Optimization of a SMES Device under Nonlinear Constraints

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***Abstract*— In this paper a method to calculate the gradients for the TEAM 22 problem is presented. Furthermore an objective function different from the one proposed in the benchmark is used where the energy requirement was handled as an equality constraint. The so derived gradients of the objective function and the constraints together with a standard optimization routine are used to solve the problem. In the last section some new results concerning the presence of local minima for the problem are given.**

I. INTRODUCTION

We consider the problem of finding an optimal configuration for a superconducting magnetic energy storage (SMES) device. The SMES device consists of an arrangement of superconducting coils which are driven by currents to store electrical energy. Due to the large current densities, one single coil produces a large magnetic stray field. A configuration of two coils with currents flowing in opposite directions diminishes the stray field considerably. The situation we consider is defined in [1] as team workshop problem 22. Two concentric solenoids should be designed in such a way that the following objectives are satisfied:

- The energy stored in the device should be $180MJ$.
- The mean stray field at 21 measurement points along the lines a and b at a distance of 10 meters should be as small as possible (Fig. 1).
- The generated magnetic field inside the solenoids must not violate a certain physical condition which guarantees superconductivity (quench condition)(Fig. 2).

The parameters that should be adjusted are the geometric quantities defining the dimensions of the coils and the two current densities(Fig. 1):

- R_1 ... mean radius of the inner coil
- R_2 ... mean radius of the outer coil
- h_1 ... half the height of the inner coil
- h_2 ... half the height of the outer coil
- d_1 ... thickness of the inner coil
- d_2 ... thickness of the outer coil
- J_1 ... current density in the inner coil
- J_2 ... current density in the outer coil

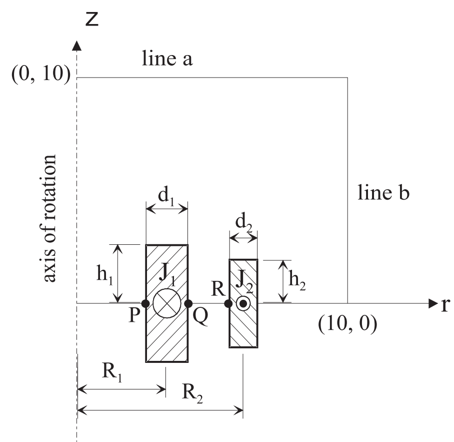

Fig. 1. Configuration of the SMES device

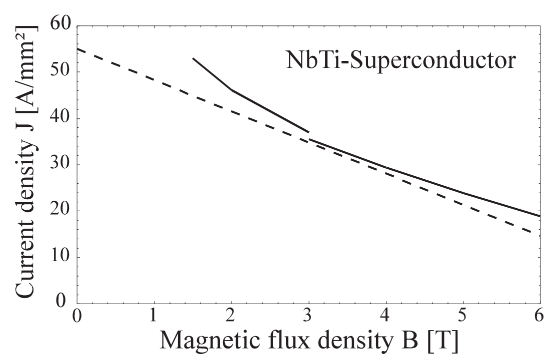

Fig. 2. Critical curve of the superconductor.

In Fig. 2 the quenching curve is plotted as a solid curve in the B-J plane. In order to maintain superconductivity, in each point of both coils the current density J and the magnetic field B must satisfy that the corresponding point in the B-J plane lies below the critical quenching curve. We use the linear approximation (dashed line in Fig. 2)

$$|J| \leq -6.4|B| + 54\,MA/m^2 \qquad (1)$$

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as quench condition which must not be violated at any point of the coils. It is sufficient to check the validity of the quenching condition in the three points P, Q and R (Fig. 1), where the magnetic field can attain its maximum.

II. FIELD AND ENERGY CALCULATION

In order to decide whether or not a certain configuration is admissible and how well the objectives are satisfied, it is necessary to calculate energy, stray field and the field in the critical points P, Q and R in terms of the parameters. Because of the linearity of the problem we used Biot-Savarts law. We define the following abbreviations

$$K_{B_r}(r,z,\rho,\zeta) = \int_{\psi=0}^{2\pi} \frac{(z-\zeta)\rho\ d\psi}{[r^2+\rho^2+(z-\zeta)^2-2r\rho\cos(\psi)]^{\frac{3}{2}}}, \quad (2)$$

$$K_{B_z}(r,z,\rho,\zeta) = \int_{\psi=0}^{2\pi} \frac{(\rho\cos(\psi)-r)\rho\ d\psi}{[r^2+\rho^2+(z-\zeta)^2-2r\rho\cos(\psi)]^{\frac{3}{2}}}, \quad (3)$$

$$K_W(r,z,\rho,\zeta) = \int_{\psi=0}^{2\pi} \frac{r\rho\ d\psi}{[r^2+\rho^2+(z-\zeta)^2-2r\rho\cos(\psi)]^{\frac{1}{2}}}. \quad (4)$$

Thus we obtain

$$B_r = \frac{\mu_0}{4\pi}\{J_1 \int_{\rho=R_1-\frac{d1}{2}}^{R_1+\frac{d_1}{2}} \int_{\zeta=-h_1}^{h_1} K_{B_r}(r,z,\rho,\zeta)\,d\zeta\,d\rho$$
$$+J_2 \int_{\rho=R_2-\frac{d_2}{2}}^{R_2+\frac{d_2}{2}} \int_{\zeta=-h_2}^{h_2} K_{B_r}(r,z,\rho,\zeta)\,d\zeta\,d\rho\} \quad (5)$$

for the component in direction of the radius of the magnetic flux density, where r and z refer to the point in which the field should be calculated, and

$$B_z = \frac{\mu_0}{4\pi}\{J_1 \int_{\rho=R_1-\frac{d1}{2}}^{R_1+\frac{d_1}{2}} \int_{\zeta=-h_1}^{h_1} K_{B_z}(r,z,\rho,\zeta)\,d\zeta\,d\rho$$
$$+J_2 \int_{\rho=R_2-\frac{d_2}{2}}^{R_2+\frac{d_2}{2}} \int_{\zeta=-h_2}^{h_2} K_{B_z}(r,z,\rho,\zeta)\,d\zeta\,d\rho\} \quad (6)$$

for the component in z-direction. The energy can be calculated using the magnetic vector potential, which leads to

$$W = \frac{\mu_0}{4}\{J_1 \int_{r=R_1-\frac{d_1}{2}}^{R_1+\frac{d_1}{2}} \int_{z=-h_1}^{h_1}$$
$$[J_1 \int_{\rho=R_1-\frac{d_1}{2}}^{R_1+\frac{d_1}{2}} \int_{\zeta=-h_1}^{h_1} K_W(r,z,\rho,\zeta)d\zeta d\rho +$$
$$J_2 \int_{\rho=R_2-\frac{d_2}{2}}^{R_2+\frac{d_2}{2}} \int_{\zeta=-h_2}^{h_2} K_W(r,z,\rho,\zeta)d\zeta d\rho]\,dr\,dz$$
$$+J_2 \int_{r=R_2-\frac{d_2}{2}}^{R_2+\frac{d_2}{2}} \int_{z=-h_2}^{h_2}$$
$$[J_1 \int_{\rho=R_1-\frac{d_1}{2}}^{R_1+\frac{d_1}{2}} \int_{\zeta=-h_1}^{h_1} K_W(r,z,\rho,\zeta)d\zeta d\rho +$$
$$J_2 \int_{\rho=R_2-\frac{d_2}{2}}^{R_2+\frac{d_2}{2}} \int_{\zeta=-h_2}^{h_2} K_W(r,z,\rho,\zeta)d\zeta d\rho]\,dr\,dz\} \quad (7)$$

III. MATHEMATICAL PROGRAMMING PROBLEM

For the optimization of the problem a sequential quadratic programming (SQP) method was applied that uses a BFGS update for the Hessian [4]. This method needs the gradient of the objective function in a point to determine a search direction. Due to the relative great number of optimization parameters the calculation of the gradient by using a finite difference approximation would take too much time. Fortunately a analytical representation of the gradient can be obtained directly by differentiating (5) - (7). We choose a formulation as a constrained optimization problem in favor of just adding a penalty term to the objective function (as it is stated in the description of the benchmark [1]) which would force the energy to be close to $180MJ$, because it turned out that the energy and the stray field respond in a completely different way to changes of the parameter configuration. While we have good sensitivity for the energy term, the stray field is quite insensitive. Therefore we would have very unequal partners in the objective function and a gradient-based method would always tend to adjust the energy to the desired value and neglect the stray field term. The optimization problem can therefore be stated as follows:

Definition 1 *minimize* $f(p) = \sum_{i=1}^{22} |B^i_{Stray}|^2$

subject to the constraints:

- $W(p) - 180MJ = 0$
- $J1 + 6.4|B_{\max}(P)| - 54 \leq 0$
- $-J2 + 6.4|B_{\max}(Q)| - 54 \leq 0$
- $J1 + 6.4|B_{\max}(R)| - 54 \leq 0$
- $\left(R_1 + \frac{d_1}{2}\right) - \left(R_2 - \frac{d_2}{2}\right) \leq 0$

where p is the vector of optimization parameters. B^i_{Stray} is measured in the i^th measurement points at the lines a and b (Fig. 1) and $B_{\max}$ is calculated in the points P, Q and R.

The quench condition is split in three inequality constraints (one for each point) because the derivative of the constraints has to be an analytical function, which would not be the case, if the quench condition is packed into one constraint referring to the maximum flux density occurring in one of the three points. The last constraint simply ensures that the two solenoids do not intersect each other. Due to the equality constraint for the energy the optimization routine searches for the minimal stray field in a seven dimensional sub-manifold in the eight dimensional parameter space, where the energy is about $180MJ$.

IV. Gradient Information

In order to get the gradient of the objective function and the constraints it is necessary to calculate the derivatives of (5)-(7) with respect to the design parameters p_i. This, however, is easily done in case of the parameters which occur in the bounds of the integrals, because differentiating with respect to these variables means only to cancel the integral and evaluate the integrand at the respective bound. Thus, for the terms occurring in the components of the magnetic flux density, which are necessary for the stray field calculation, the derivative with respect to a design parameter p is given by an expression of the following kind

$$\frac{\partial}{\partial p}\left(\int\limits_{l_v(p)}^{u_v(p)}\int\limits_{l_w}^{u_w} K(v,w)dwdv\right) = \frac{\partial u_v(p)}{\partial p}\int\limits_{l_w}^{u_w} K(u_v(p),w)dw - \frac{\partial l_v(p)}{\partial p}\int\limits_{l_w}^{u_w} K(l_v(p),w)dw, \tag{8}$$

where the upper and lower bounds for the v-integration are functions $u_v(p)$ and $l_v(p)$ respectively of the design parameter p. The integration variables v and w have been introduced because of generality since v can play the role of either ρ or ζ. In some terms for the energy calculation a specific design parameter occurs in two integrals (in the r- and the ρ-integration or in the z-and the ζ-integration), therefore we obtain

$$\frac{\partial}{\partial p}\left(\int\limits_{l_v(p)}^{u_v(p)}\int\limits_{l_w(p)}^{u_w(p)} K(v,w)dwdv\right) = \frac{\partial uv(p)}{\partial p}\int\limits_{l_w(p)}^{u_w(p)} K(u_v(p),w)dw - \frac{\partial lv(p)}{\partial p}\int\limits_{l_w(p)}^{u_w(p)} K(l_v(p),w)dw + \frac{\partial uw(p)}{\partial p}\int\limits_{l_v(p)}^{u_v(p)} K(v,u_w(p))dv - \frac{\partial lw(p)}{\partial p}\int\limits_{l_v(p)}^{u_v(p)} K(v,l_w(p))dv. \tag{9}$$

Since the kernel in (4) is symmetric with respect to the integration in r- and ρ- direction (and also symmetric in z and ζ), the first and third and the second and fourth term in (9) coincide.

The calculation of the derivative of the magnetic flux density in the points P, Q and R is much more delicate. In this case, the design parameters occur in the bounds of the integral and, as coordinates of the fieldpoint, also in the kernel. Moreover, since the fieldpoint lies at the boundary of one of the coils, the integrals in (5) and (6) are weakly singular and we have to differentiate with respect to the coordinates of the singularity. We were not able to derive the correct analytic expression for these derivatives (there seems to be some tricky kind of jump condition involved), thus we used a finite difference approximation for these components of the gradient. Note however that, for the calculation of $B_{\max}$, the coordinates $(r, 0)$ of the fieldpoint do not depend on the heights h_i, hence if we differentiate with respect to h_i, (8) is valid.

The integrals are evaluated using a Gauss-Kronrod quadrature [2] with 7-63 points, where, for the energy, the vector potential was calculated at hundred points in each coil. For the derivation of the constraints concerning the quench condition symmetry was exploited and the necessary integrals over the source points had only to be performed in one halfplane, which was not possible in case of the energy and stray field gradient, where the field points are in general not located in the plane of symmetry.

V. Optimization Results

As SQP routine we used the e04ucf program from the NAG library, which was launched from several starting points among them is the best point found with a genetic algorithm [3].

In Table I the result of the optimization run launched from one of the two proposed starting points in the benchmark is given. This point leads to an energy of $180MJ$ and a stray field of 37.55 μT and is indeed a local minimum because the Kuhn-Tucker condition [4] is satisfied. It has to be remarked that first the optimization routine terminated in the point R_1=2.062, R_2=2.72189, h_1=0.595, h_2=1.632, d_1=0.638, d_2=0.107, J_1=15.15 and J_2=-18.988 without satisfying the Kuhn-Tucker condition and was then restarted from this point with slightly modified components, namely R_1=2.06, R_2=2.718, h_1=0.596,

TABLE I
Starting Point 1

	$Start_1$	End_1
R_1[m]	2.094	2.059
R_2[m]	3.422	2.719
h_1[m]	1.027	0.596
h_2[m]	1.302	1.632
d_1[m]	0.571	0.638
d_2[m]	0.254	0.108
J_1[A/mm^2]	14.14	15.15
J_2[A/mm^2]	-19.04	-18.99
$Energy\,[MJ]$	594.75	180.00
$B_{Stray}\,[\mu T]$	25282.6	37.55

TABLE II
Starting Point 2

	$Start_2$	End_2
R_1[m]	1.577	1.616
R_2[m]	2.131	2.204
h_1[m]	0.466	0.606
h_2[m]	1.367	1.384
d_1[m]	0.732	0.772
d_2[m]	0.138	0.216
J_1[A/mm^2]	16.18	15.96
J_2[A/mm^2]	-13.51	-13.65
$Energy\,[MJ]$	114.85	180.00
$B_{Stray}\,[\mu T]$	1727.5	14.238

h_2=1.631, d_1=0.6378, d_2=0.1, J_1=15.16 and J_2=-18.899. For the entire optimization process 57 iterations with about 190 function evaluations where needed.

TABLE III
Starting Point 3

	$Start_3$	End_3
R$_1$[m]	1.3015	1.3012
R$_2$[m]	1.8000	1.8007
h$_1$[m]	1.1322	1.1325
h$_2$[m]	1.5421	1.5422
d$_1$[m]	0.5793	0.5802
d$_2$[m]	0.1959	0.1961
J$_1$[A/mm^2]	16.416	16.422
J$_2$[A/mm^2]	-18.925	-18.925
$Energy\,[MJ]$	179.99	180.00
$B_{Stray}\,[\mu T]$	9.305	8.928

In (Fig. 3) the increasing values for the stray field in some points are remarkable, the reason for this is that also the function evaluations during a line search are plotted. Furthermore it can be seen that the energy approaches the required value within very few iterations due to the fact that the optimization process tries to get into a feasible region as soon as possible.

The point End_2 in Table II was reached within 31 iterations needing 78 function evaluations and leads also to an energy of exactly $180MJ$ but a far better stray field of 14.24 μT. In this point also the Kuhn-Tucker condition is fulfilled, but this minimum is located at the border of a region where the quench-effect occurs ($B_{\max}$=5.943765 T in point P (Fig. 1)).

The point $Start_3$ in Table III is the best point up to now that has been found with a genetic algorithm. This point has a stray field of 9.3 μT. During 4 iterations with the SQP method and the analytical gradient the 'Kuhn-Tucker point' End_3 was found, which has an energy of exactly $180MJ$ and a stray field of 8.928 μT, and which is very likely to be the global optimum of the problem.

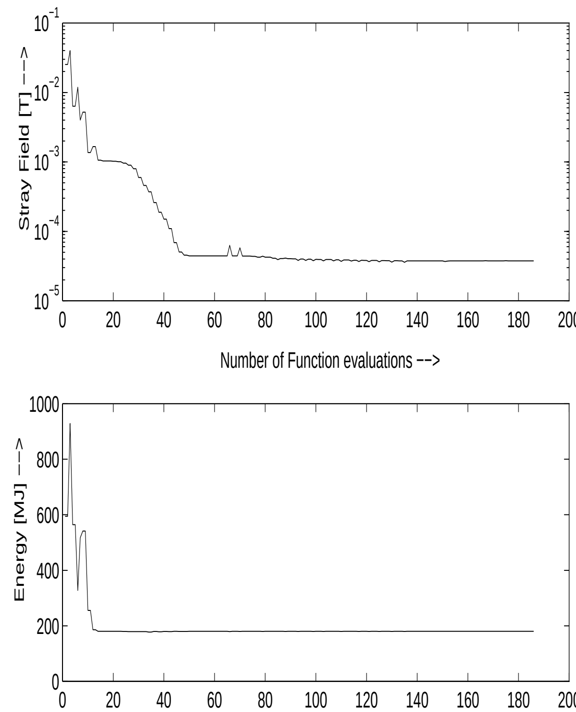

Fig. 3. Stray Field and Energy during Optimization

VI. Conclusions

In this paper we proposed a method of calculating the gradients necessary to optimize the problem with a SQP method under nonlinear constraints in an analytical way by directly differentiating the terms obtained from the Biot-Savarts law directly. This method leads to a remarkable speedup compared to a finite difference approximation of the gradients.

Furthermore the existence of local minima has been proved and a point that might be the global optimum was found.

References

[1] Ch. Magele et. al, "SMES optimization benchmark", *Proceedings of the TEAM Workshop in the Sixth Round*, March 20-21, 1996 Okayama, Japan

[2] P. Molfino, M. Nervi, M. Repetto, A. Riccio Tabassi, "A numerical procedure for the computation of the field and the vector potential generated by general shaped conductors", *Proceedings of the 6th International IGTE Symposium*, September 26-28, 1994 Graz, Austria

[3] Ch. Magele, G. Fuerntratt, B. Brandstätter, K. R. Richter, "Self adaptive fuzzy sets in multiobjective optimizations using genetic algorithms", *Proceedings of the 7th International IGTE Symposium*, September 23-25, 1996 Graz, Austria

[4] P. E. Gill, W. Murray, M. H. Wright, *Practical Optimization*, London: Academic Press, 1995