integro-differential equation (2.20).

3. Convergence Analysis

We set

$$k_\nu = k^{(\nu-1)}(0) \neq 0.$$

(Recall our assumption that k is ν-smoothing.) We assume in the following without loss of generality that $k_\nu > 0$. Otherwise, if $k_\nu < 0$ we deal with $-k$. Note that this assumption implies that $k(s) > 0$ for s small enough. We therefore usually omit the absolute-value bars if we integrate k over small intervals of the form $[0, \rho]$.

For every $\rho \in (0, R]$ let ω_ρ be a positive, regular Borel measure on $[0, \rho]$. If we do not want to stress the dependence of ω_ρ on ρ, we frequently omit the subscript. Here and in the following we use standard Landau o-symbols to denote functions vanishing at zero. Throughout the rest of the paper we use the following assumptions on the moments of the measure ω_ρ.

(A_1) There exists an integer $s \geq 0$ and there exist positive numbers C_j independent of ρ such that

$$\int_0^\rho \tau^j \, d\omega_\rho(\tau) = \rho^{s+j}(C_j + o(\rho)) \text{ for } j = 0, \dots, 2\nu + 2$$

with

$$(A_2) \qquad C_{2\nu} C_{2\nu+2} - C_{2\nu+1}^2 > 0.$$

(A_3) All roots of the polynomial $\displaystyle\sum_{l=0}^{\nu+1} \frac{C_{\nu+l+1}}{l!} x^l$ have negative real parts.

Remark 1. By assumption (A_1) we have

$$\left(\int_0^\rho \tau^{2\nu+1} \, d\omega \right)^2 = \rho^{2s+4\nu+2}\big(C_{2\nu+1}^2 + o(\rho)\big).$$

On the other hand, we have

$$\left(\int_0^\rho \tau^{2\nu+1} \, d\omega \right)^2 = \left(\int_0^\rho \tau^\nu \, \tau^{\nu+1} \, d\omega \right)^2 \leq \int_0^\rho \tau^{2\nu} \, d\omega \int_0^\rho \tau^{2\nu+2} \, d\omega$$
$$= \rho^{2s+4\nu+2}\big(C_{2\nu} C_{2\nu+2} + o(\rho)\big).$$

Hence, we find that $C_{2\nu} C_{2\nu+2} - C_{2\nu+1}^2 \geq 0$ is always satisfied. Condition (A_2) is therefore only the requirement that the inequality be strict.

Remark 2. The role of condition (A_3) will become apparent throughout the subsequent considerations. It is a stability criterion for (2.20).

Before we investigate solvability and well-posedness of the integro-differential equation (2.20) we give some asymptotic estimates which we derive from (A_1) and the definitions of the respective terms in section 1. We have:

$$k^{(l)}(s) = s^{\nu-l-1}\left(\frac{k_\nu}{(\nu-l-1)!} + o(s)\right) \text{ for } l = 0,\dots,\nu-1, \tag{3.1}$$

$$\kappa(\tau) = \tau^{\nu}\left(\frac{k_\nu}{\nu!} + o(\tau)\right), \tag{3.2}$$

$$\lambda(\tau) = \tau^{\nu+1}\left(\frac{k_\nu}{(\nu+1)!} + o(\tau)\right), \tag{3.3}$$

$$\int_0^\rho \kappa(\tau)\, d\omega(\tau) = \rho^{s+\nu}\left(C_\nu \frac{k_\nu}{\nu!} + o(\rho)\right), \tag{3.4}$$

$$\int_0^\rho \lambda(\tau)\, d\omega(\tau) = \rho^{s+\nu+1}\left(C_{\nu+1} \frac{k_\nu}{(\nu+1)!} + o(\rho)\right), \tag{3.5}$$

$$a = \rho^{s+2\nu}\left(C_{2\nu} \frac{k_\nu^2}{(\nu!)^2} + o(\rho)\right), \tag{3.6}$$

$$b = \rho^{s+2\nu+1}\left(C_{2\nu+1} \frac{k_\nu^2}{\nu!(\nu+1)!} + o(\rho)\right), \tag{3.7}$$

$$c = \rho^{s+2\nu+2}\left(C_{2\nu+2} \frac{k_\nu^2}{((\nu+1)!)^2} + o(\rho)\right). \tag{3.8}$$

From definition (2.18) it easily follows that $\tilde{k} \in \mathcal{C}[0,1]$. Using (3.1), (3.3), and (A_1), we obtain:

$$\tilde{k}^{(l)}(0) = \rho^{s+2\nu-l}\left(C_{2\nu-l} \frac{k_\nu^2}{(\nu+1)!(\nu-l-1)!} + o(\rho)\right) \text{ for } l = 0,\dots,\nu-1. \tag{3.9}$$

We now prove solvability and well-posedness for (2.20) for a general continuous right-hand side f.

Proposition 1. *Let f be continuous on $[0, 1+R)$ for some $R > 0$ and let k be ν-smoothing with $\nu > 0$. Let a, b, and c be given by (2.15) and suppose that $\tilde{g}$, $\tilde{f}$ and $\tilde{k}$ are defined by (2.16), (2.17), and (2.18), respectively. Assume moreover that assumptions (A_1) and (A_2) hold. Then there exists a $\rho_0 > 0$ such that problem (2.20) has a unique solution $u \in \mathcal{C}^1[0,1]$ for all $\rho \le \rho_0$. Moreover there exist constants $\mathcal{M}, \overline{\mathcal{M}} > 0$ independent of ρ such that*

$$\|u\|_{\mathcal{C}[0,1]} \le \frac{\mathcal{M}}{\rho^{\nu+1}} \exp\left(\frac{\mathcal{M}\, t}{\rho^{\nu+1}}\right) \|f\|_{\mathcal{C}[0,1+\rho]} \tag{3.10}$$

and

$$\|u'\|_{\mathcal{C}[0,1]} \le \frac{\overline{\mathcal{M}}}{\rho^{2\nu+2}} \exp\left(\frac{\mathcal{M}\, t}{\rho^{\nu+1}}\right) \|f\|_{\mathcal{C}[0,1+\rho]}. \tag{3.11}$$

Proof. It is easily seen from the definition (2.17) that $\tilde{f}$ is continuous on $[0, 1+R-\rho)$, and by (3.7) and (3.8) that $b \neq 0$ and $c \neq 0$ for ρ small enough. Hence, it follows by standard Picard-iteration arguments (see Burton [2, pp. 23, 24] that (2.20b) has a unique solution $u \in \mathcal{C}^1[0,1]$ provided that we can solve the linear system (2.20a) uniquely for $u(0)$. Using (3.6) and (3.8) we find

(3.12)
$$\det\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \rho^{2s+4\nu+2}\frac{k_\nu^4}{(\nu!)^2((\nu+1)!)^2}\Big(C_{2\nu}C_{2\nu+2} - C_{2\nu+1}^2 + o(\rho)\Big) > 0$$

for ρ small enough due to assumption (A_2). Thus, the matrix in (2.20a) is invertible, and solvability for (2.20b) follows.

Note that the mappings $f \mapsto (\tilde{g}, \tilde{f}(0))$, $(\tilde{g}, \tilde{f}(0)) \mapsto u(0)$, and $f \mapsto \tilde{f}$ are all linear. Therefore, the solution operator $f \mapsto u(f)$, where $u(f)$ is the solution to (2.20), is linear.

We now estimate $\tilde{g}$ and $\tilde{f}(0)$ in terms of $\|f\|_{\mathcal{C}[0,1+\rho]}$. With (2.16), (2.17), (3.4), and (3.5) we have

$$\tilde{g} \le \rho^{s+\nu}\left(C_\nu\frac{k_\nu}{\nu!} + o(\rho)\right)\|f\|_{\mathcal{C}[0,1+\rho]}$$

and

$$\tilde{f(0)} \le \rho^{s+\nu+1}\left(C_{\nu+1}\frac{k_\nu}{(\nu+1)!} + o(\rho)\right)\|f\|_{\mathcal{C}[0,1+\rho]}.$$

Thus, using (3.6), (3.7), and (3.8) we find

$$\begin{aligned} |u(0)| &= \left|\frac{c\tilde{g} - b\tilde{f}(0)}{ac-b^2}\right| \\ &\le \frac{1}{\rho^\nu}\left|\frac{\nu!}{k_\nu}\frac{C_{2\nu+2}C_\nu + C_{2\nu+1}C_{\nu+1}}{C_{2\nu+2}C_{2\nu} - C_{2\nu+1}^2} + o(\rho)\right|\|f\|_{\mathcal{C}[0,1+\rho]} \\ &\le \frac{1}{\rho^\nu}\, m_1\,\|f\|_{\mathcal{C}[0,1+\rho]} \end{aligned} \tag{3.13}$$

for some constant $m_1 > 0$ independent of ρ. With (3.8) we obtain

$$|c\,u(0)| \le \rho^{s+\nu+2}\, m_2\,\|f\|_{\mathcal{C}[0,1+\rho]} \tag{3.14}$$

for some constant $m_2 > 0$ independent of ρ.

From (3.5) and definition (2.17) we derive the following estimate:

$$\begin{aligned} \left|\int_0^t \tilde{f}(s)\,ds\right| &\le \rho^{s+\nu+1}\left(C_{\nu+1}\frac{k_\nu}{(\nu+1)!} + o(\rho)\right)\|f\|_{\mathcal{C}[0,1+\rho]} \\ &\le \rho^{s+\nu+1}\, m_3\,\|f\|_{\mathcal{C}[0,1+\rho]} \end{aligned} \tag{3.15}$$

for all $t \in [0,1]$ with some constant $m_3 > 0$ independent of ρ. Estimates (3.14) and (3.15) yield that there exists a constant $m_4 > 0$ independent of ρ such that

$$\left|\int_0^t \tilde{f}(s)\,ds + c\,u(0)\right| \le \rho^{s+\nu+1}\, m_4\,\|f\|_{\mathcal{C}[0,1+\rho]} \tag{3.16}$$

for all $t \in [0,1]$.

We set

$$K(t) = \int_0^t \tilde{k}(s)\,ds + b.$$

Using (3.5) and (3.7) we get:

$$
\begin{aligned}
|K(t)| &\leq \rho^{s+\nu+1}\left(C_{\nu+1}\,\frac{k_\nu}{(\nu+1)!}+o(\rho)\right)\|k\|_{\mathcal{C}[0,1+\rho]}+b \\
&\leq \rho^{s+\nu+1}\,m_5 \qquad (3.17)
\end{aligned}
$$

for all $t \in [0,1]$ with some constant m_5 depending on k but not on ρ. Moreover, it is obvious from (3.8) that there exists a constant $m_6 > 0$ independent of ρ such that

$$\frac{1}{c} \leq \frac{m_6}{\rho^{s+2\nu+2}} \qquad (3.18)$$

for all ρ sufficiently small. We set

$$\mathcal{M} = \max(m_4\,m_6, m_5\,m_6). \qquad (3.19)$$

Integrating (2.20b) with respect to t yields

$$\int_0^t K(t-s)\,u(s)\,ds + c\,u(t) = \int_0^t \tilde{f}(s)\,ds + c\,u(0).$$

Hence, by (3.16), (3.17), (3.18), and (3.19) we obtain

$$|u(t)| \leq \frac{\mathcal{M}}{\rho^{\nu+1}}\left(\|f\|_{\mathcal{C}[0,1+\rho]} + \int_0^t |u(s)|\,ds\right)$$

for all $t \in [0,1]$. Gronwall's inequality then implies

$$|u(t)| \leq \frac{\mathcal{M}}{\rho^{\nu+1}}\,\exp\left(\frac{\mathcal{M}\,t}{\rho^{\nu+1}}\right)\,\|f\|_{\mathcal{C}[0,1+\rho]}. \qquad (3.20)$$

Thus, (3.10) follows. From (2.20b) we conclude

$$|u'(t)| \leq \frac{1}{c}\Big((1+\|\tilde{k}\|_{\mathcal{C}[0,1]})\,\|u\|_{\mathcal{C}[0,1]} + \|\tilde{f}\|_{\mathcal{C}[0,1]}\Big)$$

for all $t \in [0,1]$. With (3.20), (3.7), (3.8), and (3.5) we find that there exists a constant $\overline{\mathcal{M}} > 0$ such that

$$|u'(t)| \leq \frac{\overline{\mathcal{M}}}{\rho^{2\nu+2}}\,\exp\left(\frac{\mathcal{M}}{\rho^{\nu+1}}\right)\,\|f\|_{\mathcal{C}[0,1+\rho]} \qquad (3.21)$$

for all $t \in [0,1]$. This completes the proof. □

The study of the homogeneous equation

$$u'(t) = -\frac{b}{c}\,u(t) - \frac{1}{c}\int_0^t \tilde{k}(t-s)\,u(s)\,ds \text{ for } t \in [0,1] \qquad (3.22)$$

will help us to obtain estimates for different corresponding inhomogeneous problems. We use the transform $y(t) = u(\rho t)$ to get the integro-differential equation

$$y'(t) = -\frac{b\rho}{c}y(t) - \frac{\rho^2}{c}\int_0^t \tilde{k}(\rho(t-s))\,y(s)\,ds \text{ for } t \in [0,\frac{1}{\rho}]. \qquad (3.23)$$

Let z_ρ denote the fundamental solution to (3.23), i.e., the solution to (3.23) with $z_\rho(0) = 1$. The following estimates for z_ρ will be useful in combination with the variation of constants formula.

Lemma 1. *Assume that conditions* (A_1), (A_2), *and* (A_3) *hold. Then there exist constants* $\tilde{C} > 0$ *(depending on* ω*) and* $m > 0$ *(depending on* k *and* ω*) such that*

$$|z_\rho(t)| \le m \quad \text{for all } t \in \Big[0, \frac{1}{\rho}\Big] \tag{3.24}$$

and

$$\|z_\rho\|_{L^1(0,\frac{1}{\rho})} \le m \tag{3.25}$$

independently of ρ *if*

$$\frac{\|k^{(\nu)}\|_{L^1(0,1)}}{k_\nu} < \tilde{C}. \tag{3.26}$$

Proof. Since $\tilde{k}$ is ν times continuously differentiable we can differentiate (3.23) with respect to t. Setting $y = z_\rho$ we obtain

$$\begin{aligned} z_\rho^{(j+1)}(t) = -\frac{b\rho}{c} z_\rho^{(j)}(t) - \sum_{l=0}^{j-1} \frac{\tilde{k}^{(l)}(0)\,\rho^{l+2}}{c}\, z^{(j-l-1)}(t) \\ - \frac{\rho^{j+2}}{c} \int_0^t \tilde{k}^{(j)}(\rho(t-s))\, z_\rho(s)\, ds \end{aligned} \tag{3.27}$$

for $j = 0, \dots, \nu$. We set

$$\alpha_l = \alpha_l(\nu) = \frac{C_{\nu+l+1}}{C_{2\nu+2}} \frac{(\nu+1)!}{l!} \text{ for } l = 0, \cdots, \nu. \tag{3.28}$$

Using (3.7), (3.8), and (3.9) we find for $j = \nu$ that

$$z_\rho^{(\nu+1)}(t) = -\sum_{l=0}^{\nu} \Big(\alpha_{\nu-l} + o(\rho)\Big)\, z^{(\nu-l)}(t) - \frac{\rho^{\nu+2}}{c} \int_0^t \tilde{k}^{(\nu)}(\rho(t-s))\, z_\rho(s)\, ds. \tag{3.29}$$

We write the $(\nu+1)$-order equation (3.29) as a system of integro-differential equations in the usual way, setting

$$\mathbf{z}_\rho = (z_\rho^{(0)}, \dots, z_\rho^{(\nu)})^T.$$

With this we obtain the system

$$\mathbf{z}_\rho'(t) = A\mathbf{z}_\rho(t) + M_\rho \mathbf{z}_\rho(t) + \int_0^t D_\rho(t-s)\, \mathbf{z}_\rho(s)\, ds \tag{3.30}$$

where:

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \dots & -\alpha_\nu \end{pmatrix}, \tag{3.31}$$

$$\|M_\rho\| \to 0 \text{ as } \rho \to 0, \tag{3.32}$$

and

$$D_\rho(t) = \begin{pmatrix} 0 & & \dots & 0 \\ \vdots & & & \\ 0 & & & \vdots \\ -\frac{\rho^{\nu+2}}{c}\tilde{k}^{(\nu)}(\rho t) & 0 & \dots & 0 \end{pmatrix}. \tag{3.33}$$

We now consider the relation between the initial conditions $z_\rho(0) = 1$ and $\mathbf{z}_\rho(0) = (z_\rho^{(0)}(0), \dots, z_\rho^{(\nu)}(0))^T$. Using (3.27) we find

$$z_\rho^{(j+1)}(0) = -\sum_{l=0}^{j} \Big(\alpha_{\nu-l} + o(\rho)\Big)\, z_\rho^{(j-l)}(0) \text{ for } j = 0, \dots, \nu.$$

It is seen by induction that there exists a constant $d > 0$ independent of ρ such that

$$|\mathbf{z}_\rho(0)| \le d \tag{3.34}$$

for all ρ sufficiently small.

We now prove the estimates for $|z_\rho(t)|$ and $\|z_\rho\|_{L^1(0,\frac{1}{\rho})}$. The eigenvalues of the matrix A are given by the roots of the polynomial,

$$q(x) = (-1)^{\nu+1}\left(\sum_{l=0}^{\nu} \alpha_l\, x^l + x^{\nu+1}\right).$$

Upon multiplying q by $\frac{C_{2\nu+2}}{(\nu+1)!}$ we see that the roots of q coincide with the roots of the polynomial,

$$\sum_{l=0}^{\nu+1} \frac{C_{\nu+l+1}}{l!}\, x^l. \tag{3.35}$$

By assumption (A_3), all roots of the polynomial (3.35) have negative real parts. Then, for sufficiently small ρ, all real parts of eigenvalues of $A_\rho = A + M_\rho$ are negative and bounded away from zero. With the notation $A_0 = A$ we define

$$B_\rho = \int_0^\infty (\exp(A_\rho t))^T \, \exp(A_\rho)\, dt$$

for $\rho \ge 0$ sufficiently small. The matrix B_ρ is well defined, symmetric, positive definite, and satisfies

$$A_\rho^T\, B_\rho + B_\rho\, A_\rho = -I$$

(cf. [2, p. 124]). Using Lebesgue's dominated convergence theorem we find $B_\rho \to B_0$ as $\rho \to 0$. From this it follows that there exist positive constants L, K, and R such that:

$$|\mathbf{x}| \ge 2L\,(\mathbf{x}^T B_\rho \mathbf{x})^{\frac{1}{2}} \tag{3.36}$$

$$|B_\rho \mathbf{x}| \le K\,(\mathbf{x}^T B_\rho \mathbf{x})^{\frac{1}{2}} \tag{3.37}$$

$$|\mathbf{x}| \le R\,(\mathbf{x}^T B_\rho \mathbf{x})^{\frac{1}{2}} \tag{3.38}$$

for all $\mathbf{x} \in \mathbb{R}^{\nu+1}$ and ρ sufficiently small. We define the Lyapunov functional

$$V_\rho\big(t, \mathbf{x}(\cdot)\big) = (\mathbf{x}^T B_\rho \mathbf{x})^{\frac{1}{2}} + \overline{K} \int_0^t \int_t^{\frac{1}{\rho}} \|D_\rho(\xi - s)\| \, d\xi \, |\mathbf{x}(s)| \, ds \tag{3.39}$$

with some $\overline{K} > 0$ chosen independently of ρ. Following the calculations in [2, p. 38] we find for the derivative of V_ρ along solutions to (3.30) that

$$\begin{aligned} \frac{d}{dt}\left[V_\rho\big(t, \mathbf{z}_\rho(t)\big)\right] \le &-\left(L - \overline{K} \int_t^{\frac{1}{\rho}} \|D_\rho(\xi - t)\| \, d\xi\right) |\mathbf{z}_\rho(t)| \\ &- (\overline{K} - K) \int_0^t \|D_\rho(t - s)\| \, |\mathbf{z}_\rho(s)| \, ds. \end{aligned} \tag{3.40}$$

We have:

$$\begin{aligned} \int_t^{\frac{1}{\rho}} \|D_\rho(\xi - t)\| \, d\xi &\le \frac{\rho^{\nu+2}}{c} \int_t^{\frac{1}{\rho}} |\tilde{k}^{(\nu)}(\rho(\xi - t))| \, d\xi \\ &= \frac{\rho^{\nu+1}}{c} \int_0^{1-\rho t} |\tilde{k}^{(\nu)}(s)| \, d \\ &\le \frac{\rho^{\nu+1}}{c} \int_0^{\rho} \lambda(\tau) \int_0^{1-\rho t} |k^{(\nu)}(s + \tau)| \, ds \, d\omega(\tau) \\ &\le \|k^{(\nu)}\|_{L^1(0,1+\rho)} \left(\frac{(\nu+1)!}{k_\nu} \frac{C_{\nu+1}}{C_{2\nu+2}} + o(\rho)\right). \end{aligned} \tag{3.41}$$

Here we used (3.5) and (3.8). We want to choose $\overline{K}$ in such a way that $\overline{K} - K > 0$ and there exists $\overline{L} > 0$ such that

$$\left(L - \overline{K} \int_t^{\frac{1}{\rho}} \|D_\rho(\xi - t)\| \, d\xi\right) \ge \overline{L} \tag{3.42}$$

for ρ sufficiently small and for all $t \in [1, \frac{1}{\rho}]$. Using the estimate (3.41) we see that $\overline{K}$ can be chosen satisfying the above inequalities if

$$K \le \overline{K} \le \frac{(L - \overline{L}) \, k_\nu \, C_{2\nu+2}}{(\nu+1)! \, \|k^{(\nu)}\|_{L^1(0,1+\rho)} \, C_{\nu+1}} + o(\rho).$$

It is now easily seen that we can find appropriate $\overline{L}$ and $\overline{K}$ if we have

$$\frac{\|k^{(\nu)}\|_{L^1(0,1)}}{k_\nu} < \frac{L \, C_{2\nu+2}}{K \, (\nu+1)! \, C_{\nu+1}} = \tilde{C}. \tag{3.43}$$

This is the constant $\tilde{C}$ referred to in the formulation of the lemma. Note that $\tilde{C}$ depends exclusively on the C_l's and hence only on the choice of ω. With (3.42) and $\overline{K} - K \ge 0$ we conclude from (3.40) that

$$\frac{d}{dt}\left[V_\rho\big(t, \mathbf{z}_\rho(t)\big)\right] \le -\overline{L} |\mathbf{z}_\rho(t)|.$$

Integrating this relation yields

$$V_\rho\big(t, \mathbf{z}_\rho(t)\big) \le V_\rho\big(0, \mathbf{z}_\rho(0)\big) - \overline{L} \int_0^t |\mathbf{z}_\rho(s)| \, ds.$$

With (3.39), (3.36), and (3.38) we obtain

$$\frac{1}{R}|\mathbf{z}_\rho(t)| \le \frac{1}{2L}|\mathbf{z}_\rho(0)| - \overline{L}\int_0^t |\mathbf{z}_\rho(s)|\,ds$$

and, using Gronwall's inequality we find

$$|\mathbf{z}_\rho(t)| \le \frac{R}{2L}e^{-R\overline{L}t}\,|\mathbf{z}_\rho(0)|$$

and

$$\|\mathbf{z}_\rho\|_{L^1(0,\frac{1}{\rho})} \le \frac{1}{2L\overline{L}}\,|\mathbf{z}_\rho(0)|.$$

For the solution z_ρ of the homogeneous problem (3.23) with initial condition $z_\rho(0) = 1$ we therefore obtain

$$|z_\rho(t)| \le |\mathbf{z}_\rho(t)| \le \frac{Rd}{2L}$$

and

$$\|z_\rho\|_{L^1(0,\frac{1}{\rho})} \le \|\mathbf{z}_\rho\|_{L^1(0,\frac{1}{\rho})} \le \frac{d}{2L\overline{L}}$$

by (3.34) for all ρ sufficiently small. Setting

$$m = \max\left(\frac{Rd}{2L}, \frac{d}{2L\overline{L}}\right)$$

proves the claim. □

For the stability estimates in Proposition 1 we did not need condition (A_3). With (A_3) holding and with Lemma 1 we can get a much better estimate for $\|u\|_{\mathcal{C}[0,1]}$.

Theorem 1. *Let f be continuous on $[0, 1+R)$ for some $R > 0$ and let k be ν-smoothing with $\nu > 0$. Let a, b, and c be given by* (2.15) *and suppose that $\tilde{g}$, $\tilde{f}$ and $\tilde{k}$ are defined by* (2.16), (2.17), *and* (2.18), *respectively. Assume moreover that assumptions (A_1), (A_2), and (A_3) hold. With $\tilde{C}$ denoting the constant in Lemma 1 and u denoting the solution to* (2.20) *we have*

$$|u(t)| \le \frac{\overline{m}}{\rho^\nu}\,\|f\|_{\mathcal{C}[0,1+\rho]} \text{ for all } t \in [0,1] \tag{3.44}$$

for some constant $\overline{m} > 0$ independent of ρ provided that

$$\frac{\|k^{(\nu)}\|_{L^1(0,1)}}{k_\nu} < \tilde{C}$$

holds.

Proof. By setting $y(t) = u(\rho t)$ we get

$$y'(t) = -\frac{b\rho}{c}\,y(t) - \frac{\rho^2}{c}\int_0^t \tilde{k}\big(\rho(t-s)\big)\,y(s)\,ds + G(t) \tag{3.45}$$

on $[0, \frac{1}{\rho}]$ with

$$G(t) = \frac{\rho}{c}\int_0^\rho \lambda(\tau)\,f(\rho t + \tau)\,d\omega(\tau). \tag{3.46}$$

Using (3.5) and (3.8) we get the estimate

$$|G(t)| \le \frac{1}{\rho^\nu}\left(\frac{(\nu+1)!\,C_\nu}{C_{2\nu+2}\,k_\nu} + o(\rho)\right)\|f\|_{\mathcal{C}[0,1+\rho]} \tag{3.47}$$

on $[0,\frac{1}{\rho}]$ for ρ sufficiently small.

We can now use the variation of constant formula (see [2, Thm. 2.3.1, p. 29]) to express y. We have

$$y(t) = z_\rho(t)\,y(0) + \int_0^t z_\rho(t-s)\,G(s)\,ds.$$

Hence, using Lemma 1 together with estimate (3.13) in the proof of Proposition 1 and (3.47) (note that $y(0)=u(0)$) we obtain:

$$\begin{aligned}
|y(t)| &\le |z_\rho(t)|\,|y(0)| + \|z_\rho\|_{L^1(0,\frac{1}{\rho})}\,\|G\|_{\mathcal{C}[0,\frac{1}{\rho}]}\\
&\le \frac{m}{\rho^\nu}\left(\frac{\nu!}{k_\nu}\frac{C_{2\nu+2}C_\nu + C_{2\nu+1}C_{\nu+1}}{C_{2\nu+2}C_{2\nu} - C_{2\nu+1}^2} + \frac{(\nu+1)!\,C_\nu}{C_{2\nu+2}\,k_\nu} + o(\rho)\right)\|f\|_{\mathcal{C}[0,1+\rho]}\\
&\le \frac{\overline{m}}{\rho^\nu}\,\|f\|_{\mathcal{C}[0,1+\rho]}
\end{aligned}$$

for some constant $\overline{m} > 0$ independent of ρ. Since $u(t) = y(\frac{t}{\rho})$, the above estimate also holds for u. $\square$

Suppose now that we are given functions f_0 and u_0 which are continuous on $[0,1+R)$ and which satisfy

$$\int_0^t k(t-s)\,u_0(s)\,ds = f_0(t) \text{ on } [0,1+R). \tag{3.48}$$

We refer to u_0 as the exact solution to (3.48) with attainable data f_0. With condition (2.7) holding, it is seen that:

$$\begin{aligned}
f_0^{(l)}(t) &= \int_0^t k^{(l)}(t-s)\,u_0(s)\,ds \text{ for } l = 0,\dots,\nu-1,\\
f_0^{(\nu)}(t) &= k^{(\nu-1)}(0)\,u_0(t) + \int_0^t k^{(\nu)}(t-s)\,u_0(s)\,ds.
\end{aligned}$$

Thus, we have $f_0 \in \mathcal{C}^{(\nu)}[0,1+\rho]$ for all $\rho < R$ and $f_0^{(l)}(0) = 0$ for $l = 0,\dots,\nu-1$ and

$$f_0^{(\nu)}(0) = k_\nu\,u_0(0). \tag{3.49}$$

Let $\tilde{g}_0$ and $\tilde{f}_0$ be constructed from f_0 by (2.16) and (2.17) respectively. We can use (3.49) to derive the following asymptotic expansions:

$$\tilde{f}_0(0) = \rho^{s+2\nu+1}\left(u_0(0)\frac{k_\nu^2}{\nu!(\nu+1)!}C_{2\nu+1} + o(\rho)\right), \tag{3.50}$$

$$\tilde{g}_0 = \rho^{s+2\nu}\left(u_0(0)\frac{k_\nu^2}{((\nu+1)!)^2}C_{2\nu} + o(\rho)\right). \tag{3.51}$$

We have the following convergence result for the approximating regularization scheme (2.20).

Theorem 2. *Let u_0 and f_0 be given satisfying* (3.48). *We assume additionally that u_0 is continuously differentiable on* $[0, 1+R)$. *Let*

$$\mu(u_0, \rho) = \max_{s,t \in [0,1+\rho],\, |s-t| \le \rho} (|u_0'(s) - u_0'(t)|).$$

Furthermore let u_ρ denote the solution to (2.20) *with data given by $\tilde{f}_0$ and $\tilde{g}_0$, and assume that condition (A_3) holds. Then there exist constants $\tilde{C} > 0$ (depending on ω) and $\overline{M} > 0$ (depending on k and ω) such that*

$$|u_\rho(t) - u_0(t)| \le \overline{M}\, \mu(u_0, \rho) \text{ for all } t \in [0, 1] \tag{3.52}$$

whenever

$$\frac{\|k^{(\nu)}\|_{L^1(0,1)}}{k_\nu} < \tilde{C}. \tag{3.53}$$

Proof. Evaluating (3.48) at the point $t + \tau$, multiplying by $\lambda(\tau)$, and integrating from 0 to ρ with respect to the measure ω yields:

$$\int_0^\rho \lambda(\tau) \int_0^{t+\tau} k(t+\tau-s)\, u_0(s)\, ds\, d\omega(\tau) = \int_0^\rho \lambda(\tau) f_0(t+\tau)\, d\omega(\tau).$$

Hence,

$$\begin{aligned}\int_{s=0}^t \left(\int_{\tau=0}^\rho \lambda(\tau)\, k(t-s+\tau)\, d\omega(\tau) \right) u_0(s)\, ds \,+ \\ \int_{\tau=0}^\rho \lambda(\tau) \left(\int_{s=t}^{t+\tau} k(t-s+\tau)\, u_0(s)\, ds \right) d\omega(\tau) = \tilde{f}_0(t).\end{aligned} \tag{3.54}$$

Since u_0 is continuously differentiable we have

$$u_0(s) = u_0(t) + u_0'(t)(s-t) + \int_t^s \big(u_0'(\xi) - u_0'(t)\big)\, d\xi.$$

With this we find:

$$\begin{aligned}&\int_0^\rho \lambda(\tau) \left(\int_{s=t}^{t+\tau} k(t-s+\tau)\, u_0(s)\, ds \right) d\omega(\tau) \\ &= u_0(t) \int_0^\rho \lambda(\tau) \left(\int_0^\tau k(s)\, ds \right) d\omega(\tau) \\ &+ u_0'(t) \int_0^\rho \lambda(\tau) \left(\int_0^\tau k(s)\, (\tau - s)\, ds \right) d\omega(\tau) \\ &+ \int_0^\rho \lambda(\tau) \left(\int_0^\tau k(\tau-s) \Big(\int_t^{t+s} \big(u_0'(\xi) - u_0'(t)\big)\, d\xi \Big)\, dz \right) d\omega(\tau).\end{aligned}$$

With (3.54), and the notation in (2.13)–(2.18), we obtain

$$\begin{aligned}&\int_0^t \tilde{k}(t-s)\, u_0(s)\, ds + b\, u_0(t) + c\, u_0'(t) = \\ &\tilde{f}_0(t) - \int_0^\rho \lambda(\tau) \left(\int_0^\tau k(\tau-s) \Big(\int_t^{t+s} \big(u_0'(\xi) - u_0'(t)\big)\, d\xi \Big)\, ds \right) d\omega(\tau).\end{aligned} \tag{3.55}$$

We set $v_\rho = u_\rho - u_0$. Since u_ρ is solution to (2.20b) with right-hand side given by $\tilde f_0$ we obtain

$$\int_0^t \tilde k(t-s)\, v_\rho(s)\, ds + b\, v_\rho(t) + c\, v_\rho'(t) = \int_0^\rho \lambda(\tau) \Big(\int_0^\tau k(\tau - s) \Big(\int_t^{t+s} \big(u_0'(\xi) - u_0'(t)\big)\, d\xi \Big)\, ds \Big)\, d\omega(\tau). \tag{3.56}$$

Before we investigate (3.56) more closely we estimate the initial value $v_\rho(0)$. Solving (2.20a) for $u_\rho(0)$ gives

$$u_\rho(0) = \frac{c\,\tilde g_0 - b\,\tilde f_0(0)}{a\,c - b^2}.$$

Hence, we have

$$\begin{aligned} |u_\rho(0) - u_0(0)| &= \left| \frac{c\,\tilde g_0 - b\,\tilde f_0(0)}{a\,c - b^2} - \frac{a\,c - b^2}{a\,c - b^2} u_0(0) \right| \\ &\le \frac{c}{a\,c - b^2} |\tilde g_0 - a\, u_0(0)| + \frac{b}{a\,c - b^2} |\tilde f_0(0) - b\, u_0(0)|. \end{aligned}$$

Using (2.16), (2.15), and $f_0(\tau) = \int_0^\tau k(s)\, u_0(\tau - s)\, ds$, we find:

$$\begin{aligned} |\tilde g_0 - a\, u_0(0)| &= \left| \int_0^\rho \kappa(\tau) \int_0^\tau k(s)\, \big(u_0(\tau - s) - u_0(0)\big) ds\, d\omega(\tau) \right| \\ &\le a \max_{\xi \in [0,\rho]} (|u_0(\xi) - u_0(0)|) \le a\, \rho\, \|u_0'\|_{\mathcal{C}[0,1+\rho]}. \end{aligned}$$

Analogously we obtain

$$|\tilde f_0(0) - b\, u_0(0)| \le b\, \rho\, \|u_0'\|_{\mathcal{C}[0,1+\rho]}.$$

Thus, we have

$$|v_\rho(0)| \le \frac{a\,c + b^2}{a\,c - b^2}\, \rho\, \|u_0'\|_{\mathcal{C}[0,1+\rho]}.$$

If we use the asymptotic expressions (3.6)–(3.8) for a, b, and c, respecitvely and assumption (A_2) it is easy to see that there exists a constant $n_1 > 0$ independent of ρ such that

$$|v_\rho(0)| \le n_1\, \rho. \tag{3.57}$$

We now come back to the analysis of (3.56). With the transformation $y_\rho(t) = v_\rho(\rho t)$ we obtain from (3.56):

$$y_\rho'(t) = -\frac{b\rho}{c} y_\rho(t) - \frac{\rho^2}{c} \int_0^t \tilde k(\rho(t-s))\, y_\rho(s)\, ds + F_\rho(t) \tag{3.58}$$

for $t \in [0, \frac{1}{\rho}]$ with

(3.59)

$$F_\rho(t) = \frac{1}{c} \int_0^\rho \lambda(\tau) \Big(\int_0^\tau k(\tau - s) \Big(\int_{\rho t}^{\rho t + s} \big(u_0'(\xi) - u_0'(\rho t)\big)\, d\xi \Big)\, ds \Big)\, d\omega(\tau).$$

We have the following estimate for F_ρ:

$$
\begin{aligned}
|F_\rho(t)| &\leq \max_{\xi\in[\rho t,\rho t+\rho]} |u_0'(\xi) - u_0'(t)| \, \frac{1}{c} \int_0^\rho \lambda(\tau) \int_0^\tau (\tau - s)\, k(s)\, ds\, d\omega(\tau) \\
&= \max_{\xi\in[\rho t,\rho t+\rho]} |u_0'(\xi) - u_0'(t)| \leq \mu(u_0,\rho) \qquad (3.60)
\end{aligned}
$$

for all $t \in [0, \frac{1}{\rho}]$.

We can now use the fundamental solution z_ρ for (3.23) and the variation of constants formula (see [2, Thm. 2.3.1, p. 29]) to obtain an estimate for the solution of the inhomogeneous problem (3.58). We have

$$
y_\rho(t) = z_\rho(t)\, y_\rho(0) + \int_0^t z_\rho(t-s)\, F_\rho(s)\, ds.
$$

Consequently, using the estimates in Lemma 1 we find:

$$
\begin{aligned}
|y_\rho(t)| &\leq |z_\rho(t)|\, |y_\rho(0)| + \|z_\rho\|_{L^1(0,\frac{1}{\rho})}\, \|F_\rho\|_{\mathcal{C}[0,\frac{1}{\rho}]} \\
&\leq \overline{m}\, (|y_\rho(0)| + \|F_\rho\|_{\mathcal{C}[0,\frac{1}{\rho}]}).
\end{aligned}
$$

We can now use (3.57), (3.60), and the fact that $y_\rho(0) = v_\rho(0)$ to conclude that

$$
|y_\rho(t)| \leq \overline{m}\, \big(n_1\, \rho + \mu(u_0,\rho)\big) \leq \overline{M}\, \mu(u_0,\rho)
$$

for some constant $\overline{M} > 0$ independent of ρ and for all $t \in [0, \frac{1}{\rho}]$. With $|v_\rho(t)| = |y_\rho(\frac{t}{\rho})|$ the proof is complete. □

Remark 3. Since u_0' is uniformly continuous on $[0, 1+\rho]$, it is clear that $\mu(u_0,\rho) \to 0$ as $\rho \to 0$. If the exact solution $u_0 \in \mathcal{C}^{1,\alpha}[0, 1+\rho]$, that is, the first derivative is Hölder continuous of order $0 < \alpha \leq 1$, then we have $\mu(u_0,\rho) = O(\rho^\alpha)$ and we get the estimate

$$
|u_\rho(t) - u_0(t)| \leq m\, \rho^\alpha \qquad (3.61)
$$

if (3.53) is satisfied.

Remark 4. In the above proof we considered stability properties of the *homogeneous* system (3.30) and we used the variation of constants formula to estimate v_ρ. In the convergence proof for the corresponding zero-order method in [6] the inhomogeneous problem is transformed into a system by differentiation, thus yielding the requirement that the exact solution u_0 must be $\mathcal{C}^\nu$-smooth. By our technique we could avoid such high regularity assumptions on u_0.

In the case of noisy data we have the following convergence result.

Theorem 3. *Suppose the notation and conditions of Theorems 1 and 2 hold. Let a family of functions $\{f_\delta\}_{\delta>0} \subset \mathcal{C}[0, 1+\rho_0]$ be given satisfying*

$$
|f_\delta(t) - f_0(t)| \leq \delta
$$

for all $t \in [0, 1+\rho_0]$ with some $\rho_0 > 0$. Moreover denote by u_ρ^δ the solution to (2.20) with data $\tilde{f}_\delta$ and $\tilde{g}_\delta$ defined by means of $f = f_\delta$. Then there exists $\rho = \rho(\delta)$ such that

$$
|u^\delta_{\rho(\delta)}(t) - u_0(t)| \to 0 \quad \text{as } \delta \to 0 \qquad (3.62)
$$

uniformly in $t \in [0,1]$.

If additionally $u_0 \in \mathcal{C}^{1,\alpha}[0,1+\rho_0]$ *for some* $0 < \alpha \le 1$ *then there exists a constant* $M > 0$ *such that for the choice*

$$\rho(\delta) = \delta^{\frac{1}{\alpha+\nu}} \tag{3.63}$$

we have

$$|u^\delta_{\rho(\delta)}(t) - u_0(t)| \le M\, \delta^{\frac{\alpha}{\alpha+\nu}} \tag{3.64}$$

for all $t \in [0,1]$.

Proof. Using Theorem 1 and Theorem 2 we get

$$\begin{aligned} |u^\delta_\rho(t) - u_0(t)| &\le |u^\delta_\rho(t) - u_\rho(t)| + |u_\rho(t) - u_0(t)| \\ &\le \frac{\overline{m}}{\rho^\nu}\,\delta + \overline{M}\,\mu(u_0,\rho). \end{aligned}$$

For the choice $\rho(\delta) = \delta^{\frac{1}{2\nu}}$ we obtain $\mu(u_0,\rho(\delta)) \to 0$ and $\rho(\delta)^{-\nu}\,\delta \to 0$ as $\delta \to 0$. This proves the first assertion.

If u_0' is Hölder continuous with exponent α we have $\mu(u_0,\rho) \le l\,\rho^\alpha$ with some constant $l > 0$. Thus, we have

$$|u^\delta_\rho(t) - u_0(t)| \le \frac{\overline{m}}{\rho^\nu}\,\delta + \overline{M}\,l\,\rho^\alpha \le \tilde{M}\left(\frac{\delta}{\rho^\nu} + \rho^\alpha\right)$$

where $\tilde{M} = \max(\overline{m}, \overline{M}l)$. If we set $\rho(\delta) = \delta^{\frac{1}{\alpha+\nu}}$ we get

$$|u^\delta_\rho(t) - u_0(t)| \le 2\tilde{M}\delta^{\frac{\alpha}{\alpha+\nu}}$$

for all $t \in [0,1]$. Setting $M = 2\tilde{M}$ completes the proof. □

4. Numerical Experiments

For all presented numerical examples we use $\omega_l = 1$ for $l = 0,\dots,r$ (uniform weights). The plots in Figure 1 show results for the first-order sequential method for different noise-levels in the data. We chose $k(t) = t^2$, i.e., k is 3-smoothing. The data are given by $f_0(t) = t^3/3 - t^5/30$. To the data we added artificial 0-1 distributed Gaussian noise scaled with a factor $\sigma\|f\|_\infty$. The plots in Figure 1 show the results for $\sigma = 0$, $\sigma = 0.005$, $\sigma = 0.01$, and $\sigma = 0.02$. The lengths of the future intervals are $\rho = 0.05$, $\rho = 0.4$, $\rho = 0.5$, and $\rho = 0.55$, respectively. We used $N = 600$ time steps on the time interval $[0,3]$. The plots show the exact solution as dashed line and the reconstructed solutions from noisy data as solid lines. It is seen that the first-order method gives stable results in the presence of data noise.

The plots in Figure 2 show a comparison between first- and zero-order sequential regularization and Tikhonov regularization. We used again the kernel $k(t) = t^2$ ($\nu = 2$), but with different data $f_0(t) = -1/32\sin 4t - 1/8t$ for $t \in [0,3]$. With these data there is more variation in the exact solution u_0 which makes it more difficult to reconstruct. We used $N = 600$ collocation points and $r = 80$ future points which means $\rho = 0.4$ for the length of the future interval. The noise level was 2%. For the zero-order method, the same number of future points was chosen. The regularization parameter for the Tikhonov method was chosen experimentally to produce “best looking” result. It is apparent that the first-order method produces smoother

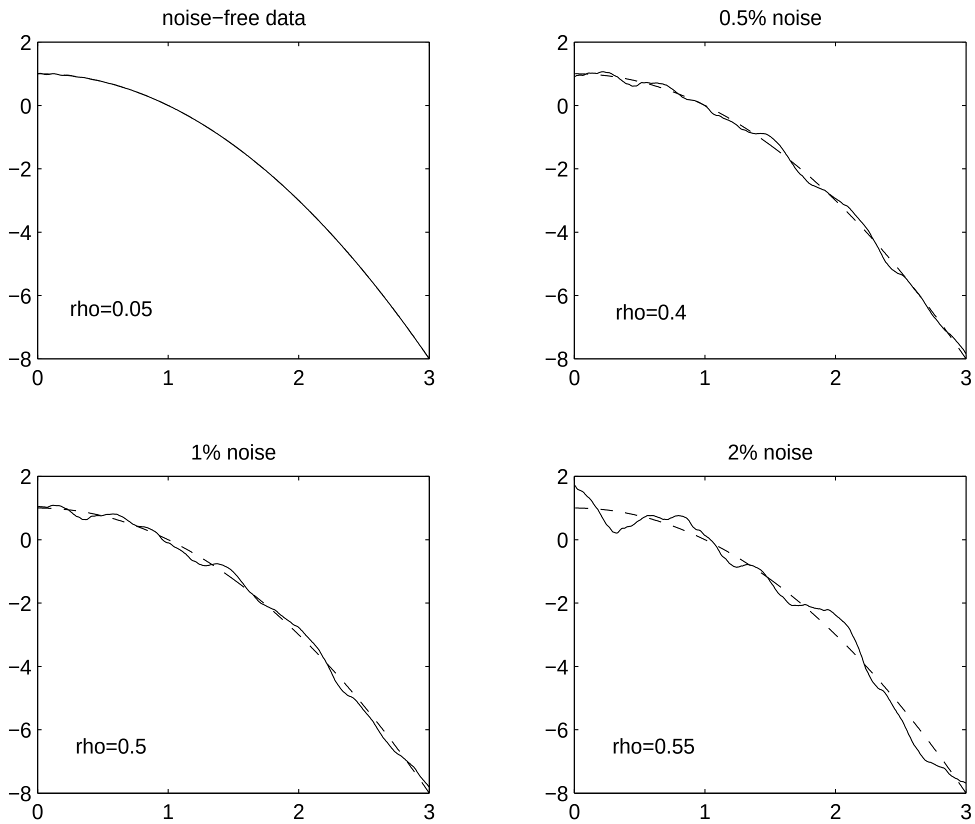

FIGURE 1. Solutions for first-order sequential regularization with different noise-levels.

results than the zero-order method with comparable supremum-norm errors. Tikhonov's method performs better than the sequential methods apart from the last part of the time interval where the solution is obviously over-regularized. For the chosen discretization (600 time steps) computation by the first-order method is faster by a factor 10 (2.3 seconds vs 21.4 seconds) when compared to Tikhonov's method. Moreover, if further data points are added, the already computed first-order solution can be extended with minimal numerical effort. For Tikhonov regularization the whole system matrix has to be reassambled and the solution has to be recomputed from the initial time zero onwards.

In Figure 3 we compare first- and zero-order sequential regularization for the case that the future interval is chosen too large. With the same specifications as in Figure 2, but with $\rho = 0.9$ we see that the zero-order method over-regularizes strongly, resulting in a strongly damped but still not very smooth solution. On the other hand the first-order method produces a solution with a qualitatively correct amplitude but with a delayed phase.

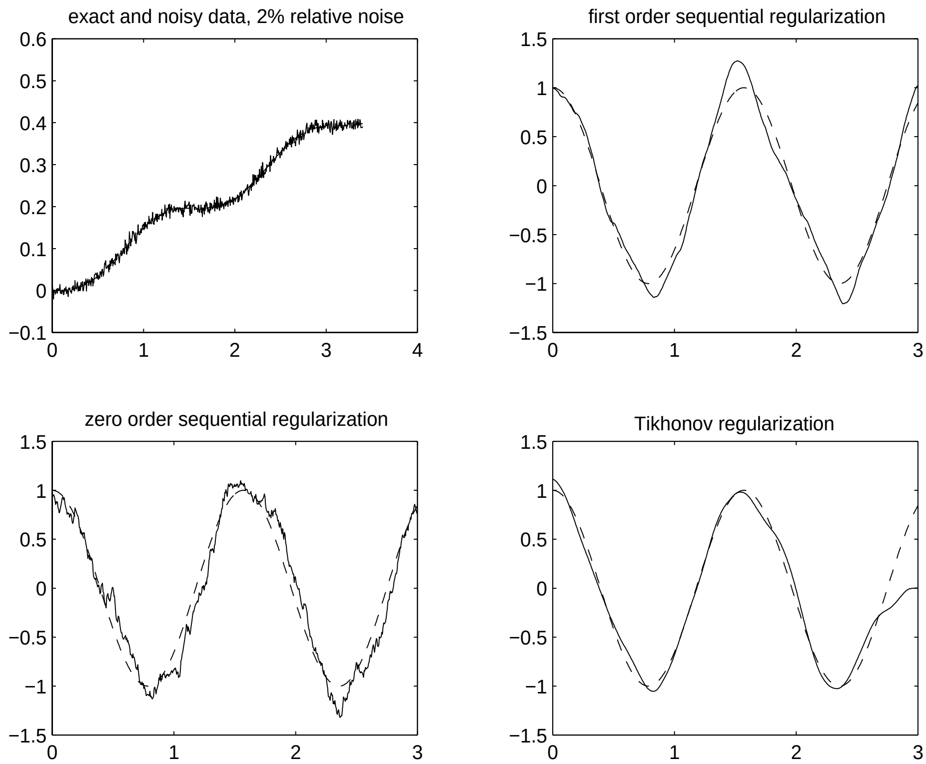

FIGURE 2. Comparison between different regularization methods for noisy data.

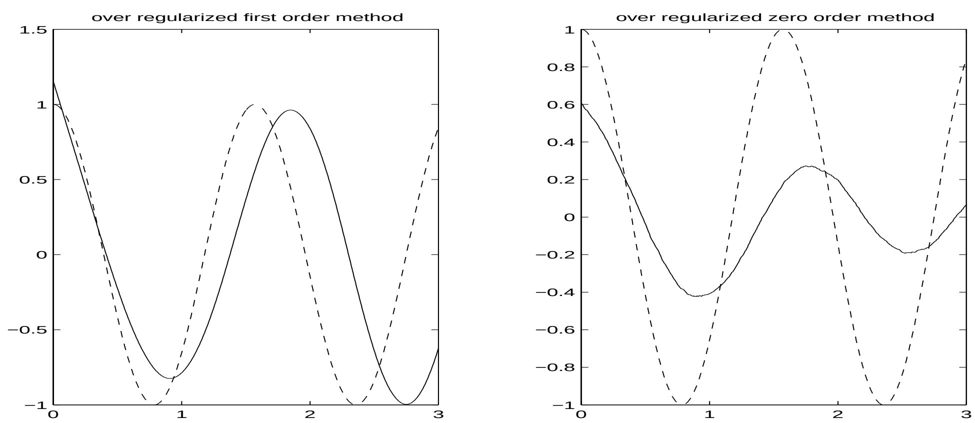

FIGURE 3. Over-regularization for first- and zero-order method.

In Figure 4 the stars ($*$) and circles (o) show the locations of the roots of the polynomial

$$\sum_{l=0}^{nu+1} \frac{C_{\nu+l+1}}{l!}\, x^l \tag{4.1}$$

for $\nu = 3$ and $\nu = 4$, respectively, in the complex plane. It is seen that for $\nu = 3$, all roots have negative real parts. For $\nu = 4$, one pair of roots has crossed the imaginary axis, implying that assumption (A_3) is not satisfied in this case. The instability effect of condition (A_3) not being satisfied is also

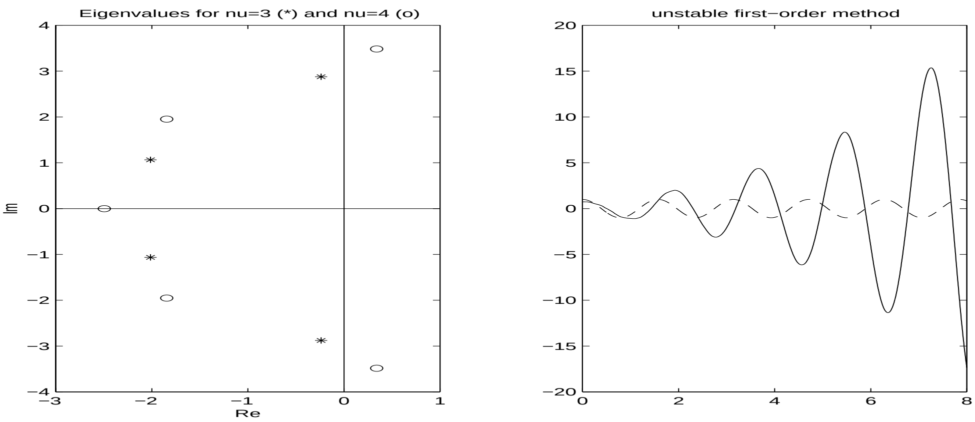

FIGURE 4. Location of roots of the polynomial (4.1) for $\nu = 3$ $(*)$ and $\nu = 4$ (o). Solution for $\nu = 4$ in case of noisy data.

indicated in Figure 4. Here we chose $k(t) = t^3$ (i.e., $\nu = 4$) on the interval $[0, 8]$ with 1% relative noise on the data. We set $\rho = 1$ for the length of the future interval. It is seen that the solution of the first-order method behaves like an unstable oscillator. For $n = 4$, a sophisticated construction gives a stabilizing weight ω for the zero-order method (cf. [6]). The construction of a stabilizing sequential regularization method for ν-smoothing kernels with $\nu \geq 5$ is still an open problem.

5. Conclusions

Starting with Beck's linear function specification method we found a family of approximating Volterra integro-differential equations of the second kind for the stable solution of ill-posed Volterra equations of the first kind. We proved stability estimates and convergence if the data-noise goes to zero. A convergence rate depending on smoothness properties of the exact solution was derived. Numerical experiments showed that the method produces smoother results than the well investigated sequential predictor-corrector method. The second-kind Volterra intergro-differential equation can be solved sequentially in time which makes the algorithm much faster than for example Tikhonov regularization.

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