ON MONOIDS AND DOMAINS WHOSE MONADIC SUBMONOIDS ARE KRULL

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Abstract. A submonoid S of a given monoid H is called monadic if it is a divisor-closed submonoid of H generated by one element (i.e., there is some (non-zero) $b \in H$ such that S is the smallest divisor-closed submonoid of H such that $b \in S$). In this paper we study monoids and domains whose monadic submonoids are Krull monoids. These monoids resp. domains are called monadically Krull. Every Krull monoid is a monadically Krull monoid, but the converse is not true. We provide several types of counterexamples and present a few characterizations for monadically Krull monoids. Furthermore, we show that rings of integer-valued polynomials over factorial domains are monadically Krull. Finally, we investigate the connections between monadically Krull monoids and generalizations of SP-domains.

1. Introduction

The main goal of this paper is to study so called monadically Krull monoids (i.e. monoids where every divisor-closed submonoid generated by one element is a Krull monoid). Studying monoids "monadically" (i.e. investigating properties that are satisfied by all divisor-closed submonoids generated by one element) is reasonable, since some types of monoids are better situated in the "local" than in the "global" situation. On the other hand it turns out that there are a lot of monoid theoretical properties that are satisfied by the monoid if and only if they are satisfied "monadically" (e.g., being atomic, completely integrally closed, factorial). However, the Krull property does not behave like this (as pointed out in this work), and thus monadically Krull monoids are of special interest. Being monadically Krull is related to "weak factorization properties" that have been studied in a series of papers (see [8, 9, 23, 24, 25]). Moreover, some recent work in studying monoids "monadically" has been done in [14, 22]. Investigating monadically Krull monoids was also motivated by a problem that we want to discuss in more detail. Let R be a (possibly noncommutative) ring and let $\mathcal{C}$ be a class of finitely generated (right) R-modules which is closed under finite direct sums, direct summands, and isomorphisms. Then the set $\mathcal{V}(\mathcal{C})$ of isomorphism classes of modules is a commutative semigroup with operation induced by the direct sum. This semigroup encodes all possible information about direct sum decompositions of modules in $\mathcal{C}$ (see [5, 11]). If the endomorphism ring of each module in $\mathcal{C}$ is semilocal, then $\mathcal{V}(\mathcal{C})$ is a Krull monoid ([10, Theorem 3.4]). Moreover, every reduced Krull monoid can be realized by such a monoid of modules ([12]). Thus the (global) property that $\mathcal{V}(\mathcal{C})$ is Krull follows from a family of local data, namely that all $\mathrm{End}_R(M)$ are semilocal. Furthermore, the assumption that $\mathrm{End}_R(M)$ is semilocal implies that the smallest divisor-closed submonoid of $\mathcal{V}(\mathcal{C})$ generated by the class of M (denoted by $\mathrm{add}(M)$) is a Krull monoid ([4, 5, 6]).

In the second section we will discuss the most important terminology. We give a brief introduction to finitary ideal systems to simplify and unify the terminology about various types of ideals (e.g. ring ideals and t-ideals).

In the third section we will prove that several interesting properties (like being completely integrally closed, being atomic or being an FF-monoid) can be characterized by using the divisor-closed submonoids generated by one element. Moreover, we provide another characterization of Krull monoids. The main result in this section is a characterization of monadically Krull monoids. It turns out that the monadically

2000 *Mathematics Subject Classification.* 13A15, 13F05, 20M11, 20M12.
Key words and phrases. monadically, integer-valued, Krull monoid, Mori set, SP-domain.
This work was supported by the Austrian Science Fund FWF, Project Number P21576-N18.

Krull monoids are precisely the atomic, completely integrally closed monoids where special sets of atoms are finite up to associates.

In the fourth section we deal with the question whether every monadically Krull monoid is already a Krull monoid. We provide several counterexamples. First we present a ring theoretical counterexample and later we will introduce a counterexample in the monoid setting that is substantially stronger. The second example will show that radical factorial FF-monoids (they are always monadically Krull) also need not be Krull. By the way we answer some questions that have been raised in the literature in the negative. In [9] it has been shown that every atomic IDPF-domain that contains a field of characteristics zero is already completely integrally closed. We will point out that such a domain is not necessarily a Krull domain. Furthermore, we deal with the problem whether the t-dimension of a t-SP-monoid (which is some sort of generalized Krull monoid) is bounded by one and show that t-SP-monoids whose height-one prime t-ideals are divisorial are not necessarily Krull. Moreover, it is well known that an integral domain is a Prüfer domain that does not have non-zero idempotent prime ideals if and only if each of its primary ideals is a power of its radical and its set of prime ideals satisfies the ACC (for example see [29, Corollary 5.5]). We show that the "t-analogue" of this statement is not true in the monoid setting.

In the fifth section we investigate rings of integer-valued polynomials. We prove that rings of integer-valued polynomials over factorial domains are monadically Krull. Using this result we are able to provide a large class of monadically Krull domains that are t-Prüfer domains and that fail to be Krull.

In the last section we deal with the question whether every radical factorial FF-domain of Krull dimension one is already a Krull domain. Although we could not solve this problem so far, we will present partial solutions. For example we will construct a BF-domain that is an SP-domain but not a Krull domain (note that every radical factorial domain of Krull dimension one is an SP-domain, see [29, Proposition 3.11]). The counterexamples in this section are based on a construction used in [19]. Furthermore, we investigate how far SP-domains (and their generalizations) are from being monadically Krull by studying a special necessary property that pops up in the characterization of monadically Krull (in [23] this special property is called pseudo-IDPF).

2. Preliminaries

In the following, a monoid is a commutative semigroup (multiplicatively written if not stated otherwise) that possesses an identity and (if not stated otherwise) a zero element different from the identity such that every non-zero element is cancellative. A quotient monoid of a monoid H is a monoid containing H as a submonoid where every non-zero element is invertible and that is minimal with respect to this property.

Let H be a monoid, K a quotient monoid of H and $X \subseteq H$. Set $H^\bullet = H\backslash\{0\}$.

- For $A, B \subseteq K$ let $(A :_K B) = \{z \in K \mid zB \subseteq A\}$, $A^{-1} = (H :_K A)$ and $A_v = (A^{-1})^{-1}$.
- X is called (H-)divisor-closed if for all $x \in H$ and $y \in H^\bullet$ such that $xy \in X$ it follows that $x \in X$.
- By $[X]_H$ (resp. $[\![X]\!]_H$) we denote the smallest (divisor-closed) submonoid of H that contains X.
- If $a \in H$, set $[\![a]\!]_H = [\![\{a\}]\!]_H$.
- X is called an (H-)Mori set if for every $F \subseteq X$ there exists some finite $E \subseteq F$ such that $E^{-1} = F^{-1}$.

Note that $[\![a]\!]_H = \{b \in H \mid b|_H a^n \text{ for some } n \in \mathbb{N}\} \cup \{0\}$ for all $a \in H^\bullet$.

Since we will use a slightly different version of ideal systems than those dealt with in [21], we will recall the definition. The ideal systems in this work will always be ideal systems in the sense of [21] (but not conversely). Let $\mathbb{P}(H)$ be the power set of H and $r : \mathbb{P}(H) \to \mathbb{P}(H)$ be a map. The map r is called a (finitary) ideal system on H if the following properties are satisfied for all $X, Y \subseteq H$ and $c \in H$.

- $XH \cup \{0\} \subseteq r(X)$.
- $r(cX) = cr(X)$.
- If $X \subseteq r(Y)$, then $r(X) \subseteq r(Y)$.
- $(r(X) = \bigcup_{E \subseteq X, |E| < \infty} r(E).)$

Note that if $H^\bullet \neq H^\times$, then $v : \mathbb{P}(H) \to \mathbb{P}(H)$ defined by $v(X) = X_v$ for all $X \subseteq H$ is an ideal system on H and $t : \mathbb{P}(H) \to \mathbb{P}(H)$ defined by $t(X) = \bigcup_{E \subseteq X, |E| < \infty} E_v$ for all $X \subseteq H$ is a finitary ideal system on H. If R is an integral domain, then $d : \mathbb{P}(R) \to \mathbb{P}(R)$ defined by $d(X) = (X)_R$ for all $X \subseteq R$ is a finitary ideal system on R. Furthermore, $s : \mathbb{P}(H) \to \mathbb{P}(H)$ defined by $s(X) = XH$ if $\emptyset \neq X \subseteq H$ and $s(\emptyset) = \{0\}$ is a finitary ideal system on H.
In the following we will use most of the definitions and notations in [15] and [21] without further reference. Especially, we will freely use the following terms: “BF-monoid”, “FF-monoid”, “ACCP”, “atomic”, “factorial”, “Krull”, “completely integrally closed”, “valuation monoid”, “v-closed”, “root-closed” and “GCD-monoid”. A monoid is called a Mori monoid if it is v-noetherian in the terminology of [15].
Observe that Mori sets defined in this work differ from those introduced in [28]. Note that if $S \subseteq H$ is a divisor-closed submonoid and H is a Krull monoid (a Mori monoid, a completely integrally closed monoid), then S has the same property by [15, Proposition 2.4.4.2].

3. Monadic properties and Mori sets

First we present a simple characterization of being a Mori set. Using this result it is straightforward to prove that H is a Mori monoid if and only if H is a Mori set.

Lemma 3.1. *Let H be a monoid and $X \subseteq H$. Then X is not a Mori set if and only if there is some $(a_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}}$ such that $\{a_i \mid i \in [1, n + 1]\}^{-1} \subsetneqq \{a_i \mid i \in [1, n]\}^{-1}$ for all $n \in \mathbb{N}$.*

Proof. “$\Rightarrow$”: Let X be not a Mori set. Then there is some $F \subseteq X$ such that for every finite $E \subseteq F$ it follows that $F^{-1} \subsetneqq E^{-1}$. There exists some $a_1 \in F$. Now let $n \in \mathbb{N}$ and $(a_i)_{i=1}^n \in F^{[1,n]}$. Then $F^{-1} \subsetneqq \{a_i \mid i \in [1, n]\}^{-1}$, and thus $F \not\subseteq \{a_i \mid i \in [1, n]\}_v$. Consequently, there exists some $a_{n+1} \in F \setminus \{a_i \mid i \in [1, n]\}_v$, and thus $\{a_i \mid i \in [1, n + 1]\}^{-1} \subsetneqq \{a_i \mid i \in [1, n]\}^{-1}$. Hence there is some $(a_i)_{i\in\mathbb{N}} \in F^{\mathbb{N}}$ such that $\{a_i \mid i \in [1, n + 1]\}^{-1} \subsetneqq \{a_i \mid i \in [1, n]\}^{-1}$ for all $n \in \mathbb{N}$. “$\Leftarrow$”: Let $F = \{a_i \mid i \in \mathbb{N}\}$. Assume that there is some finite $E \subseteq F$ such that $E^{-1} = F^{-1}$. Then there exists some $n \in \mathbb{N}$ such that $E \subseteq \{a_i \mid i \in [1, n]\}$. This implies that $F^{-1} \subseteq \{a_i \mid i \in [1, n+1]\}^{-1} \subsetneqq \{a_i \mid i \in [1, n]\}^{-1} \subseteq E^{-1} = F^{-1}$, a contradiction. □

Next we specify Krull monoids by using Mori sets. Note that the equivalence of 1 and 4 is well known.

Proposition 3.2. *Let H be a monoid. The following conditions are equivalent:*

1. *H is a Krull monoid.*
2. *H is atomic, completely integrally closed and $\mathcal{A}(H)$ is a Mori set.*
3. *H is completely integrally closed and there is some Mori set $F \subseteq H$ such that $H = [F \cup H^\times]_H$.*
4. *H is completely integrally closed and every t-maximal t-ideal of H is divisorial.*

Proof. **1.** $\Rightarrow$ **2.**: Clear. **2.** $\Rightarrow$ **3.**: Set $F = \mathcal{A}(H)$. Since H is atomic we have $H = [F \cup H^\times]$. **3.** $\Rightarrow$ **4.**: Let $F \subseteq H$ be a Mori set such that $H = [F \cup H^\times]_H$, P a t-maximal t-ideal of H and $x \in P^\bullet$. Then there are some $\varepsilon \in H^\times$ and $(\alpha_e)_{e\in F} \in \mathbb{N}_0^{(F)}$ such that $x = \varepsilon \prod_{e\in F} e^{\alpha_e}$. Therefore, there exists some $e \in F$ such that $e \in P$ and $x \in eH$. It follows that $x \in \{e\}_t \subseteq (P \cap F)_t$. Consequently, $P \subseteq (P \cap F)_t$. There is some finite $E \subseteq P \cap F$ such that $E^{-1} = (P \cap F)^{-1}$. This implies that $P \subseteq (P \cap F)_t \subseteq (P \cap F)_v = E_v = E_t \subseteq (P \cap F)_t \subseteq P$, hence $P = E_v$, and thus P is divisorial. **4.** $\Rightarrow$ **1.**: Let I be a non-zero t-ideal of H. It is sufficient to show that I is t-invertible. Since H is completely integrally closed, it follows that $(II^{-1})_v = H$. Assume that $(II^{-1})_t \subsetneqq H$. Then there exists some t-maximal t-ideal P of H such that $(II^{-1})_t \subseteq P$. We have $H = (II^{-1})_v \subseteq ((II^{-1})_t)_v \subseteq P_v = P$, a contradiction. Consequently, $(II^{-1})_t = H$. □

Now we provide a few minor results about Mori sets and divisor-closed submonoids to prepare for the main result in this section.

Lemma 3.3. *Let H be a monoid, $S \subseteq H$ a divisor-closed submonoid and $X \subseteq S$ a subset. If X is an H-Mori set, then X is an S-Mori set.*

Proof. Let K be a quotient monoid of H and $L \subseteq K$ the quotient monoid of S. First we show that for every $Y \subseteq S$ it follows that $(H :_L Y) = (S :_L Y)$. Let $Y \subseteq S$. "$\subseteq$": Let $x \in (H :_L Y)$, then $xY \subseteq H$. Since $S \subseteq H$ is divisor-closed it follows that $xY \subseteq H \cap L = S$, hence $x \in (S :_L Y)$. "$\supseteq$": Trivial. Now let X be an H-Mori set and $F \subseteq X$. Then there exists some finite $E \subseteq F$ such that $(H :_K F) = (H :_K E)$. This implies that $(S :_L F) = (H :_L F) = (H :_K F) \cap L = (H :_K E) \cap L = (H :_L E) = (S :_L E)$, hence X is an S-Mori set. □

Let H be a monoid, $x \in H$ and $n \in \mathbb{N}$ and let $\mathbb{A}$ be some property that can be stated in the language of monoids (e.g. atomic, Krull, Mori).

- Set $\mathcal{D}_n(x) = \{u \in \mathcal{A}(H) \mid u \,|_H\, x^n\}$.
- A submonoid $S \subseteq H$ is called monadic if $S = [\![a]\!]$ for some $a \in H^\bullet$.
- We say that H is monadically $\mathbb{A}$ (or H is a monadically $\mathbb{A}$ monoid) if every monadic submonoid of H satisfies $\mathbb{A}$.
- The property $\mathbb{A}$ is said to be monadic (for H) if H has property $\mathbb{A}$ if and only if every monadic submonoid of H has property $\mathbb{A}$.
- If H is an integral domain we say that H is a $\mathbb{A}$ domain if H satisfies $\mathbb{A}$ as a monoid.

Note that H is monadically $\mathbb{A}$ if and only if $[\![E]\!]$ satisfies $\mathbb{A}$ for all non-empty finite $E \subseteq H^\bullet$.

Proposition 3.4. *Let H be a monoid and K a quotient monoid of H.*

1. *$H^\times = [\![a]\!]^\times$ and $\mathcal{A}([\![a]\!]) = \mathcal{A}(H) \cap [\![a]\!] = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n(a)$ for all $a \in H^\bullet$.*
2. *H is atomic if and only if H is monadically atomic.*
3. *H is completely integrally closed if and only if H is monadically completely integrally closed.*
4. *H is an FF-monoid if and only if H is a monadically FF-monoid.*

Proof. **1.** Let $a \in H^\bullet$. Clearly, $[\![a]\!]^\times \subseteq H^\times$. If $\varepsilon \in H^\times$, then $\varepsilon\varepsilon^{-1} = 1 \in [\![a]\!]$, hence $\varepsilon, \varepsilon^{-1} \in [\![a]\!]$, and thus $\varepsilon \in [\![a]\!]^\times$. If $x \in \mathcal{A}([\![a]\!])$ and $b, c \in H$ are such that $x = bc$, then $b, c \in [\![a]\!]$, hence $b \in [\![a]\!]^\times = H^\times$ or $c \in [\![a]\!]^\times = H^\times$. Finally, if $x \in \mathcal{A}(H) \cap [\![a]\!]$ and $b, c \in [\![a]\!]$ are such that $x = bc$, then $b \in H^\times = [\![a]\!]^\times$ or $c \in H^\times = [\![a]\!]^\times$. Obviously, $\mathcal{A}(H) \cap [\![a]\!] = \bigcup_{n\in\mathbb{N}}\{u \in \mathcal{A}(H) \mid u \,|_H\, a^n\} = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n(a)$.
2. This is an immediate consequence of 1.
3. "$\Rightarrow$": Trivial "$\Leftarrow$": Let $x \in K^\bullet$ be almost integral over H. There exists some $c \in H^\bullet$ such that $cx^n \in H$ for all $n \in \mathbb{N}$ and there are some $a, b \in H^\bullet$ such that $x = \frac{a}{b}$. Let $L \subseteq K$ be the quotient monoid of $[\![abc]\!]$. Then $c \in [\![abc]\!]^\bullet$, $x \in L$ and $cx^n \in H \cap L = [\![abc]\!]$ for all $n \in \mathbb{N}$. Since $[\![abc]\!]$ is completely integrally closed we have $x \in [\![abc]\!] \subseteq H$.
4. "$\Rightarrow$": This follows from 1 and [15, Theorem 1.5.6.2]. "$\Leftarrow$": Let $x \in H^\bullet$. It is an easy consequence of 1 that $f : \{y[\![x]\!] \mid y \in [\![x]\!], y \,|_{[\![x]\!]}\, x\} \to \{yH \mid y \in [\![x]\!], y \,|_{[\![x]\!]}\, x\}$ defined by $f(I) = IH$ is a bijective map. Since $\{y[\![x]\!] \mid y \in [\![x]\!], y \,|_{[\![x]\!]}\, x\}$ is finite, we have $\{yH \mid y \in H, y \,|_H\, x\} = \{yH \mid y \in [\![x]\!], y \,|_{[\![x]\!]}\, x\}$ is finite. □

Let H be a monoid and K a quotient monoid of H.

- H is called seminormal if for all $x \in K$ such that $x^2, x^3 \in H$ we have $x \in H$.
- H is called weakly factorial if every $x \in H^\bullet \setminus H^\times$ is a finite product of primary elements of H (i.e. of elements $x \in H^\bullet$ such that xH is primary).
- H is called radical factorial if every $x \in H^\bullet$ is a finite product of radical elements of H (i.e. of elements $x \in H^\bullet$ such that $\sqrt{xH} = xH$).

We leave to the reader to prove that "satisfying the ACCP", "seminormal", "root-closed", "atomic and weakly factorial", "atomic and radical factorial", "being a BF-monoid", "being a valuation monoid", "being a GCD-monoid" and "factorial" are also monadic properties for H. We do not know whether "weakly factorial", "radical factorial" and "v-closed" are monadic properties. If H is weakly factorial (resp. radical factorial), then H is monadically weakly factorial (resp. monadically radical factorial) and the following holds.

Remark 3.5. *Let H be a monoid that is monadically v-closed. Then H is v-closed.*

Proof. Let K be a quotient monoid of H, $\emptyset \neq E \subseteq H^\bullet$ finite and $x \in K^\bullet$ such that $xE \subseteq E_v$. There are some $y, z \in H^\bullet$ such that $x = \frac{y}{z}$. Set $S = [\![E \cup xE \cup \{y, z\}]\!]$. Observe that S is a monadic submonoid of H, and thus S is v_S-closed. Let $L \subseteq K$ be the quotient monoid of S. Clearly, $x \in L$ and it follows by [15, Proposition 2.4.2.3] that $xE \subseteq E_v \cap S \subseteq (E_{v_S})_v \cap S = E_{v_S}$, hence $x \in S$. Consequently, $x \in H$. □

Now we present the main result in this section. It connects monadically Krull monoids with concepts that are well known in the literature.

Theorem 3.6. *Let H be a monoid. The following conditions are equivalent:*

1. *H is a monadically Krull monoid.*
2. *H is atomic and completely integrally closed and $\{uH \mid u \in \mathcal{A}([\![a]\!])\}$ is finite for all $a \in H^\bullet$.*
3. *H is a completely integrally closed FF-monoid and for all $a \in H^\bullet$, $\mathcal{A}([\![a]\!]) \subseteq \mathcal{D}_k(a)$ for some $k \in \mathbb{N}$.*
4. *H is atomic and completely integrally closed and $\mathcal{A}([\![a]\!])$ is an H-Mori set for all $a \in H^\bullet$.*

Proof. **1.** $\Rightarrow$ **2.**: By Propositions 3.4.2 and 3.4.3 we have H is atomic and completely integrally closed. Let $a \in H^\bullet$. If $P \in \mathfrak{X}([\![a]\!])$, then there is some $b \in P^\bullet$, hence there are some $c \in H$ and $n \in \mathbb{N}$ such that $bc = a^n$. It follows that $c \in [\![a]\!]$, hence $a^n \in P$, and thus $a \in P$. Therefore, $a \in P$ for all $P \in \mathfrak{X}([\![a]\!])$. It follows by [15, Proposition 2.2.4.2] and [15, Theorem 2.2.5.2] that $\mathfrak{X}([\![a]\!])$ is finite. It can be easily deduced from [15, Theorem 2.7.14] that $\{u[\![a]\!] \mid u \in \mathcal{A}([\![a]\!])\}$ is finite. It follows by Proposition 3.4.1 that $f : \{u[\![a]\!] \mid u \in \mathcal{A}([\![a]\!])\} \to \{uH \mid u \in \mathcal{A}([\![a]\!])\}$ defined by $f(I) = IH$ is bijective, hence $\{uH \mid u \in \mathcal{A}([\![a]\!])\}$ is finite.
2. $\Rightarrow$ **3.**: Let $a \in H^\bullet$. It follows that $\{uH \mid u \in \mathcal{D}_1(a)\} \subseteq \{uH \mid u \in \mathcal{A}([\![a]\!])\}$ is finite which implies (together with the fact that H is atomic) that H is an FF-monoid. On the other hand there is some finite $E \subseteq \mathcal{A}([\![a]\!])$ such that $\{uH \mid u \in \mathcal{A}([\![a]\!])\} = \{uH \mid u \in E\}$. There is some $k \in \mathbb{N}$ such that $E \subseteq \mathcal{D}_k(a)$. Let $u \in \mathcal{A}([\![a]\!])$. Then some $\varepsilon \in H^\times$ and some $v \in E$ exist such that $u = \varepsilon v$. Since $v \in \mathcal{D}_k(a)$, we immediately obtain that $u \in \mathcal{D}_k(a)$.
3. $\Rightarrow$ **4.**: Let $a \in H^\bullet$ and $F \subseteq \mathcal{A}([\![a]\!])$. There is some $k \in \mathbb{N}$ such that $F \subseteq \mathcal{D}_k(a)$ and since H is an FF-monoid we have $\{uH \mid u \in F\} \subseteq \{uH \mid u \in \mathcal{D}_k(a)\}$ is finite. Consequently, there is some finite $E \subseteq F$ such that $\{uH \mid u \in F\} = \{uH \mid u \in E\}$. If $x \in K$, then $x \in E^{-1}$ if and only if $xuH \subseteq H$ for all $u \in E$ if and only if $xuH \subseteq H$ for all $u \in F$ if and only if $x \in F^{-1}$. Therefore, $E^{-1} = F^{-1}$.
4. $\Rightarrow$ **1.**: Let $a \in H^\bullet$. By Propositions 3.4.2 and 3.4.3 it follows that $[\![a]\!]$ is atomic and completely integrally closed. Lemma 3.3 implies that $\mathcal{A}([\![a]\!])$ is an $[\![a]\!]$-Mori set. By Proposition 3.2 we obtain that $[\![a]\!]$ is a Krull monoid. □

Using the terminology in [23] we obtain by Theorem 3.6 that H is a monadically Krull monoid if and only if it is an atomic, completely integrally closed IDPF-monoid if and only if it is a completely integrally closed FF-monoid that is a pseudo-IDPF monoid. We will see later that monadically Krull monoids are not necessarily Krull monoids. Especially, we have that monadically Mori monoids are not necessarily Mori monoids. The next result shows that if the Mori property is satisfied by a bigger class of divisor-closed submonoids, then the monoid itself satisfies the Mori property.

Proposition 3.7. *Let H be a monoid. Then H is a Mori monoid if and only if $[\![X]\!]$ is a Mori monoid for every denumerable subset $X \subseteq H^\bullet$.*

Proof. Let K be a quotient monoid of H. "$\Rightarrow$": Trivial. "$\Leftarrow$": Assume that H is not a Mori monoid. Then H is not a Mori set. By Lemma 3.1 there exists some $(a_i)_{i\in\mathbb{N}} \in H^{\mathbb{N}}$ such that $(H :_K \{a_i \mid i \in [1, n+1]\}) \subsetneq (H :_K \{a_i \mid i \in [1, n]\})$ for all $n \in \mathbb{N}$. Therefore, there exist some $(x_i)_{i\in\mathbb{N}} \in H^{\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}} \in (H^\bullet)^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$ we have $a_{n+1}\frac{x_n}{y_n} \in K \setminus H$ and $a_i \frac{x_n}{y_n} \in H$ for all $i \in [1, n]$. Let $S = [\![\{a_n x_n y_n \mid n \in \mathbb{N}\}]\!]$ and let $L \subseteq K$ be the quotient monoid of S. Then $(a_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}}$ and $(\frac{x_i}{y_i})_{i\in\mathbb{N}} \in L^{\mathbb{N}}$. Moreover, we have for all $n \in \mathbb{N}$ that $a_{n+1}\frac{x_n}{y_n} \in L \setminus S$ and $a_i \frac{x_n}{y_n} \in H \cap L = S$ for all $i \in [1, n]$. This implies that $(S :_L \{a_i \mid i \in [1, n+1]\}) \subsetneq (S :_L \{a_i \mid i \in [1, n]\})$ for all $n \in \mathbb{N}$. It follows by Lemma 3.1 that S is not an S-Mori set, and thus S is not a Mori monoid, a contradiction. □

4. Counterexamples

It is of interest to know whether every monadically Krull monoid is already a Krull monoid. In this section we prove that this is not necessarily true and show that even strong improvements of monadically Krull can fail to be Krull. For technical reasons we will consider (multiplicatively written) monoids that do not posses a zero element in this section. Moreover, we will use monoids that are additively written (the zero element is their identity and they do not posses an "additive" analogue of a "multiplicative" zero element). All terminology that has been introduced so far can be adapted in an obvious way for these types of monoids. Observe that the label "quotient monoid" will be replaced by "quotient group" for monoids without a zero element. Note that a monoid is root closed if and only if it is integrally closed in terms of [16]. We want to thank F. Kainrath who led our attention to the integral domain constructed in the next example.

Example 4.1. *There exists a monadically Krull domain that is not a Krull domain.*

Proof. Let R be an integrally closed noetherian domain, $(X_i)_{i\in\mathbb{N}_0}$ a sequence of independent indeterminates over R and K a field of quotients of $R[\{X_i \mid i\in\mathbb{N}_0\}]$. For $n\in\mathbb{N}_0$ set $S_n = R[\{\prod_{i=0}^n X_i^{a_i} \mid (a_i)_{i=0}^n \in \mathbb{N}_0^{[0,n]}, a_0 \geq \sum_{i=1}^n \frac{a_i}{2^i}\}]$. Let $S=\bigcup_{n\in\mathbb{N}} S_n$. We show that S is a monadically Krull domain that is not a Krull domain. Note that S is a subring of $R[\{X_i \mid i\in\mathbb{N}_0\}]$ and K is a field of quotients of S.
First we show that S is not a Krull domain. For $n\in\mathbb{N}_0$ set $a_n = X_0^{n+1}(\prod_{i=1}^n X_i^{2^i})X_{n+1}^{2^{n+1}-1}$. By Lemma 3.1 it is sufficient to show that $X_{k+1}\in\{a_i \mid i\in[0,k]\}^{-1}\setminus\{a_i \mid i\in[0,k+1]\}^{-1}$ for all $k\in\mathbb{N}_0$. Let $k\in\mathbb{N}_0$ and $i\in[0,k]$. Since $\sum_{j=1}^i \frac{2^j}{2^j}+\frac{2^{i+1}-1}{2^{i+1}}+\frac{1}{2^{k+1}} = i+1-\frac{1}{2^{i+1}}+\frac{1}{2^{k+1}} \leq i+1$ it follows that $X_{k+1}a_i = X_0^{i+1}(\prod_{j=1}^i X_j^{2^j})X_{i+1}^{2^{i+1}-1}X_{k+1}\in S$, and thus $X_{k+1}\in\{a_j \mid j\in[0,k]\}^{-1}$. Since $\sum_{j=1}^k \frac{2^j}{2^j}+\frac{2^{k+1}+1}{2^{k+1}}+\frac{2^{k+2}-1}{2^{k+2}} = k+2+\frac{1}{2^{k+2}} > k+2$, we have $X_{k+1}a_{k+1} = X_0^{k+2}(\prod_{j=1}^k X_j^{2^j})X_{k+1}^{2^{k+1}+1}X_{k+2}^{2^{k+2}-1}\notin S$, hence $X_{k+1}\notin\{a_j \mid j\in[0,k+1]\}^{-1}$.
Next we prove that S_l is a divisor-closed subring of S that is noetherian and integrally closed for all $l\in\mathbb{N}$. Let $l\in\mathbb{N}$. Clearly, S_l is a subring of S. Let $f,g\in S^\bullet$ be such that $fg\in S_l$. There is some $m\in\mathbb{N}$ such that $m\geq l$ and $f,g\in S_m$. Since $fg\in R[\{X_i \mid i\in[0,l]\}]$, it follows that $f\in R[\{X_i \mid i\in[0,l]\}]$, hence $f\in S_m\cap R[\{X_i \mid i\in[0,l]\}] = S_l$. Therefore, S_l is a divisor-closed subring of S.
Set $T=\{\prod_{i=0}^l X_i^{a_i} \mid (a_i)_{i=0}^l\in\mathbb{N}_0^{[0,l]}, a_0\geq\sum_{i=1}^l\frac{a_i}{2^i}\}$, $U=\{\prod_{i=0}^l X_i^{a_i} \mid (a_i)_{i=0}^l\in\mathbb{N}_0^{[0,l]}, l\geq a_0\geq\sum_{i=1}^l\frac{a_i}{2^i}\}$ and $V=[U]$. Then T is a submonoid of S_l. We show that T is root closed and $T=V$. Let $L\subseteq K$ be a quotient group of T. Let $x\in L$ and $r\in\mathbb{N}$ be such that $x^r\in T$. Since $\{\prod_{i=0}^l X_i^{a_i} \mid (a_i)_{i=0}^l\in\mathbb{N}_0^{[0,l]}\}$ is a root closed submonoid of L that contains T it follows that there is some $(b_i)_{i=0}^l\in\mathbb{N}_0^{[0,l]}$ such that $x=\prod_{i=0}^l X_i^{b_i}$. This implies that $\prod_{i=0}^l X_i^{rb_i}\in T$, and thus $rb_0\geq\sum_{i=1}^l\frac{rb_i}{2^i}$. Consequently, $b_0\geq\sum_{i=1}^l\frac{b_i}{2^i}$, hence $x\in T$. Therefore, T is root closed. It remains to prove by induction on k that for all $k\in\mathbb{N}$ and all $(a_i)_{i=1}^l\in\mathbb{N}_0^{[1,l]}$ such that $k\geq\sum_{i=1}^l\frac{a_i}{2^i}$ it follows that $X_0^k\prod_{i=1}^l X_i^{a_i}\in V$. If $k=1$, then the assertion is clear. Now let $k\in\mathbb{N}$ and $(a_i)_{i=1}^l\in\mathbb{N}_0^{[1,l]}$ be such that $k+1\geq\sum_{i=1}^l\frac{a_i}{2^i}$. Case 1: There is some $j\in[1,l]$ such that $a_j>2^j$: We have $k\geq\sum_{i=1,i\neq j}^l\frac{a_i}{2^i}+\frac{a_j-2^j}{2^j}$. It follows by the induction hypothesis that $X_0^k(\prod_{i=1,i\neq j}^l X_i^{a_i})X_j^{a_j-2^j}\in V$. Therefore, $X_0^{k+1}\prod_{i=1}^l X_i^{a_i} = X_0X_j^{2^j}X_0^k(\prod_{i=1,i\neq j}^l X_i^{a_i})X_j^{a_j-2^j}\in V$. Case 2: $a_j\leq 2^j$ for all $j\in[1,l]$: We have $\sum_{i=1}^l\frac{a_i}{2^i}\leq l$. If $l\geq k+1$, then $X_0^{k+1}\prod_{i=1}^l X_i^{a_i}\in V$, by definition. Now let $l<k+1$. Since $X_0\in V$, it follows that $X_0^{k+1}\prod_{i=1}^l X_i^{a_i} = X_0^{k+1-l}X_0^l\prod_{i=1}^l X_i^{a_i}\in V$. By [17, Corollary 15.12] it follows that S_l is noetherian and integrally closed.
Let $a\in S^\bullet$. There is some $s\in\mathbb{N}$ such that $a\in S_s$. Since S_s is a divisor-closed subring of S it follows that $[\![a]\!]_S$ is a divisor-closed submonoid of S_s. Since S_s is a Krull domain this implies that $[\![a]\!]_S$ is a Krull monoid. Consequently, S is a monadically Krull domain. □

It has been pointed out in [9] that every atomic IDPF-domain (this notion has been introduced in [23]) that contains a field of characteristics zero is already completely integrally closed. Now let the domain

R in the last example be a field of characteristics zero. Then the domain S in the last example is an atomic IDPF-domain that contains a field of characteristics zero and yet S is not a Krull domain. Let H be a monoid and r a finitary ideal system on H. Let $\mathcal{J} \subseteq \mathbb{P}(H)$. We say that $\mathcal{J}$ possesses a length function if there exists some map $\lambda : \mathcal{J} \to \mathbb{N}_0$ such that $\lambda(J) < \lambda(I)$ for all $I, J \in \mathcal{J}$ such that $I \subsetneq J$. Note if $\mathcal{J}$ possesses a length function, then $\mathcal{J}$ satisfies the ACC. Moreover, if R is an integral domain, then the set of non-zero ideals of R possesses a length function if and only if R is a noetherian domain and $\dim(R) \leq 1$. Observe that a monoid H is a BF-monoid if and only if $\{xH \mid x \in H^\bullet\}$ possesses a length function. Moreover, possessing a length function is in some sense the same as satisfying a strong version of the ACC. We use it in the next example to highlight that it is not only a BF-monoid but also that the set of radicals of principal ideals possesses a length function. Next we introduce several other types of generalizations of the Krull property to study the following example in detail.

- H is called an r-SP-monoid if every r-ideal of H is a finite r-product of radical r-ideals of H.
- H is called an r-Prüfer monoid if every non-zero r-finitely generated r-ideal of H is r-invertible.
- An r-ideal I of H is called r-cancellative if I is cancellative in the r-ideal semigroup of H.

Clearly, H is a Krull monoid if and only if it is a Mori monoid that is a t-Prüfer monoid. Note that if H is a radical factorial monoid, then for all $a \in H^\bullet$ we have $\mathcal{A}(\llbracket a \rrbracket) \subseteq \mathcal{D}_1(a)$. Using Theorem 3.6 and [29, Proposition 2.4] it is straightforward to prove that every radical factorial FF-monoid is a monadically Krull monoid (but even a Krull monoid needs not be radical factorial, see [29, Example 4.3]). In this light we will sharpen our first counterexample in the monoid setting and prove that even radical factorial FF-monoids can fail to be Krull. A sequence $(x_i)_{i\in\mathbb{N}_0}$ of integers is called formally infinite if $\{i \in \mathbb{N}_0 \mid x_i \neq 0\}$ is finite. If H is additively written, $k \in \mathbb{N}$ and $I \subseteq H$, then set $kI = \{\sum_{i=1}^k a_i \mid (a_i)_{i=1}^k \in I^{[1,k]}\}$.

Example 4.2. *Let G be a free abelian group with basis $(e_i)_{i\in\mathbb{N}_0}$. For $x \in G$, let $(x_i)_{i\in\mathbb{N}_0} \in \mathbb{Z}^{(\mathbb{N}_0)}$ denote the unique formally infinite sequence such that $x = \sum_{i\in\mathbb{N}_0} x_i e_i$. Set $H = \{x \in G \mid x_0 \geq x_i \geq 0$ for all $i \in \mathbb{N}_0\}$. Then H is a submonoid of G, G is a quotient group of H and the following is true:*

1. *$v\text{-spec}(H)^\bullet = \mathfrak{X}(H)$, $t\text{-spec}(H)^\bullet = \mathfrak{X}(H) \cup \{H\backslash H^\times\}$, every non-empty t-ideal of H is t-cancellative and $(kP)_t$ is P-primary for all $k \in \mathbb{N}$ and $P \in t\text{-spec}(H)^\bullet$.*
2. *H is a t-SP-monoid, $t\text{-dim}(H) = 2$, H is an FF-monoid, $\{\sqrt{y+H} \mid y \in H\}$ possesses a length function and every radical element of H is either an atom or a unit.*

In particular, H is a radical factorial monoid that is neither a Mori monoid nor a t-Prüfer monoid.

Proof. Clearly, H is a submonoid of G and $H^\times = \{0\}$. Let $K \subseteq G$ be the quotient group of H and $i \in \mathbb{N}_0$. Obviously, $e_0, e_0 + e_i \in H$, hence $e_i = e_0 + e_i - e_0 \in K$. Therefore, $G = K$ is a quotient group of H. For $r \in \mathbb{N}_0^{\mathbb{N}_0}$ set $I_r = \{x \in G \mid x_0 \geq x_{j+1} + r_{2j+1}, x_j \geq r_{2j}$ for all $j \in \mathbb{N}_0\}$. Set $\mathfrak{I} = \{r \in \mathbb{N}_0^{\mathbb{N}_0} \mid |\{j \in \mathbb{N}_0 \mid r_{2j} \neq 0\}| < \infty, r_0 \geq r_{2j+1} + r_{2j+2}$ for all $j \in \mathbb{N}_0\}$ and $\mathfrak{L} = \{r \in \mathbb{N}_0^{\mathbb{N}_0} \mid |\{j \in \mathbb{N}_0 \mid r_{2j} \neq 0\}| < \infty, r_0 = \max\{r_{2j+1} + r_{2j+2} \mid j \in \mathbb{N}_0\}\}$. For $i \in \mathbb{N}_0$, let $s^{(i)} \in \mathbb{N}_0^{\mathbb{N}_0}$ be defined by $s_j^{(i)} = 1$ if $j \in \{0, i\}$ and $s_j^{(i)} = 0$ if $j \in \mathbb{N}_0\backslash\{0, i\}$. If $r, s \in \mathbb{N}_0^{\mathbb{N}_0}$ and $k \in \mathbb{N}_0$, then we set $r + s = (r_i + s_i)_{i\in\mathbb{N}_0}$, $kr = (kr_i)_{i\in\mathbb{N}_0}$ and $r \leq s$ if $r_j \leq s_j$ for all $j \in \mathbb{N}_0$.
Claim 1: For all $x, y \in H$ it follows that $x \in \sqrt{y+H}$ if and only if $\{i \in \mathbb{N}_0 \mid y_i > 0\} \subseteq \{i \in \mathbb{N}_0 \mid x_i > 0\}$ and $\{i \in \mathbb{N}_0 \mid y_0 > y_i\} \subseteq \{i \in \mathbb{N}_0 \mid x_0 > x_i\}$. Let $x, y \in H$. Observe that $x \in \sqrt{y+H}$ if and only if there is some $k \in \mathbb{N}$ such that $kx_i \geq y_i$ and $k(x_0 - x_i) \geq y_0 - y_i$ for all $i \in \mathbb{N}_0$. "$\Rightarrow$": Let $k \in \mathbb{N}$ be such that $kx_j \geq y_j$ and $k(x_0 - x_j) \geq y_0 - y_j$ for all $j \in \mathbb{N}_0$. Let $i \in \mathbb{N}_0$. If $y_i > 0$, then $kx_i \geq y_i > 0$, and thus $x_i > 0$. If $y_0 > y_i$, then $k(x_0 - x_i) \geq y_0 - y_i > 0$, hence $x_0 > x_i$. "$\Leftarrow$": Let $\{i \in \mathbb{N}_0 \mid y_i > 0\} \subseteq \{i \in \mathbb{N}_0 \mid x_i > 0\}$ and $\{i \in \mathbb{N}_0 \mid y_0 > y_i\} \subseteq \{i \in \mathbb{N}_0 \mid x_0 > x_i\}$. Set $k = 1 + \max\{y_i \mid i \in \mathbb{N}_0\}$. Then $k \in \mathbb{N}$ and it is clear that $kx_i \geq y_i$ and $k(x_0 - x_i) \geq y_0 - y_i$ for all $i \in \mathbb{N}_0$.
Claim 2: For all $\emptyset \neq A \subseteq H$ we have $A_v = \{x \in G \mid x_0 \geq x_i + \min\{a_0 - a_i \mid a \in A\}$ and $x_i \geq \min\{a_i \mid a \in A\}$ for all $i \in \mathbb{N}\}$. Let $\emptyset \neq A \subseteq H$. Set $m^{(i)} = \min\{a_0 - a_i \mid a \in A\}$ and $n^{(i)} = \min\{a_i \mid a \in A\}$ for all $i \in \mathbb{N}$. Observe that $A^{-1} = \{x \in G \mid x + a \in H$ for all $a \in A\} = \{x \in G \mid$ for all $a \in A$ and $i \in \mathbb{N}$, $x_0 + a_0 \geq x_i + a_i \geq 0\} = \{x \in G \mid x_0 + m^{(i)} \geq x_i$ and $x_i + n^{(i)} \geq 0$ for all $i \in \mathbb{N}\}$. "$\subseteq$": Let $x \in A_v$ and

$i \in \mathbb{N}$. Set $y = m^{(i)}e_i$ and $z = -n^{(i)}e_i$. We have $y, z \in A^{-1}$, hence $x + y \in H$ and $x + z \in H$. Therefore, $x_0 + y_0 \geq x_i + y_i$ and $x_i + z_i \geq 0$. This implies that $x_0 \geq x_i + m^{(i)}$ and $x_i \geq n^{(i)}$. "$\supseteq$": Let $x \in G$ be such that $x_0 \geq x_j + m^{(j)}$ and $x_j \geq n^{(j)}$ for all $j \in \mathbb{N}$. Let $y \in A^{-1}$ and $i \in \mathbb{N}$. Then $x_0 \geq x_i + m^{(i)}$, $y_0 + m^{(i)} \geq y_i$, $x_i \geq n^{(i)}$ and $y_i + n^{(i)} \geq 0$, hence $x_0 + y_0 + m^{(i)} \geq x_i + y_i + m^{(i)}$ and $x_i + y_i + n^{(i)} \geq n^{(i)}$. Therefore, $x_0 + y_0 \geq x_i + y_i \geq 0$. This implies that $x + y \in H$. Consequently, $x \in A_v$.
As usual we denote by $\mathcal{I}_t(H)^\bullet$ (resp. $\mathcal{I}_v(H)^\bullet$) the set of non-empty t-ideals of H (resp. the set of non-empty divisorial ideals of H).
Claim 3: $\mathcal{I}_t(H)^\bullet = \{I_r \mid r \in \mathfrak{I}\}$ and $\mathcal{I}_v(H)^\bullet = \{I_r \mid r \in \mathfrak{L}\}$. First we prove that $\mathcal{I}_t(H)^\bullet = \{I_r \mid r \in \mathfrak{I}\}$. "$\subseteq$": Let $I \in \mathcal{I}_t(H)^\bullet$. For $i \in \mathbb{N}_0$ set $r_{2i} = \min\{y_i \mid y \in I\}$ and $r_{2i+1} = \min\{y_0 - y_{i+1} \mid y \in I\}$. There is some sequence $(z^{(j)})_{j\in\mathbb{N}_0} \in I^{\mathbb{N}_0}$ such that $z_i^{(2i)} = r_{2i}$ and $z_0^{(2i+1)} - z_{i+1}^{(2i+1)} = r_{2i+1}$ for all $i \in \mathbb{N}_0$. If $j \in \mathbb{N}_0$, then since $z_0^{(0)} - z_{j+1}^{(0)} \geq r_{2j+1}$ and $z_{j+1}^{(0)} \geq r_{2j+2}$ we obtain that $r_0 = z_0^{(0)} \geq r_{2j+1} + r_{2j+2}$. Moreover, $|\{j \in \mathbb{N}_0 \mid r_{2j} \neq 0\}| \leq |\{j \in \mathbb{N}_0 \mid z_j^{(0)} \neq 0\}| < \infty$. Therefore, $r \in \mathfrak{I}$. It remains to show that $I = I_r$. "$\subseteq$": Trivial. "$\supseteq$": Let $x \in I_r$. Set $E = \{i \in \mathbb{N} \mid x_i \neq 0 \text{ or } z_i^{(0)} \neq 0\}$. Then E is finite. It is sufficient to prove that $x_0 - x_j \geq \min(\{z_0^{(2i-1)} - z_j^{(2i-1)}, z_0^{(2i)} - z_j^{(2i)} \mid i \in E\} \cup \{z_0^{(0)} - z_j^{(0)}\})$ and $x_j \geq \min(\{z_j^{(2i-1)}, z_j^{(2i)} \mid i \in E\} \cup \{z_j^{(0)}\})$ for all $j \in \mathbb{N}$, because then $x \in (\{z^{(2i-1)}, z^{(2i)} \mid i \in E\} \cup \{z^{(0)}\})_v$ by claim 2, hence $x \in I$. Let $j \in \mathbb{N}$. Case 1a: $x_j \neq 0$. It follows that $x_0 - x_j \geq r_{2j-1} = z_0^{(2j-1)} - z_j^{(2j-1)}$. Case 1b: $x_j = 0$. We have $x_0 - x_j = x_0 \geq r_0 = z_0^{(0)} \geq z_0^{(0)} - z_j^{(0)}$. Case 2a: $j \in E$. It follows that $x_j \geq r_{2j} = z_j^{(2j)}$. Case 2b: $j \notin E$. We have $x_j = 0 = z_j^{(0)}$. "$\supseteq$": Let $r \in \mathfrak{I}$ and $x \in (I_r)_t$. Then there is some finite $\emptyset \neq A \subseteq I_r$ such that $x \in A_v$. It is an immediate consequence of claim 2 that $x_0 \geq x_j + r_{2j-1}$ and $x_j \geq r_{2j}$ for all $j \in \mathbb{N}$. Since A is finite, there is some $l \in \mathbb{N}$ such that $x_l = 0$ and $a_l = 0$ for all $a \in A$. It follows by claim 2 that $x_0 \geq x_l + \min\{a_0 - a_l \mid a \in A\} = \min\{a_0 \mid a \in A\} \geq r_0$. Consequently, $x_0 \geq x_{j+1} + r_{2j+1}$ and $x_j \geq r_{2j}$ for all $j \in \mathbb{N}_0$, and thus $x \in I_r$. Observe that $\sum_{i\in\mathbb{N}_0} r_{2i}e_i \in I_r$, hence $I_r \in \mathcal{I}_t(H)^\bullet$.
Next we show that $\mathcal{I}_v(H)^\bullet = \{I_r \mid r \in \mathfrak{L}\}$. "$\subseteq$": Let $I \in \mathcal{I}_v(H)^\bullet$. As in the preceding part of the proof there is some $r \in \mathfrak{I}$ such that $I = I_r$, $r_{2i} = \min\{y_i \mid y \in I\}$ and $r_{2i+1} = \min\{y_0 - y_{i+1} \mid y \in I\}$ for all $i \in \mathbb{N}_0$. Set $s = \max\{r_{2i+1} + r_{2i+2} \mid i \in \mathbb{N}_0\}$. It remains to show that $s \geq r_0$, because then $r \in \mathfrak{L}$. Set $x = se_0 + \sum_{i\in\mathbb{N}} r_{2i}e_i$. If $i \in \mathbb{N}$, then $x_0 = s \geq r_{2i-1} + r_{2i} = x_i + r_{2i-1}$ and $x_i = r_{2i}$. Therefore, claim 2 implies that $x \in (I_r)_v = I_r$, and thus $s = x_0 \geq r_0$. "$\supseteq$": Let $r \in \mathfrak{L}$ and $x \in (I_r)_v$. It follows by claim 2 that $x_0 \geq x_i + r_{2i-1}$ and $x_i \geq r_{2i}$ for all $i \in \mathbb{N}$. There is some $j \in \mathbb{N}$ such that $r_0 = r_{2j-1} + r_{2j}$, hence $x_0 \geq x_j + r_{2j-1} \geq r_{2j} + r_{2j-1} = r_0$. This implies that $x \in I_r$.
Claim 4: For all $a, b \in \mathfrak{I}$, $I_{a+b} = (I_a + I_b)_t$ and $I_a \subseteq I_b$ if and only if $b \leq a$. Let $a, b \in \mathfrak{I}$. Set $y^{(0)} = \sum_{i\in\mathbb{N}_0} a_{2i}e_i$, $z^{(0)} = \sum_{i\in\mathbb{N}_0} b_{2i}e_i$ and for $j \in \mathbb{N}$ set $y^{(j)} = \sum_{i\in\mathbb{N}_0, i\neq j} a_{2i}e_i + (a_0 - a_{2j-1})e_j$ and $z^{(j)} = \sum_{i\in\mathbb{N}_0, i\neq j} b_{2i}e_i + (b_0 - b_{2j-1})e_j$. Observe that $y^{(j)} \in I_a$ and $z^{(j)} \in I_b$ for all $j \in \mathbb{N}_0$. "$\subseteq$": Let $x \in I_{a+b}$. We prove that $x_0 \geq x_j + \min\{y_0^{(l)} + z_0^{(l)} - y_j^{(l)} - z_j^{(l)} \mid l = 0 \text{ or } l \in \mathbb{N}, x_l \neq 0\}$ and $x_j \geq \min\{y_j^{(l)} + z_j^{(l)} \mid l = 0 \text{ or } l \in \mathbb{N}, x_l \neq 0\}$ for all $j \in \mathbb{N}$, because then $x \in \{y^{(l)} + z^{(l)} \mid l = 0 \text{ or } l \in \mathbb{N}, x_l \neq 0\}_v$ by claim 2, hence $x \in (I_a + I_b)_t$. Let $j \in \mathbb{N}$. Clearly, $x_j \geq a_{2j} + b_{2j} = y_j^{(0)} + z_j^{(0)}$. Case 1: $x_j \neq 0$. We have $x_j + y_0^{(j)} + z_0^{(j)} - y_j^{(j)} - z_j^{(j)} = x_j + a_0 + b_0 - (a_0 - a_{2j-1}) - (b_0 - b_{2j-1}) = x_j + a_{2j-1} + b_{2j-1} \leq x_0$. Case 2: $x_j = 0$. It follows that $x_j + y_0^{(0)} + z_0^{(0)} - y_j^{(0)} - z_j^{(0)} = a_0 + b_0 - a_{2j} - b_{2j} \leq a_0 + b_0 \leq x_0$. "$\supseteq$": Obviously, $I_a + I_b \subseteq I_{a+b}$ and $a + b \in \mathfrak{I}$. Therefore, claim 3 implies that $(I_a + I_b)_t \subseteq I_{a+b}$.
Clearly, if $b \leq a$, then $I_a \subseteq I_b$. Now let $I_a \subseteq I_b$. Note that $y^{(i)} \in I_b$ for all $i \in \mathbb{N}_0$, hence $y_0^{(i)} \geq y_{j+1}^{(i)} + b_{2j+1}$ and $y_j^{(i)} \geq b_{2j}$ for all $i, j \in \mathbb{N}_0$. If $j \in \mathbb{N}_0$, then $y_0^{(j+1)} \geq y_{j+1}^{(j+1)} + b_{2j+1}$ and $y_j^{(0)} \geq b_{2j}$, hence $a_0 \geq a_0 - a_{2j+1} + b_{2j+1}$ and $a_{2j} \geq b_{2j}$. Consequently, $a_i \geq b_i$ for all $i \in \mathbb{N}_0$, and thus $b \leq a$.
Claim 5: $t\text{-spec}(H)^\bullet = \{I_{s^{(i)}} \mid i \in \mathbb{N}_0\}$ and $\mathfrak{X}(H) = \{I_{s^{(i)}} \mid i \in \mathbb{N}\}$. First we show that $t\text{-spec}(H)^\bullet = \{I_{s^{(i)}} \mid i \in \mathbb{N}_0\}$. "$\subseteq$": Let $P \in t\text{-spec}(H)^\bullet$. By claim 3 there is some $r \in \mathfrak{I}$ such that $P = I_r$. Case 1: $r_j = 0$ for all $j \in \mathbb{N}$. Since $P \neq H$, we have $r_0 \neq 0$. This implies that $r = ks^{(0)}$ for some $k \in \mathbb{N}$, and thus

$P=(kI_{s^{(0)}})_t$ by claim 4. Therefore, $P=I_{s^{(0)}}$. Case 2: $r_j\neq 0$ for some $j\in\mathbb{N}$. Let $a\in\mathbb{N}_0^{\mathbb{N}_0}$ be defined by $a_i=r_i$ if $i\in\mathbb{N}_0$, $i\neq j$ and $a_i=r_i-1$ otherwise. Then $a\in\mathfrak{I}$, $r\leq s^{(j)}+a$ and $r\nleq a$. Therefore, claim 4 implies that $(I_{s^{(j)}}+I_a)_t\subseteq P$ and $I_a\nsubseteq P$, hence $I_{s^{(j)}}\subseteq P$. Note that $s^{(j)}\leq r$, and thus $P=I_{s^{(j)}}$ by claim 4. "$\supseteq$": Observe that $I_{s^{(2i)}}=\{x\in H\mid x_i\geq 1\}$ and $I_{s^{(2i+1)}}=\{x\in H\mid x_0\geq x_{i+1}+1\}$ for all $i\in\mathbb{N}_0$. Using this and claim 3 it is straightforward to prove that $I_{s^{(i)}}\in t\text{-spec}(H)^\bullet$ for all $i\in\mathbb{N}_0$.

Next we show that $\mathfrak{X}(H)=\{I_{s^{(i)}}\mid i\in\mathbb{N}\}$. "$\subseteq$": Let $P\in\mathfrak{X}(H)$. Then $P\in t\text{-spec}(H)^\bullet$, hence $P=I_{s^{(i)}}$ for some $i\in\mathbb{N}_0$. Since $s^{(0)}<s^{(1)}$, it follows by claim 4 that $I_{s^{(1)}}\subsetneq I_{s^{(0)}}$, and thus $i\in\mathbb{N}$. "$\supseteq$": Let $i\in\mathbb{N}$ and $P\in t\text{-spec}(H)^\bullet$ be such that $P\subseteq I_{s^{(i)}}$. There is some $j\in\mathbb{N}_0$ such that $P=I_{s^{(j)}}$. It follows by claim 4 that $s^{(i)}\leq s^{(j)}$, hence $s^{(i)}=s^{(j)}$. Therefore, $P=I_{s^{(i)}}$, and thus $I_{s^{(i)}}\in\mathfrak{X}(H)$.

Claim 6: I_r is a radical t-ideal of H for all $r\in\mathfrak{I}$ such that $r_0=1$. Let $r\in\mathfrak{I}$ be such that $r_0=1$. By claim 4 and claim 5 we have $\sqrt{I_r}=\bigcap_{P\in t\text{-spec}(H)^\bullet,P\supseteq I_r}P=\bigcap_{i\in\mathbb{N}_0,s^{(i)}\leq r}I_{s^{(i)}}=\{x\in G\mid x_0\geq x_{j+1}+\max\{s^{(i)}_{2j+1}\mid i\in\mathbb{N}_0,s^{(i)}\leq r\},x_j\geq\max\{s^{(i)}_{2j}\mid i\in\mathbb{N}_0,s^{(i)}\leq r\}\text{ for all }j\in\mathbb{N}_0\}=\{x\in G\mid x_0\geq x_{j+1}+r_{2j+1},x_j\geq r_{2j}\text{ for all }j\in\mathbb{N}_0\}=I_r$.

1. By claim 3 and claim 5 we have $v\text{-spec}(H)^\bullet=t\text{-spec}(H)^\bullet\cap\mathcal{I}_v(H)^\bullet=\{I_{s^{(i)}}\mid i\in\mathbb{N}_0\}\cap\{I_r\mid r\in\mathfrak{L}\}=\{I_{s^{(i)}}\mid i\in\mathbb{N}\}=\mathfrak{X}(H)$ and $t\text{-spec}(H)^\bullet=\mathfrak{X}(H)\cup\{I_{s^{(0)}}\}=\mathfrak{X}(H)\cup\{H\setminus H^\times\}$. Let $A,B,C\in\mathcal{I}_t(H)^\bullet$ be such that $(A+B)_t=(A+C)_t$. By claim 3 there exist some $a,b,c\in\mathfrak{I}$ such that $A=I_a$, $B=I_b$ and $C=I_c$. It follows by claim 4 that $I_{a+b}=(A+B)_t=(A+C)_t=I_{a+c}$, and thus $a+b=a+c$ by claim 4. Consequently, $b=c$, hence $B=C$. Now let $k\in\mathbb{N}$ and $P\in t\text{-spec}(H)^\bullet$. By claim 4 we have $(kI_{s^{(2i)}})_t=I_{ks^{(2i)}}=\{x\in H\mid x_i\geq k\}$ and $(kI_{s^{(2i+1)}})_t=I_{ks^{(2i+1)}}=\{x\in H\mid x_0\geq x_{i+1}+k\}$ for all $i\in\mathbb{N}_0$. Using this it is straightforward to prove that $(kP)_t$ is P-primary.

2. It follows by 1 that $t\text{-dim}(H)=2$, since $H\setminus H^\times$ is not divisorial. Let $I\in\mathcal{I}_t(H)^\bullet$. By claim 3 there is some $r\in\mathfrak{I}$ such that $I=I_r$. For $i\in[1,r_0]$ and $j\in\mathbb{N}_0$ set $r^{(i)}_j=1$, if ((j is even and $r_j\geq i$) or (j is odd and $r_{j+1}<i\leq r_j+r_{j+1}$)) and set $r^{(i)}_j=0$ otherwise. Observe that $(r^{(i)})_{i=1}^{r_0}\in\{a\in\mathfrak{I}\mid a_0=1\}^{[1,r_0]}$ and $r=\sum_{i=1}^{r_0}r^{(i)}$. By claim 4 we have $I=(\sum_{i=1}^{r_0}I_{r^{(i)}})_t$. Moreover, $I_{r^{(i)}}$ is a radical t-ideal for all $i\in[1,r_0]$ by claim 6. Therefore, H is a t-SP-monoid. Set $F=\{x\in G\mid x_i\geq 0\text{ for all }i\in\mathbb{N}_0\}$. Obviously, F is a free abelian monoid and $H\subseteq F$ is a submonoid. Consequently, H is an FF-monoid.

Set $\mathcal{M}=\{\sqrt{y+H}\mid y\in H\}$ and let $I\in\mathcal{M}$. Then $I=\sqrt{x+H}$ for some $x\in H$. Set $m=|\{i\in\mathbb{N}_0\mid x_i>0\}|$ and $l=(m+1)^2$. Let $\mathcal{K}\subseteq\mathcal{M}$ be a chain such that $\min(\mathcal{K})=I$ (where $\min(\mathcal{K})$ denotes the smallest element of $\mathcal{K}$ with respect to inclusion). There is some sequence $(x^{(L)})_{L\in\mathcal{K}}\in H^{\mathcal{K}}$ such that $J=\sqrt{x^{(J)}+H}$ for all $J\in\mathcal{K}$. Let $f:\mathcal{K}\to[0,m]\times[0,m]$ be defined by $f(J)=(|\{i\in\mathbb{N}_0\mid x^{(J)}_i>0\}|,|\{i\in\mathbb{N}_0\mid x^{(J)}_0>x^{(J)}_i>0\}|)$. Using claim 1 and the fact that $\min(\mathcal{K})=I$ it is easy to prove that f is well-defined. We show that f is injective. Let $J,L\in\mathcal{K}$ be such that $f(J)=f(L)$. Without restriction let $J\subseteq L$. By claim 1 we have $\{i\in\mathbb{N}_0\mid x^{(L)}_i>0\}\subseteq\{i\in\mathbb{N}_0\mid x^{(J)}_i>0\}$ and $\{i\in\mathbb{N}_0\mid x^{(L)}_0>x^{(L)}_i>0\}\subseteq\{i\in\mathbb{N}_0\mid x^{(J)}_0>x^{(J)}_i>0\}$. Since $f(J)=f(L)$, this implies that $\{i\in\mathbb{N}_0\mid x^{(J)}_i>0\}=\{i\in\mathbb{N}_0\mid x^{(L)}_i>0\}$ and $\{i\in\mathbb{N}_0\mid x^{(J)}_0>x^{(J)}_i>0\}=\{i\in\mathbb{N}_0\mid x^{(L)}_0>x^{(L)}_i>0\}$. Consequently, $\{i\in\mathbb{N}_0\mid x^{(J)}_0>x^{(J)}_i\}=\{i\in\mathbb{N}_0\mid x^{(L)}_0>x^{(L)}_i\}$, and thus $J=\sqrt{x^{(J)}+H}=\sqrt{x^{(L)}+H}=L$ by claim 1. Since f is injective we have $|\mathcal{K}|\leq l$. Let $\lambda:\mathcal{M}\to\mathbb{N}_0$ be defined by $\lambda(J)=\max\{|\mathcal{K}|\mid\mathcal{K}\subseteq\mathcal{M}$ is a chain and $\min(\mathcal{K})=J\}$. Using the previous it is easy to prove that λ is a well-defined map and $\lambda(J)<\lambda(L)$ for all $J,L\in\mathcal{M}$ such that $L\subsetneq J$. Consequently, $\mathcal{M}$ possesses a length function.

Now let y be a radical element of H. There is some $k\in\mathbb{N}$ such that $y_k=0$. Set $x=2e_0+e_k+\sum_{i\in\mathbb{N},y_i>0}e_i$. It follows by claim 1 that $x\in\sqrt{y+H}=y+H$, hence $x_0-y_0\geq x_i-y_i\geq 0$ for all $i\in\mathbb{N}_0$. Therefore, $2-y_0=x_0-y_0\geq x_k-y_k=1$, and thus $y_0\leq 1$. Consequently, $y\in\mathcal{A}(H)\cup H^\times$.

Since $H\setminus H^\times$ is a t-ideal it follows that $\mathcal{C}_t(H)$ is trivial, and thus we have H is radical factorial by [29, Proposition 3.10.2]. Moreover, since $t\text{-dim}(H)=2$ we have H is not a Krull monoid. Therefore, H is not a Mori monoid by [29, Proposition 2.6]. It follows by [29, Proposition 3.9] and [29, Corollary 3.14] that H is not a t-Prüfer monoid. □

Note that if H is a discrete valuation monoid (i.e. an atomic valuation monoid H with $H^\bullet \neq H^\times$), then every radical element of H is either an atom or a unit. The last example also shares this property with discrete valuation monoids. An integral domain is called an almost Krull domain if all its localizations at prime ideals are Krull domains. The following question has been raised by Pirtle (see [27]): Is every almost Krull domain whose height-one prime ideals are divisorial already a Krull domain? Arnold and Matsuda answered Pirtle's question in the negative (see [3]). Note that our last example is of similar type, since it shows that a (radical factorial) t-SP-monoid whose height-one prime t-ideals are divisorial is not necessarily a Krull monoid. This also answers the questions raised after Proposition 2.6 in [29] in the negative. Finally, Example 4.2 shows that being a t-Prüfer monoid is not a monadic property and being "primary r-ideal inclusive" in [29, Corollary 5.3 and Theorem 5.4] cannot be omitted.

5. Connections with rings of integer-valued polynomials

In this section we investigate the connections between rings of integer-valued polynomials and monadically Krull monoids. In particular, we continue our search for examples of monadically Krull domains that are not Krull. As in section four, we will consider additively written monoids that do not posses an "additive" analogue of a "multiplicative" zero element.

Let R be an integral domain, K a field of quotients of R and X an indeterminate over K. If $a, b \in R$, then we write $a \simeq_R b$ if $b = ac$ for some $c \in R^\times$. Set $\mathrm{Int}(R) = \{f \in K[X] \mid f(c) \in R \text{ for all } c \in R\}$, called the ring of integer-valued polynomials over R. Observe that $R \subseteq R[X]$ and $R \subseteq \mathrm{Int}(R)$ are divisor-closed, $\mathrm{Int}(R)^\times = R[X]^\times = R^\times$ and $\mathcal{A}(\mathrm{Int}(R)) \cap R = \mathcal{A}(R[X]) \cap R = \mathcal{A}(R)$. Especially, if $R[X]$ is monadically Krull or $\mathrm{Int}(R)$ is monadically Krull, then R is monadically Krull.

Now let R be factorial and Q a system of representatives of prime elements of R. Recall that $R[X]$ is factorial. If $T \subseteq R$, then let $\mathrm{GCD}_R(T)$ be the set of all greatest common divisors of T (in R). If $g \in R[X]\setminus R$, then g is called primitive if $\mathrm{GCD}_{R[X]}(g, c) = R[X]^\times$ for all $c \in R^\bullet$ (equivalently: $\mathrm{GCD}_R(\{a_i \mid i \in [0, k]\}) = R^\times$ for all $k \in \mathbb{N}_0$ and $(a_i)_{i=0}^k \in R^{[0,k]}$ such that $g = \sum_{i=0}^k a_i X^i$). If $q \in Q$, then let $v_q : R \to \mathbb{N}_0 \cup \{\infty\}$ denote the q-adic valuation of R. Let $d_Q : \mathrm{Int}(R)^\bullet \to R^\bullet$ be defined by $d_Q(g) = \prod_{p \in Q} p^{\min\{v_p(g(c)) \mid c \in R\}}$ for all $g \in \mathrm{Int}(R)^\bullet$. Set $d = d_Q$. Note that $d(g) \in \mathrm{GCD}_R(\{g(c) \mid c \in R\})$ and $\frac{g}{d(g)} \in \mathrm{Int}(R)$ for all $g \in \mathrm{Int}(R)^\bullet$.

If M is a set and $l \in \mathbb{N}$, then a finite sequence $(a_i)_{i=1}^l \in M^l$, will be denoted by $\underline{a}$ (i.e. $\underline{a} = (a_i)_{i=1}^l$). Let $n \in \mathbb{N}$, $\underline{f} \in (\mathrm{Int}(R)^\bullet)^n$ and $\underline{x} \in \mathbb{N}_0^n\setminus\{\underline{0}\}$. Then $\underline{x}$ is called $\underline{f}$-irreducible if for all $\underline{y}, \underline{z} \in \mathbb{N}_0^n$ such that $\underline{x} = \underline{y} + \underline{z}$ and $d(\prod_{i=1}^n f_i^{x_i}) = d(\prod_{i=1}^n f_i^{y_i}) d(\prod_{i=1}^n f_i^{z_i})$ it follows that $\underline{y} = \underline{0}$ or $\underline{z} = \underline{0}$ (this definition does not depend on the choice of Q). In the next Lemma we will use [15, Definition 1.5.2] and Dickson's Theorem (see [15, Theorem 1.5.3]) without further citation.

Lemma 5.1. *Let R be a factorial domain, $n \in \mathbb{N}$ and $\underline{f} \in (\mathrm{Int}(R)^\bullet)^n$. Then $\{\underline{x} \in \mathbb{N}_0^n \mid \underline{x}$ is $\underline{f}$-irreducible$\}$ is finite.*

Proof. Let Q be a system of representatives of prime elements of R and $T = \{w \in R \mid (\prod_{i=1}^n f_i)(w) \neq 0\}$. We prove that $\min\{v_q(g(w)) \mid w \in R\} = \min\{v_q(g(w)) \mid w \in T\}$ for all $q \in Q$ and $g \in \mathrm{Int}(R)^\bullet$. Let $q \in Q$ and $g \in \mathrm{Int}(R)^\bullet$. Then $\min\{v_q(g(w)) \mid w \in R\} = v_q(g(v))$ for some $v \in R$. Observe that there is some $k \in \mathbb{N}$ such that $v_q(g(v)) = v_q(g(v + q^l))$ for all $l \in \mathbb{N}_{\geq k}$. Since $R\setminus T$ is finite, there is some $m \in \mathbb{N}_{\geq k}$ such that $v + q^m \in T$. We have $\min\{v_q(g(w)) \mid w \in R\} = v_q(g(v + q^m))$, and thus $\min\{v_q(g(w)) \mid w \in R\} = \min\{v_q(g(w)) \mid w \in T\}$.
Set $P = \{p \in Q \mid \min\{v_p((\prod_{i=1}^n f_i)(w)) \mid w \in R\} > 0\}$. Clearly, P is finite. If $p \in P$, then there is some finite $S_p \subseteq T$ such that $\mathrm{Min}(\{(v_p(f_i(w)))_{i=1}^n \mid w \in T\}) = \{(v_p(f_i(w)))_{i=1}^n \mid w \in S_p\}$. Set $S = \bigcup_{p \in P} S_p$. Then S is finite. For $\gamma \in S^P$ set $\Omega_\gamma = \{\underline{u} \in \mathbb{N}_0^n \mid \sum_{i=1}^n (v_p(f_i(w)) - v_p(f_i(\gamma(p))))u_i \geq 0$ for all $p \in P$ and $w \in S\}$. If $\gamma \in S^P$, then Ω_γ is an additive monoid and by [15, Theorem 2.7.14] and [15, Proposition 1.1.7.2] we have $\mathcal{A}(\Omega_\gamma)$ is finite. It suffices to show that $\{\underline{x} \in \mathbb{N}_0^n \mid \underline{x}$ is $\underline{f}$-irreducible$\} \subseteq \bigcup_{\gamma \in S^P} \mathcal{A}(\Omega_\gamma)$. Let $\underline{x} \in \mathbb{N}_0^n$ be $\underline{f}$-irreducible. There is some $\delta \in S^P$ such that $\min\{\sum_{i=1}^n v_p(f_i(w))x_i \mid w \in S\} = \sum_{i=1}^n v_p(f_i(\delta(p)))x_i$

for all $p \in P$, hence $\underline{x} \in \Omega_\delta\backslash\{\underline{0}\}$. Let $\underline{u} \in \Omega_\delta$. If $p \in P$, then $\min\{v_p((\prod_{i=1}^n f_i^{u_i})(w)) \mid w \in R\} = \min\{\sum_{i=1}^n v_p(f_i(w))u_i \mid w \in T\} = \min\{\sum_{i=1}^n v_p(f_i(w))u_i \mid w \in S\} = \sum_{i=1}^n v_p(f_i(\delta(p)))u_i$, and if $p \in Q\backslash P$, then $\min\{v_p((\prod_{i=1}^n f_i^{u_i})(w)) \mid w \in R\} = 0$. Let $\underline{y}, \underline{z} \in \Omega_\delta$ be such that $\underline{x} = \underline{y} + \underline{z}$. If $p \in P$, then $\min\{v_p((\prod_{i=1}^n f_i^{x_i})(w)) \mid w \in R\} = \sum_{i=1}^n v_p(f_i(\delta(p)))x_i = \sum_{i=1}^n v_p(f_i(\delta(p)))y_i + \sum_{i=1}^n v_p(f_i(\delta(p)))z_i = \min\{v_p((\prod_{i=1}^n f_i^{y_i})(w)) \mid w \in R\} + \min\{v_p((\prod_{i=1}^n f_i^{z_i})(w)) \mid w \in R\}$. This implies that $d_Q(\prod_{i=1}^n f_i^{x_i}) = d_Q(\prod_{i=1}^n f_i^{y_i})d_Q(\prod_{i=1}^n f_i^{z_i})$, hence $\underline{y} = \underline{0}$ or $\underline{z} = \underline{0}$. Therefore, $\underline{x} \in \mathcal{A}(\Omega_\delta)$. □

Now we present the main result of this section.

Theorem 5.2. *Let R be a factorial domain. Then* $\mathrm{Int}(R)$ *is monadically Krull.*

Proof. Let K be a field of quotients of R, X an indeterminate over K, Q a system of representatives of prime elements of R and $d = d_Q$. Set $S = R[X]$ and $T = \mathrm{Int}(R)$. It is well known that T is atomic and completely integrally closed (see [7, Propositions VI.2.1 and VI.2.9]). By Theorem 3.6 we need to prove that $\{yT \mid y \in \mathcal{A}([\![g]\!]_T)\}$ is finite for all $g \in T^\bullet$. Let $g \in T^\bullet$. Some $a, b \in R^\bullet$, $n \in \mathbb{N}$, $\underline{v} \in \mathbb{N}_0^n$ and $\underline{f} \in (\mathcal{A}(S)\backslash R)^n$ exist such that $g = \frac{a}{b}\prod_{i=1}^n f_i^{v_i}$ and $f_j \not\simeq_S f_k$ for all different $j, k \in [1, n]$. By Lemma 5.1 it is sufficient to show that $\{yT \mid y \in \mathcal{A}([\![g]\!]_T)\} \subseteq \{yT \mid y \in \mathcal{A}(R), y \,|_R\, d(g)\} \cup \{\frac{\prod_{i=1}^n f_i^{\alpha_i}}{d(\prod_{i=1}^n f_i^{\alpha_i})}T \mid \underline{\alpha} \in \mathbb{N}_0^n, \underline{\alpha}$ is $\underline{f}$-irreducible$\}$. Let $y \in \mathcal{A}([\![g]\!]_T)$. Then $y \in \mathcal{A}(T)$ and $y \,|_T\, g^l$ for some $l \in \mathbb{N}$.
Case 1: $y \in R$. We have $y \in \mathcal{A}(R)$ and $y \,|_R\, d(g^l) = d(g)^l$. Therefore, $y \,|_R\, d(g)$.
Case 2: $y \notin R$. Some primitive $t \in S$ and some $c, e \in R^\bullet$ exist such that $\mathrm{GCD}_S(c, et) = S^\times$ and $y = \frac{et}{c}$. This implies that $c \,|_R\, d(t)$. Observe that $y = \frac{ed(t)}{c} \cdot \frac{t}{d(t)}$, $\frac{ed(t)}{c} \in T$ and $\frac{t}{d(t)} \in T\backslash T^\times$. Consequently, $y \simeq_T \frac{t}{d(t)}$. There are some $w \in S$ and $u \in R^\bullet$ such that $y\frac{w}{u} = g^l$. Therefore, $etwb^l = cua^l\prod_{i=1}^n f_i^{lv_i}$ and since t is primitive it follows that $t \,|_S\, \prod_{i=1}^n f_i^{lv_i}$. Hence, there is some $\underline{\alpha} \in \mathbb{N}_0^n\backslash\{\underline{0}\}$ such that $t \simeq_S \prod_{i=1}^n f_i^{\alpha_i}$. This implies that $y \simeq_T \frac{\prod_{i=1}^n f_i^{\alpha_i}}{d(\prod_{i=1}^n f_i^{\alpha_i})}$, and thus $yT = \frac{\prod_{i=1}^n f_i^{\alpha_i}}{d(\prod_{i=1}^n f_i^{\alpha_i})}T$. Let $\underline{\beta}, \underline{\gamma} \in \mathbb{N}_0^n$ be such that $\underline{\alpha} = \underline{\beta} + \underline{\gamma}$ and $d(\prod_{i=1}^n f_i^{\alpha_i}) = d(\prod_{i=1}^n f_i^{\beta_i})d(\prod_{i=1}^n f_i^{\gamma_i})$. Note that $\frac{\prod_{i=1}^n f_i^{\beta_i}}{d(\prod_{i=1}^n f_i^{\beta_i})}, \frac{\prod_{i=1}^n f_i^{\gamma_i}}{d(\prod_{i=1}^n f_i^{\gamma_i})} \in T$ and $\frac{\prod_{i=1}^n f_i^{\beta_i}}{d(\prod_{i=1}^n f_i^{\beta_i})} \cdot \frac{\prod_{i=1}^n f_i^{\gamma_i}}{d(\prod_{i=1}^n f_i^{\gamma_i})} = \frac{\prod_{i=1}^n f_i^{\alpha_i}}{d(\prod_{i=1}^n f_i^{\alpha_i})} \in \mathcal{A}(T)$. Therefore, $\frac{\prod_{i=1}^n f_i^{\beta_i}}{d(\prod_{i=1}^n f_i^{\beta_i})} \in T^\times$ or $\frac{\prod_{i=1}^n f_i^{\gamma_i}}{d(\prod_{i=1}^n f_i^{\gamma_i})} \in T^\times$, hence $\underline{\beta} = \underline{0}$ or $\underline{\gamma} = \underline{0}$. Consequently, $\underline{\alpha}$ is $\underline{f}$-irreducible. □

Theorem 5.2 is also interesting from a purely factorization theoretical point of view, since it provides a class of Krull monoids whose arithmetic is not fully understood by now. The arithmetic of the Krull monoids involved may also differ from the arithmetic of monadic submonoids of principal orders in algebraic number fields.

Corollary 5.3. *Let R be a factorial domain. Then* $\mathrm{Int}(R)$ *is an FF-domain.*

Proof. This follows from Theorems 3.6 and 5.2. □

In [13] it has been shown that $\mathrm{Int}(\mathbb{Z})$ is an FF-domain. Corollary 5.3 is a generalization of this result. By Theorem 5.2, [7, Theorem VI.1.7] and [7, Remark VI.2.10] we obtain that $\mathrm{Int}(\mathbb{Z})$ and $\mathrm{Int}(\mathbb{Z}_{(p)})$ for $p \in \mathbb{P}$ are monadically Krull domains and Prüfer domains (and thus t-Prüfer domains) that are no Krull domains.

6. Further remarks

In section four we showed that a radical factorial FF-monoid is not necessarily a Krull monoid. So far we do not know whether every radical factorial, 1-dimensional FF-domain is a Krull domain. In this last section we investigate special types of examples that have been introduced in [19] to construct atomic Prüfer domains that are no Dedekind domains. We study these examples in greater detail and generality to obtain an interesting class of examples that are not "too far away" from being examples of radical factorial 1-dimensional FF-domains that are not Krull.

Let H be a monoid. We say that H is a weakly Krull monoid if $\bigcap_{P\in\mathfrak{X}(H)} H_P = H$ and $\{P \in \mathfrak{X}(H) \mid a \in P\}$ is finite for all $a \in H^\bullet$. Note that H is a Krull monoid if and only if H is a weakly Krull monoid and H_P is a Krull monoid for all $P \in \mathfrak{X}(H)$. It follows from Example 4.2 that being a weakly Krull monoid is not a monadic property (since the monoid in this example is monadically Krull, hence monadically weakly Krull, but by [29, Proposition 2.6] it fails to be weakly Krull). By [2, Theorem 1] and [1, Theorem 5.1] we have H is an FF-monoid if and only if H is atomic and $\{uH \mid u \in \mathcal{A}(H), u\,|_H\, x\}$ is finite for all $x \in H^\bullet$ (such monoids are called IDF-monoids, for example see [23]) if and only if H is a BF-monoid and $\{uH \mid u \in \mathcal{A}(H), u\,|_H\, x\}$ is finite for all $x \in H^\bullet$. Clearly, if H is a BF-monoid, then H satisfies the ACCP. First we start with a simple Lemma that might be of independent interest. It gives a hint how to construct monoids H where $\{uH \mid u \in \mathcal{A}(H), u\,|_H\, x\}$ is finite for all $x \in H^\bullet$, but that fail to be FF-monoids.

Lemma 6.1. *Let K be a monoid, S a submonoid of K that is an FF-monoid, $T \subseteq K$ a submonoid of K that is a valuation monoid and $H = S \cap T$. Then $\{uH \mid u \in \mathcal{A}(H), u\,|_H\, x\}$ is finite for all $x \in H^\bullet$.*

Proof. Let $x \in H^\bullet$ and $\mathcal{D}(x) = \{u \in \mathcal{A}(H) \mid u\,|_H\, x\}$. Then $\{uS \mid u \in \mathcal{D}(x)\} \subseteq \{uS \mid u \in S$ and $u\,|_S\, x\}$. Since S is an FF-monoid, it follows that $\{uS \mid u \in \mathcal{D}(x)\}$ is finite. Let $v, w \in \mathcal{A}(H)$ be such that $vS = wS$. We have $vT \subseteq wT$ or $wT \subseteq vT$. Therefore, $vH = vS \cap vT \subseteq wS \cap wT = wH$ or $wH = wS \cap wT \subseteq vS \cap vT = vH$, hence $vH = wH$. Consequently, $f : \{uH \mid u \in \mathcal{D}(x)\} \to \{uS \mid u \in \mathcal{D}(x)\}$ defined by $f(I) = IS$ for all $I \in \{uH \mid u \in \mathcal{D}(x)\}$ is an injective map. This implies that $\{uH \mid u \in \mathcal{D}(x)\}$ is finite. □

Proposition 6.2. *Let H be a monoid, K a quotient monoid of H, $\mathcal{U}$ a set of overmonoids of H that are FF-monoids and $\mathcal{V}$ a set of overmonoids of H that are valuation monoids such that $H = \bigcap_{S\in\mathcal{U}\cup\mathcal{V}} S$. Let $(\mathcal{N}_T)_{T\in\mathcal{V}} \in \mathbb{P}(\mathcal{U})^{\mathcal{V}}$ be such that $T \cap \bigcap_{S\in\mathcal{N}_T} S$ is atomic for all $T \in \mathcal{V}$ and $\{T \in \mathcal{V} \mid S \in \mathcal{N}_T\}$ is finite for all $S \in \mathcal{U}$. If $\{S \in \mathcal{U} \cup \mathcal{V} \mid a \notin S^\times\}$ is finite for all $a \in H^\bullet$, then H is an FF-monoid.*

Proof. Let $\{S \in \mathcal{U} \cup \mathcal{V} \mid a \notin S^\times\}$ be finite for all $a \in H^\bullet$ and $\mathcal{M} = \mathcal{U} \cup \{T \cap \bigcap_{S\in\mathcal{N}_T} S \mid T \in \mathcal{V}\}$.
Claim 1: For all $U \in \mathcal{M}$ it follows that U is an FF-monoid. Let $U \in \mathcal{M}$ and $T \in \mathcal{V}$ be such that $U = T \cap \bigcap_{S\in\mathcal{N}_T} S$. We show that $\bigcap_{S\in\mathcal{N}_T} S$ is an FF-monoid. If $\mathcal{N}_T = \emptyset$, then $\bigcap_{S\in\mathcal{N}_T} S = K$, hence $\bigcap_{S\in\mathcal{N}_T} S$ is an FF-monoid. Since $\{S \in \mathcal{N}_T \mid a \notin S^\times\}$ is finite for all $a \in H^\bullet$, we have $\{S \in \mathcal{N}_T \mid a \notin S^\times\}$ is finite for all $a \in K^\bullet$, hence $\{S \in \mathcal{N}_T \mid a \notin S^\times\}$ is finite for all $a \in (\bigcap_{S\in\mathcal{N}_T} S)^\bullet$. Therefore, [2, Theorem 2] implies that $\bigcap_{S\in\mathcal{N}_T} S$ is an FF-monoid. It follows by Lemma 6.1 that U is an FF-monoid.
Claim 2: For every $a \in H^\bullet$, $\{S \in \mathcal{M} \mid a \notin S^\times\}$ is finite. Let $a \in H^\bullet$. We have $\{T \in \mathcal{V} \mid a \notin (T \cap \bigcap_{S\in\mathcal{N}_T} S)^\times\} \subseteq \{T \in \mathcal{V} \mid a \notin T^\times\} \cup \bigcup_{S\in\mathcal{U}, a\notin S^\times} \{T \in \mathcal{V} \mid S \in \mathcal{N}_T\}$, and thus $\{T \in \mathcal{V} \mid a \notin (T \cap \bigcap_{S\in\mathcal{N}_T} S)^\times\}$ is finite. Therefore, $\{S \in \mathcal{M} \mid a \notin S^\times\} = \{S \in \mathcal{U} \mid a \notin S^\times\} \cup \{T \cap \bigcap_{S\in\mathcal{N}_T} S \mid T \in \mathcal{V}, a \notin (T \cap \bigcap_{S\in\mathcal{N}_T} S)^\times\}$ is finite.
Since $H = \bigcap_{S\in\mathcal{M}} S$, it follows by claim 1, claim 2 and [2, Theorem 2] that H is an FF-monoid. □

Proposition 6.3. *Let K be a monoid, H a submonoid of K and Λ a set of intermediate monoids of H and K such that $\{S \in \Lambda \mid a \notin S^\times\}$ is finite for all $a \in H^\bullet$ and $H \cap \bigcap_{S\in\Lambda} S^\times = H^\times$.*

1. *If S satisfies the ACCP for all $S \in \Lambda$, then H satisfies the ACCP.*
2. *If S is a BF-monoid for all $S \in \Lambda$, then H is a BF-monoid.*

Proof. **1.** Let S satisfy the ACCP for all $S \in \Lambda$. Let $(a_i)_{i\in\mathbb{N}} \in H^{\mathbb{N}}$ be such that $a_iH \subseteq a_{i+1}H$ for all $i \in \mathbb{N}$. Without restriction let $a_1 \neq 0$. Let $\mathcal{A} = \{S \in \Lambda \mid a_1 \notin S^\times\}$. Since $\mathcal{A}$ is finite there is some $r \in \mathbb{N}$ such that $a_kS = a_rS$ for all $S \in \mathcal{A}$ and $k \in \mathbb{N}_{\geq r}$. It is sufficient to show that $a_kH = a_rH$ for all $k \in \mathbb{N}_{\geq r}$. Let $k \in \mathbb{N}_{\geq r}$ and $T \in \Lambda$. If $a_1 \in T^\times$, then $a_r, a_k \in T^\times$, and thus $\frac{a_r}{a_k} \in T^\times$. If $a_1 \notin T^\times$, then $a_rT = a_kT$, hence $\frac{a_r}{a_k} \in T^\times$. Consequently, $\frac{a_r}{a_k} \in H \cap \bigcap_{S\in\Lambda} S^\times = H^\times$, and thus $a_rH = a_kH$.
2. Let S be a BF-monoid for all $S \in \Lambda$ and set $\mathcal{M} = \{(S\setminus S^\times) \cap H \mid S \in \Lambda\}$. It follows by [15, Proposition 1.3.2] that $\bigcap_{n\in\mathbb{N}} (S\setminus S^\times)^n = \{0\}$ for all $S \in \Lambda$. Therefore, $\bigcap_{n\in\mathbb{N}} M^n = \{0\}$ for all $M \in \mathcal{M}$. Let $a \in H^\bullet\setminus H^\times$. Then $\{M \in \mathcal{M} \mid a \in M\} \subseteq \{(S\setminus S^\times) \cap H \mid S \in \Lambda, a \notin S^\times\}$, hence $\{M \in \mathcal{M} \mid a \in M\}$ is

finite. Since $a \notin H^\times$, there is some $T \in \Lambda$ such that $a \notin T^\times$, hence $(T \setminus T^\times) \cap H \in \{M \in \mathcal{M} \mid a \in M\}$. Consequently, [15, Theorem 1.3.4] implies that H is a BF-monoid. □

Corollary 6.4. *Let H be a monoid and $\mathcal{M} \subseteq s\text{-spec}(H)$ such that $\bigcup_{M \in \mathcal{M}} M = H \setminus H^\times$ and $\{M \in \mathcal{M} \mid a \in M\}$ is finite for all $a \in H^\bullet$.*

1. *If H_M satisfies the ACCP for all $M \in \mathcal{M}$, then H satisfies the ACCP.*
2. *If H_M is a BF-monoid for all $M \in \mathcal{M}$, then H is a BF-monoid.*

Proof. Let $\Lambda = \{H_M \mid M \in \mathcal{M}\}$. We have $H \cap \bigcap_{S \in \Lambda} S^\times = H \cap \bigcap_{M \in \mathcal{M}} (H_M \setminus M_M) = \bigcap_{M \in \mathcal{M}} (H \setminus M) = H \setminus \bigcup_{M \in \mathcal{M}} M = H^\times$. Let $a \in H^\bullet \setminus H^\times$. Then $\{S \in \Lambda \mid a \notin S^\times\} = \{H_M \mid M \in \mathcal{M}, a \in M\}$ is finite. Consequently, the assertions follow from Proposition 6.3. □

If S is an integral domain and $R \subseteq S$ is a subring, then let $cl_S(R)$ denote the integral closure of R in S. We say that $M \in \max(S)$ is critical if for each finite $E \subseteq M$, there exists $Q \in \max(S)$ such that $E \subseteq Q^2$.

Proposition 6.5. *Let R be a Dedekind domain that is not a field, K a field of quotients of R, L/K a field extension, $S = cl_L(R)$, $(L_i)_{i \in \mathbb{N}}$ a sequence of intermediate fields of K and L such that $L_i \subseteq L_{i+1}$ and $[L_i : K] < \infty$ for all $i \in \mathbb{N}$ and $L = \bigcup_{j \in \mathbb{N}} L_j$. Let $(\mathcal{A}_i)_{i \in \mathbb{N}} \in \mathbb{P}(\max(R))^{\mathbb{N}}$ be such that $\mathcal{A}_i \subseteq \mathcal{A}_{i+1}$, for all $i \in \mathbb{N}$. Set $\mathcal{A} = \bigcup_{i \in \mathbb{N}} \mathcal{A}_i$ and $\mathcal{N} = \{M \in \max(S) \mid M \cap R \in \mathcal{A}\}$. For $i \in \mathbb{N}$ set $S_i = cl_{L_i}(R)$ and $\mathcal{N}_i = \{M \in \max(S) \mid M \cap R \in \mathcal{A}_i\}$.*

1. *S_i is a Dedekind domain for all $i \in \mathbb{N}$, S is a 1-dimensional Prüfer domain and $S = \bigcup_{i \in \mathbb{N}} S_i$.*
2. *Let for all $i \in \mathbb{N}$ and $P \in \max(S_i)$ such that $P \cap R \in \mathcal{A}_i$ be $P \nsubseteq Q^2$ for all $Q \in \max(S_{i+1})$. Then for all $M \in \mathcal{N}$ we have M is not critical and if $\mathcal{A} = \max(R)$, then S is an SP-domain.*
3. *Let for all $i \in \mathbb{N}$ and $P \in \max(S_i)$ such that $P \cap R \in \mathcal{A}_i$ be $\sqrt[S_{i+1}]{PS_{i+1}} \in \max(S_{i+1})$. Then for all $a \in S^\bullet$ it follows that $\{M \in \mathcal{N} \mid a \in M\}$ is finite and if $\mathcal{A} = \max(R)$, then S is weakly Krull.*
4. *Let $\bigcup_{P \in \mathcal{A}} P = R \setminus R^\times$ and let for all $i \in \mathbb{N}$ and $P \in \max(S_i)$ such that $P \cap R \in \mathcal{A}_i$ be $PS_{i+1} \in \max(S_{i+1})$. Then S is a BF-domain and if $|\max(S) \setminus \mathcal{N}| \leq 1$, then S is an FF-domain.*
5. *If there is some sequence $(M_i)_{i \in \mathbb{N}}$ such that $M_i \in \max(S_i)$ and $M_{i+1} \cap S_i = M_i$ for all $i \in \mathbb{N}$ and $\{j \in \mathbb{N} \mid M_j S_{j+1} \notin \max(S_{j+1})\}$ is infinite, then S is not a Dedekind domain.*

Proof. **1.** Clearly, S is 1-dimensional and $S = \bigcup_{i \in \mathbb{N}} S_i$. By the Theorem of Krull-Akizuki we have S_i is a Dedekind domain for all $i \in \mathbb{N}$. Since L/K is an algebraic field extension we have S is a Prüfer domain.
2. Claim 1: For all $j \in \mathbb{N}$ and $M \in \mathcal{N}_j$ it follows that $M \cap S_j \nsubseteq (M \cap S_k)^2$ for all $k \in \mathbb{N}_{\geq j}$. Let $j \in \mathbb{N}$ and $M \in \mathcal{N}_j$. We show by induction on k that $M \cap S_j \nsubseteq (M \cap S_k)^2$ for all $k \in \mathbb{N}_{\geq j}$. Obviously, $M \cap S_j \nsubseteq (M \cap S_j)^2$. Now let $k \in \mathbb{N}_{\geq j}$ be such that $M \cap S_j \nsubseteq (M \cap S_k)^2$. Since $(M \cap S_j)S_k \subseteq M \cap S_k$, there is some ideal I of S_k such that $(M \cap S_j)S_k = (M \cap S_k)I$. Since $M \cap S_j \nsubseteq (M \cap S_k)^2$, it follows that $I \nsubseteq M \cap S_k$, hence $IS_{k+1} \nsubseteq M \cap S_{k+1}$. We have $M \cap S_k \in \max(S_k)$, $M \cap S_{k+1} \in \max(S_{k+1})$ and $(M \cap S_k) \cap R = M \cap R \in \mathcal{A}_j \subseteq \mathcal{A}_k$, and thus $(M \cap S_k)S_{k+1} \nsubseteq (M \cap S_{k+1})^2$. Since $(M \cap S_{k+1})^2$ is $M \cap S_{k+1}$-primary it follows that $(M \cap S_j)S_{k+1} = (M \cap S_k)S_{k+1}IS_{k+1} \nsubseteq (M \cap S_{k+1})^2$, hence $M \cap S_j \nsubseteq (M \cap S_{k+1})^2$.
Claim 2: For every $M \in \max(S)$ we have $M^2 = \bigcup_{i \in \mathbb{N}} (M \cap S_i)^2$. Let $M \in \max(S)$. "$\subseteq$": Let $x \in M^2$. There exist some $r \in \mathbb{N}$ and $(x_i)_{i=1}^r, (y_i)_{i=1}^r \in M^{[1,r]}$ such that $x = \sum_{i=1}^r x_i y_i$. There is some $l \in \mathbb{N}$ such that $x_i, y_i \in S_l$ for all $i \in [1, r]$. Consequently, $x \in (M \cap S_l)^2$. "$\supseteq$": Trivial.
Now let $Q \in \mathcal{N}$. There is some $j \in \mathbb{N}$ such that $Q \in \mathcal{N}_j$. Assume that there is some $M \in \max(S)$ such that $Q \cap S_j \subseteq M^2$. Then $Q \cap S_j = M \cap S_j$ and $M \cap R = Q \cap R \in \mathcal{A}_j$, hence $M \in \mathcal{N}_j$. It follows by claim 2 that there exists some $k \in \mathbb{N}_{\geq j}$ such that $M \cap S_j \subseteq (M \cap S_k)^2$ which is a contradiction to claim 1. Consequently, $(Q \cap S_j)S \nsubseteq M^2$ for all $M \in \max(S)$. Since $(Q \cap S_j)S$ is a finitely generated ideal of S we have Q is not critical.
Now let $\mathcal{A} = \max(R)$. Then $\mathcal{N} = \max(S)$, hence every $M \in \max(S)$ is not critical. It follows by 1 that S is a 1-dimensional Prüfer domain. Consequently, S is an SP-domain by [26, Corollary 2.2].
3. Claim: For all $i \in \mathbb{N}$ we have $f_i : \mathcal{N}_i \to \{M \cap S_i \mid M \in \mathcal{N}_i\}$ defined by $f_i(M) = M \cap S_i$ is a bijective map. Let $i \in \mathbb{N}$. Obviously, f_i is a surjective map. Let $M, Q \in \mathcal{N}_i$ be such that $M \cap S_i = Q \cap S_i$. We

show by induction on j that for all $j \in \mathbb{N}_{\geq i}$, $M \cap S_j = Q \cap S_j$. Let $j \in \mathbb{N}_{\geq i}$. The assertion holds for $j = i$. Now let $M \cap S_j = Q \cap S_j$. We have $(M \cap S_j)S_{j+1} \subseteq M \cap S_{j+1}$, hence $\sqrt[S_{j+1}]{(M \cap S_j)S_{j+1}} = M \cap S_{j+1}$. Analogously $\sqrt[S_{j+1}]{(Q \cap S_j)S_{j+1}} = Q \cap S_{j+1}$, hence $M \cap S_{j+1} = Q \cap S_{j+1}$. Finally, it follows that $M = \bigcup_{j \in \mathbb{N}_{\geq i}} (M \cap S_j) = \bigcup_{j \in \mathbb{N}_{\geq i}} (Q \cap S_j) = Q$.
Let $a \in S^\bullet$. There is some $l \in \mathbb{N}$ such that $a \in S_l$. Obviously, there is some surjective map from $\{M \cap S_l \mid M \in \mathcal{N}, a \in M\}$ to $\{M \cap R \mid M \in \mathcal{N}, a \in M\}$. Since $\{M \cap S_l \mid M \in \mathcal{N}, a \in M\} \subseteq \{Q \in \max(S_l) \mid a \in Q\}$ and $\{Q \in \max(S_l) \mid a \in Q\}$ is finite we have $\{M \cap R \mid M \in \mathcal{N}, a \in M\}$ is finite. Therefore, there is some $k \in \mathbb{N}_{\geq l}$ such that $\{M \cap R \mid M \in \mathcal{N}, a \in M\} \subseteq \mathcal{A}_k$. Since $f_k(\{M \in \mathcal{N}_k \mid a \in M\}) = \{M \cap S_k \mid M \in \mathcal{N}_k, a \in M\} \subseteq \{Q \in \max(S_k) \mid a \in Q\}$, it follows by the claim that $\{M \in \mathcal{N} \mid a \in M\} = \{M \in \mathcal{N}_k \mid a \in M\}$ is finite.
Now let $\mathcal{A} = \max(R)$. Then $\mathcal{N} = \max(S) = \mathfrak{X}(S)$ by 1, and thus S is a weakly Krull domain.
4. It follows by 3 that $\{M \in \mathcal{N} \mid a \in S\}$ is finite for all $a \in S^\bullet$. By [18, Proposition 4] we have $\bigcup_{M \in \mathcal{N}} M = S \setminus S^\times$. Let $M \in \mathcal{N}$. By 2 it follows that M is not critical, hence $M \neq M^2$. Since M^2 is M-primary we have $M_M^2 \neq M_M$. Therefore, 1 implies that S_M is a valuation domain, and thus M_M is a principal ideal of S_M. This implies that S_M is a Dedekind domain, hence S_M is an FF-domain and a BF-domain. Consequently, Corollary 6.4.2 implies that S is a BF-domain. Now let $|\max(S) \setminus \mathcal{N}| \leq 1$. Set $\mathcal{U} = \{S_M \mid M \in \mathcal{N}\}$ and $\mathcal{V} = \{S_M \mid M \in \max(S) \setminus \mathcal{N}\}$. Every $T \in \mathcal{U}$ is an FF-domain and by 1 we have that every $T \in \mathcal{V}$ is a valuation domain. Obviously, $\mathcal{U} \cup \mathcal{V} = \{S_M \mid M \in \max(S)\}$, hence $\bigcap_{T \in \mathcal{U} \cup \mathcal{V}} T = S$. For $T \in \mathcal{V}$ set $\mathcal{N}_T = \mathcal{U}$. Since $|\mathcal{V}| \leq 1$, we have $T \cap \bigcap_{U \in \mathcal{N}_T} U = S$ is atomic for all $T \in \mathcal{V}$. It follows that $\{T \in \mathcal{U} \cup \mathcal{V} \mid a \notin T^\times\}$ is finite for all $a \in S^\bullet$, and thus Proposition 6.2 implies that S is an FF-domain.
5. Let $(M_i)_{i \in \mathbb{N}}$ be such that $M_i \in \max(S_i)$ and $M_{i+1} \cap S_i = M_i$ for all $i \in \mathbb{N}$ and $\{j \in \mathbb{N} \mid M_j S_{j+1} \notin \max(S_{j+1})\}$ is infinite. Let $M = \bigcup_{i \in \mathbb{N}} M_i$. Observe that $M \in \max(S)$. Assume that S is a Dedekind domain, then there is some finite $E \subseteq M$ such that $M = (E)_S$. There is some $i \in \mathbb{N}$ such that $E \subseteq M_i$. There is some $j \in \mathbb{N}_{\geq i}$ such that $M_j S_{j+1} \notin \max(S_{j+1})$, and thus there are some $Q, Q' \in \max(S_{j+1})$ such that $M_j S_{j+1} \subseteq QQ'$. This implies that $M = QS = Q'S$ and $M^2 = QSQ'S = M$, hence $M = \{0\}$, a contradiction. □

Proposition 6.6. *Let R be a Dedekind domain such that $\max(R)$ is infinite, K a field of quotients of R, $(\mathcal{A}_i)_{i \in \mathbb{N}}, (\mathcal{B}_i)_{i \in \mathbb{N}}, (\mathcal{C}_i)_{i \in \mathbb{N}} \in \mathbb{P}(\max(R))^{\mathbb{N}}$ such that $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$ are finite and $\mathcal{A}_i \subseteq \mathcal{A}_{i+1}$, $\mathcal{B}_i \subseteq \mathcal{B}_{i+1}$, $\mathcal{C}_i \subseteq \mathcal{C}_{i+1}$ for all $i \in \mathbb{N}$. Set $\mathcal{A} = \bigcup_{i \in \mathbb{N}} \mathcal{A}_i$, $\mathcal{B} = \bigcup_{i \in \mathbb{N}} \mathcal{B}_i$ and $\mathcal{C} = \bigcup_{i \in \mathbb{N}} \mathcal{C}_i$. Assume that $\mathcal{A} \cap \mathcal{B} = \mathcal{A} \cap \mathcal{C} = \mathcal{B} \cap \mathcal{C} = \emptyset$ and let R/M be finite for all $M \in \max(R)$. Then there exists some sequence $(L_i)_{i \in \mathbb{N}}$ of extension fields of K such that $L_1 = K$, $L_i \subseteq L_{i+1}$, $[L_i : K] < \infty$ and $S_i = cl_{L_i}(R)$ for all $i \in \mathbb{N}$ and such that the following conditions are satisfied:*

1. *For all $i \in \mathbb{N}$ and $M \in \max(S_i)$ such that $M \cap R \in \mathcal{A}_i$ we have $MS_{i+1} \in \max(S_{i+1})$.*
2. *For all $i \in \mathbb{N}$ and $M \in \max(S_i)$ such that $M \cap R \in \mathcal{B}_i$ we have $MS_{i+1} \notin \max(S_{i+1})$ and $M \not\subseteq Q^2$ for all $Q \in \max(S_{i+1})$.*
3. *For all $i \in \mathbb{N}$ and $M \in \max(S_i)$ such that $M \cap R \in \mathcal{C}_i$ we have $MS_{i+1} \notin \max(S_{i+1})$ and $\sqrt[S_{i+1}]{MS_{i+1}} \in \max(S_{i+1})$.*

Proof. This follows by induction from [16, Theorem 42.5]. □

By [29, Example 4.3] there is some Dedekind domain R such that $\max(R)$ is countable, R/M is finite for all $M \in \max(R)$ and $\mathrm{Pic}(R)$ is torsion-free. Let $M : \mathbb{N}_0 \to \max(R)$ be a bijection. Note that since $\mathrm{Pic}(R)$ is torsion-free we obtain that $\bigcup_{M \in \max(R) \setminus \{M_0\}} M = R \setminus R^\times$ (since every non-unit of R is contained in at least two different maximal ideals of R). For $j \in \mathbb{N}$ set $\mathcal{A}_j = \{M_i \mid i \in [1, j]\}$.
First set $\mathcal{B}_j = \{M_0\}$ and $\mathcal{C}_j = \emptyset$ for all $j \in \mathbb{N}$. Let $(L_i)_{i \in \mathbb{N}}$ be the sequence in Proposition 6.6, $L = \bigcup_{i \in \mathbb{N}} L_i$ and $S = cl_L(R)$. Then S is an SP-domain that is a BF-domain but not Krull by Proposition 6.5.
Next set $\mathcal{B}_j = \emptyset$ and $\mathcal{C}_j = \{M_0\}$ for all $j \in \mathbb{N}$. Let $(L_i)_{i \in \mathbb{N}}$ be the sequence in Proposition 6.6, $L = \bigcup_{i \in \mathbb{N}} L_i$ and $S = cl_L(R)$. Then S is a completely integrally closed FF-domain that is a weakly Krull domain but not a Krull domain by Proposition 6.5.

Proposition 6.7. *Let R be a Prüfer domain, K a field of quotients of R, L/K an algebraic field extension and $S = cl_L(R)$.*

1. *If for all intermediate fields $K \subseteq M \subseteq L$ such that $[M : K] < \infty$ it follows that $\mathrm{Pic}(cl_M(R))$ is a torsion group, then $\mathrm{Pic}(S)$ is a torsion group.*
2. *If for all $a \in L$ and $n \in \mathbb{N}$ there is some $b \in L$ such that $b^n = a$, then $\mathrm{Pic}(S)$ is torsion-free.*

Proof. **1.** Let I be an invertible ideal of S. Then there are some $m \in \mathbb{N}$ and some sequence $(a_i)_{i=1}^m \in I^{[1,m]}$ such that $I = \sum_{i=1}^m a_iS$. Set $M = K(\{a_i \mid i \in [1,m]\})$, $T = cl_M(R)$ and $J = \sum_{i=1}^m a_iT$. Note that $\{a_i \mid i \in [1,m]\} \subseteq M \cap S = T$, and thus J is a non-zero finitely generated ideal of T. Since T is a Prüfer domain we have J is an invertible ideal of T. Since $\mathrm{Pic}(cl_M(R))$ is a torsion group, there are some $n \in \mathbb{N}$ and $a \in T$ such that $J^n = aT$. Therefore, $I^n = J^nS = aS$.
2. Let I be an invertible ideal of S, $n \in \mathbb{N}$ and $a \in S$ such that $I^n = aS$. There is some $b \in L$ such that $b^n = a$. Observe that $b \in S$ and $I^n = b^nS$. Let $M \in \max(S)$. Since S is a Prüfer domain it follows that S_M is a valuation domain, hence there is some $c \in S$ such that $I_M = cS_M$. This implies that $c^nS_M = I_M^n = b^nS_M$, and thus there exists some $\varepsilon \in S_M^\times$ such that $c^n = \varepsilon b^n$. Since S_M is a valuation domain it follows that $bS_M \subseteq cS_M$ or $cS_M \subseteq bS_M$. Case 1: $bS_M \subseteq cS_M$. There exists some $\nu \in S_M$ such that $b = c\nu$. This implies that $b^n = c^n\nu^n = \varepsilon b^n\nu^n$, hence $1 = \varepsilon\nu^n$. Consequently, $\nu \in S_M^\times$, and thus $I_M = cS_M = c\nu S_M = bS_M$. Case 2: $cS_M \subseteq bS_M$. There is some $\nu \in S_M$ such that $c = b\nu$. We have $c^n = b^n\nu^n = \varepsilon^{-1}c^n\nu^n$, hence $\nu^n = \varepsilon$. This implies that $\nu \in S_M^\times$, and thus $I_M = cS_M = b\nu S_M = bS_M$. Therefore, $I_Q = bS_Q$ for all $Q \in \max(S)$, hence $I = bS$. □

Let H be a monoid. So far we said little about the additional property that popped up in Theorem 3.6.3 (i.e. that for every $a \in H^\bullet$, $\mathcal{A}([\![a]\!]) \subseteq \mathcal{D}_k(a)$ for some $k \in \mathbb{N}$). Note that this additional property is equivalent to the notion of being pseudo-IDPF introduced in [23]. Since we are interested in studying monadically Krull monoids and their specializations, we investigate how to control the r-ideal class group of an r-SP-monoid to obtain this additional property. This is reasonable since there are non-trivial situations using the construction in Proposition 6.5 where SP-domains that are BF-domains can show up (as pointed out before). Moreover, Proposition 6.7 indicates that the class group of domains in this construction can behave nicely. If G is an abelian group, then let $\exp(G)$ be the exponent of G (i.e. if 1 is the identity of G, then $\exp(G) = \inf(\{n \in \mathbb{N} \mid x^n = 1 \text{ for all } x \in G\})$). The group G is called bounded if $\exp(G) < \infty$.

Proposition 6.8. *Let H be a monoid and r a finitary ideal system on H such that H is an r-SP-monoid.*

1. *If $\mathcal{C}_r(H)$ is finite, then for all $a \in H^\bullet$, $\mathcal{A}([\![a]\!]) \subseteq \mathcal{D}_k(a)$ for some $k \in \mathbb{N}$.*
2. *If H is an FF-monoid and $\mathcal{C}_r(H)$ is bounded, then for all $a \in H^\bullet$, $\mathcal{A}([\![a]\!]) \subseteq \mathcal{D}_k(a)$ for some $k \in \mathbb{N}$.*
3. *If H is r-Prüfer and $\mathcal{C}_r(H)$ is bounded, then for all $a \in H^\bullet$, $\mathcal{A}([\![a]\!]) \subseteq \mathcal{D}_k(a)$ for some $k \in \mathbb{N}$.*

Proof. **1.** Let $a \in H^\bullet$. Set $k = |\mathcal{C}_r(H)|$. We prove that $\mathcal{A}([\![a]\!]) \subseteq \mathcal{D}_k(a)$. Let $u \in \mathcal{A}([\![a]\!])$. There are some $l, s \in \mathbb{N}$ and some sequence $(I_i)_{i=1}^s$ of proper radical r-ideals of H such that $a^l \in uH = (\prod_{i=1}^s I_i)_r$. Observe that $a^l \in I_i$ for all $i \in [1,s]$, hence $a \in I_i$ for all $i \in [1,s]$. This implies that $a^s \in uH$. If $s \leq k$, then $a^k \in uH$, and thus $u \in \mathcal{D}_k(a)$. Now let $s > k$. There is some $\emptyset \neq E \subseteq [1,s]$ such that $|E| \leq k$ and $(\prod_{i\in E} I_i)_r$ is principal. Since $uH = (\prod_{i=1}^s I_i)_r \subseteq (\prod_{i\in E} I_i)_r \subsetneq H$, we have $uH = (\prod_{i\in E} I_i)_r$. Therefore, $a^k \in a^{|E|}H \subseteq (\prod_{i\in E} I_i)_r = uH$. Consequently, $u \in \mathcal{D}_k(a)$.
2. Let H be an FF-monoid, $\mathcal{C}_r(H)$ bounded and $a \in H^\bullet$. Set $l = \exp(\mathcal{C}_r(H))$, $\mathcal{M} = \{I \mid I \text{ is an } r\text{-invertible radical } r\text{-ideal of } H, a \in I\}$ and $\mathcal{N} = \{bH \mid b \in H, a^l \in bH\}$. Let $f : \mathcal{M} \to \mathcal{N}$ be defined by $f(I) = (I^l)_r$ for all $I \in \mathcal{M}$. If $I \in \mathcal{M}$, then there is some $b \in H$ such that $(I^l)_r = bH$. Set $J = aI^{-1}$. Then $J \in \mathcal{I}_r(H)$ and $aH = (IJ)_r$. This implies that $a^l \in a^lH = (I^lJ^l)_r = b(J^l)_r \subseteq bH$, and thus f is well-defined. Now let $I, J \in \mathcal{M}$ be such that $f(I) = f(J)$. It follows that $I = \sqrt{(I^l)_r} = \sqrt{f(I)} = \sqrt{f(J)} = \sqrt{(J^l)_r} = J$. Therefore, f is injective. Since H is an FF-monoid we have $|\mathcal{M}| \leq |\mathcal{N}| < \infty$. Set $k = l|\mathcal{M}|$. We show that $\mathcal{A}([\![a]\!]) \subseteq \mathcal{D}_k(a)$. Let $u \in \mathcal{A}([\![a]\!])$. There are some $m, n \in \mathbb{N}$, some sequence $(\alpha_i)_{i=1}^n \in \mathbb{N}^{[1,n]}$ and some sequence $(I_i)_{i=1}^n$ of distinct proper radical r-ideals of H such that $a^m \in uH = (\prod_{i=1}^n I_i^{\alpha_i})_r$. Note that

$I_i \in \mathcal{M}$ for all $i \in [1,n]$, hence $n \leq |\mathcal{M}|$. If $\alpha_j > l$ for some $j \in [1,n]$, then $uH \subseteq (I_j^{\alpha_j})_r \subseteq (I_j^l)_r \subseteq I \subsetneq H$, and thus $uH = (I_j^l)_r = (I_j^{\alpha_j})_r$ which implies that $I_j = H$, a contradiction. Therefore, $a^l \in (I_j^{\alpha_j})_r$ for all $j \in [1,n]$. It follows that $a^k \in (\prod_{i=1}^n I_i^{\alpha_i})_r = uH$, hence $u \in \mathcal{D}_k(a)$.

3. Let H be an r-Prüfer monoid, $\mathcal{C}_r(H)$ bounded and $a \in H^\bullet$. Set $m = \exp(\mathcal{C}_r(H))$ and $k = 2m$. It is sufficient to show that $\mathcal{A}(\llbracket a \rrbracket) \subseteq \mathcal{D}_k(a)$. Let $u \in \mathcal{A}(\llbracket a \rrbracket)$. By [29, Proposition 3.9], [29, Theorem 3.13] and [29, Theorem 3.3.2] there are some $l \in \mathbb{N}$ and some ascending sequence $(I_i)_{i=1}^l$ of proper radical r-ideals of H such that $uH = (\prod_{i=1}^l I_i)_r$. Set $F = I_l$. Then F is r-invertible. Clearly, $uH \subseteq (F^l)_r$ and there is some $b \in H \backslash H^\times$ such that $(F^m)_r = bH$. Assume that $l > k$. We have $uH \subseteq (F^l)_r \subseteq (F^k)_r = b^2 H \subseteq bH$. Since $u \in \mathcal{A}(H)$, this implies that $uH = b^2 H = bH$, hence $b \in H^\times$, a contradiction. Therefore, $l \leq k$, and thus $a^k \in a^l H \subseteq (\prod_{i=1}^l I_i)_r = uH$. Consequently, $u \in \mathcal{D}_k(a)$. □

Acknowledgement. We want to thank A. Geroldinger, F. Halter-Koch, F. Kainrath and the referee for their comments and suggestions.

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