# RADICAL FACTORIAL MONOIDS AND DOMAINS

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ABSTRACT. In this paper we study variations and generalizations of SP-domains (i.e. domains where every ideal is a finite product of radical ideals) with respect to monoids and finitary ideal systems $r$. In particular we consider so called $r$-SP-monoids and investigate monoids where every principal ideal is a finite product of radical principal ideals (which we call radical factorial monoids). We clarify the connections between radical factorial monoids and $r$-SP-monoids and characterize the GCD-monoids that are radical factorial. Furthermore, we specify the factorial monoids using radical factorial monoids and some of their variations. As a byproduct of these investigations we present a characterization of strongly $r$-discrete $r$-Prüfer monoids.

## 0. INTRODUCTION

Recall that a Dedekind domain is an integral domain where each of its ideals is a finite product of prime ideals. Moreover, a factorial domain is an integral domain where each of its principal ideals is a finite product of principal prime ideals. These concepts have been generalized into various directions. It is well known that an ideal of a commutative ring with identity is a prime ideal if and only if it is primary and radical. Therefore, it is natural to ask how integral domains (resp. monoids) can be characterized if each of its ideals (principal ideals) is a finite product of primary ideals resp. radical ideals (primary principal ideals resp. radical principal ideals). Most of these generalizations (like Q-rings, SP-domains and weakly factorial domains) have already been studied (see [2, 3, 9, 11, 15, 16]). As far as we know the monoids where every principal ideal is a finite product of radical principal ideals have not been investigated so far (we call them radical factorial monoids). Note that factorial monoids are closely connected with the $t$-operation. Consequently, it is of interest to look at the $t$-analogue of Dedekind domains and SP-domains. More generally we introduce and study so called $r$-SP-monoids with respect to a finitary ideal system $r$ and reveal their relations with radical factorial monoids.

In the first section we recall the most important facts about ideal systems. We define the most significant ideal systems and provide much of the terminology that is used in the succeeding sections.

In the second section we investigate radical factorial monoids. In particular we show that radical factorial monoids are completely integrally closed. We characterize the radical factorial monoids that are Krull monoids and specify the GCD-monoids that are radical factorial. The main result in this section is a characterization result for factorial monoids using the radical factorial property and similar concepts. Finally we prove that polynomial rings over radical factorial GCD-domains are radical factorial.

In the third section we deal with finitary ideal systems $r$ and $r$-SP-monoids. We investigate the connections between $r$-SP-monoids and radical factorial monoids and improve the results of [11, 15]. The first main result in this section characterizes the $r$-ideals that are finite $r$-products of radical $r$-ideals in the context of almost $r$-Dedekind monoids. As a consequence we specify the almost $r$-Dedekind monoids that are $r$-SP-monoids. We introduce and study monoids that are primary $r$-ideal inclusive. The second main result in this section shows that $r$-SP-monoids that are primary $r$-ideal inclusive are already almost $r$-Dedekind monoids.

The fourth section contains several counterexamples. We show that none of the conditions of the main

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result in section two can be cancelled and we give a few non-trivial examples of radical factorial monoids. Moreover, we present an example of a noetherian domain that is not primary $t$-ideal inclusive.
In the last section we generalize parts of [4] and [10] to monoids and finitary ideal systems. The main result in this section specifies the strongly $r$-discrete $r$-Prüfer monoids.

## 1. Ideal systems

In the following a monoid is a commutative (multiplicatively written) semigroup that possesses a zero element and an indentity (which is not the zero element), where every non-zero element is cancellative. A quotient monoid of a monoid is an extension monoid, where every non-zero element is invertible and that is minimal with respect to this property.

Let $H$ be a monoid and $K$ a quotient monoid of $H$. If $A, B, C \subseteq K$, then set $(A :_C B) = \{z \in C \mid zB \subseteq A\}$, $A^{-1} = (H :_K A)$ and $A^\bullet = A\backslash\{0\}$. By $\mathbb{P}(H)$ we denote the power set of $H$. Next we give a brief description of ideal systems. An elaboration of ideal systems can be found in [8]. Let $r : \mathbb{P}(H) \to \mathbb{P}(H)$ be a map. We call $r$ an ideal system on $H$ if the following conditions are satisfied for all $X, Y \subseteq H$ and $c \in H$.

- $XH \cup \{0\} \subseteq r(X)$.
- $r(cX) = cr(X)$.
- If $X \subseteq r(Y)$, then $r(X) \subseteq r(Y)$.

We call $r$ a finitary ideal system on $H$ if the following property is additionally satisfied for all $X \subseteq H$.

- $r(X) = \bigcup_{E \subseteq X, |E|<\infty} r(E)$.

Observe that $s : \mathbb{P}(H) \to \mathbb{P}(H)$ defined by $s(X) = XH$ if $\emptyset \neq X \subseteq H$ and by $s(X) = \{0\}$ if $X = \emptyset$ is a finitary ideal system on $H$. Furthermore, if $H^\bullet \neq H^\times$, then $v : \mathbb{P}(H) \to \mathbb{P}(H)$ defined by $v(X) = (X^{-1})^{-1}$ for all $X \subseteq H$ is an ideal system on $H$. Let $t : \mathbb{P}(H) \to \mathbb{P}(H)$ be defined by $t(X) = \bigcup_{E \subseteq X, |E|<\infty} E_v$ for all $X \subseteq H$. If $H^\bullet \neq H^\times$, then $t$ is a finitary ideal system on $H$. If $H$ is an integral domain, then $d : \mathbb{P}(H) \to \mathbb{P}(H)$ defined by $d(X) = (X)_H$ for all $X \subseteq H$ is a finitary ideal system on $H$.
Let $r$ be an ideal system on $H$. For $X \subseteq H$ we set $X_r = r(X)$. Note that $\{0\}_r = \{0\}$. Set $\mathcal{I}_r(H) = \{X_r \mid X \subseteq H\}$ and $\mathcal{I}_{r,f}(H) = \{E_r \mid E \subseteq H, |E| < \infty\}$. We have $\mathcal{I}_v(H) \subseteq \mathcal{I}_r(H) \subseteq \mathcal{I}_s(H)$. If $r$ is finitary, then $\mathcal{I}_t(H) \subseteq \mathcal{I}_r(H)$. Especially if $H$ is an integral domain that is not a field, then $\mathcal{I}_v(H) \subseteq \mathcal{I}_t(H) \subseteq \mathcal{I}_d(H) \subseteq \mathcal{I}_s(H)$. Let $r$-max$(H)$ be the set of all maximal elements of $\{X \subsetneqq H \mid X_r = X\}$ and $r$-spec$(H)$ the set of all prime $r$-ideals of $H$. It is well known that $r$-max$(H) \subseteq r$-spec$(H)$. We call $H$ $r$-local if $H\backslash H^\times \in r$-max$(H)$ (equivalently: $|r\text{-max}(H)| = 1$). If $Y \subseteq \mathcal{I}_s(H)$, then set $Y^\bullet = Y\backslash\{\{0\}\}$. By $\mathfrak{X}(H)$ we denote the set of all minimal elements of $s$-spec$(H)^\bullet$.
Let $I \in \mathcal{I}_s(H)$. Set $\sqrt{I} = \{x \in H \mid x^k \in I \text{ for some } k \in \mathbb{N}\}$, called the radical of $I$. Set $\mathcal{I}_{\sqrt{r}}(H) = \{J \in \mathcal{I}_r(H) \mid \sqrt{J} = J\}$ and $\mathcal{I}_{\sqrt{r},f}(H) = \mathcal{I}_{\sqrt{r}}(H) \cap \mathcal{I}_{r,f}(H)$. Let $\mathcal{P}(I)$ be the set of all minimal elements of $\{P \in s\text{-spec}(H) \mid I \subseteq P\}$, called the set of prime divisors of $I$. Note that if $r$ is finitary and $I \in \mathcal{I}_r(H)$, then $\mathcal{P}(I) \subseteq r$-spec$(H)$. If $I \in \mathcal{I}_r(H)$, then $I$ is called $r$-invertible if $(II^{-1})_r = H$. Let $\mathcal{I}_r^*(H)$ be the set of $r$-invertible $r$-ideals of $H$. Observe that $\mathcal{I}_r^*(H)$ forms a monoid (without a zero element) under $r$-multiplication. There exists some quotient group $\mathcal{F}_r(H)^\times$ of $\mathcal{I}_r^*(H)$ such that $\{aH \mid a \in K^\bullet\}$ is a subgroup of $\mathcal{F}_r(H)^\times$. Set $\mathcal{C}_r(H) = \mathcal{F}_r(H)^\times/\{aH \mid a \in K^\bullet\}$, called the $r$-class group of $H$. If $r$ is finitary and $T \subseteq H^\bullet$ multiplicatively closed, then there exists an unique finitary ideal system $T^{-1}r$ on $T^{-1}H$ such that $(T^{-1}X)_{T^{-1}r} = T^{-1}X_r$ for all $X \subseteq H$. If $T = H\backslash Q$ for some $Q \in s$-spec$(H)$, then set $r_Q = T^{-1}r$.

## 2. Radical factorial monoids

Let $H$ be a monoid and $x \in H^\bullet$.

- $x$ is called radical if $\sqrt{xH} = xH$ and $x$ is called primary if $xH$ is primary.
- $H$ is called radical factorial if every $y \in H^\bullet$ is a finite product of radical elements of $H$ (equivalently: Every principal ideal of $H$ is a finite product of radical principal ideals of $H$).

- $H$ is called primary if $H^\bullet \neq H^\times$ and every $y \in H^\bullet \backslash H^\times$ is primary.
- $H$ is called weakly factorial if every $y \in H^\bullet \backslash H^\times$ is a finite product of primary elements of $H$.

As usual one defines the notions of Krull monoids, Mori monoids, discrete valuation monoids and completely integrally closed monoids (for instance see [8, Definition 16.4.], [7, Definition 2.1.9.] and [7, Definition 2.3.1.] or [14]). Let $\mathcal{A}(H)$ denote the set of atoms of $H$ and let $\mathcal{B}(H)$ be the set of radical elements of $H$. Obviously, if $H$ is atomic and $\mathcal{A}(H) \subseteq \mathcal{B}(H)$, then $H$ is radical factorial. Note that if $H$ is radical factorial, then $\mathcal{A}(H) \subseteq \mathcal{B}(H)$. Clearly every primary monoid is weakly factorial. Moreover, an integral domain is primary if and only if it is local and 1-dimensional. Set $\Omega(H) = \{\prod_{i=1}^n x_i \mid n \in \mathbb{N}, (x_i)_{i=1}^n \in \mathcal{B}(H)^{[1,n]}\}$ and $\mathfrak{V}(H) = \bigcup_{y \in H^\bullet} \mathcal{P}(yH)$. A subset $Z \subseteq H$ is called divisor-closed if for all $x \in H$ and $y \in H^\bullet$ such that $xy \in Z$ it follows that $x \in Z$.

**Lemma 2.1.** *Let $H$ be a monoid and $T \subseteq H^\bullet$ multiplicatively closed.*

1. *$\mathcal{B}(H) \subseteq \mathcal{B}(T^{-1}H)$.*
2. *If $H$ is radical factorial, then $T^{-1}H$ is radical factorial.*

*Proof.* **1.** Let $x \in \mathcal{B}(H)$. It follows that $x(T^{-1}H) = T^{-1}(xH) = T^{-1}(\sqrt{xH}) = \sqrt[T^{-1}H]{T^{-1}(xH)} = \sqrt[T^{-1}H]{x(T^{-1}H)}$, hence $x \in \mathcal{B}(T^{-1}H)$.
**2.** This is an easy consequence of 1.. □

**Lemma 2.2.** *Let $H$ be a monoid, $I \in \mathcal{I}_s(H)$ and $P \in s\text{-spec}(H)$ such that $I \subseteq P$.*

1. *If $I$ is primary and $x \in H$ such that $\sqrt{I} = xH$, then there exists some $r \in \mathbb{N}$ such that $I = x^r H$.*
2. *We have $P \in \mathcal{P}(I)$ if and only if $P_P = (\sqrt{I})_P$.*

*Proof.* **1.** Let $I$ be primary and $x \in H$ such that $\sqrt{I} = xH$. There exists some smallest $r \in \mathbb{N}$ such that $x^r \in I$. It remains to show that $I \subseteq x^r H$. Let $y \in I$. Case 1: $y \in \bigcap_{n \in \mathbb{N}} x^n H$: Clearly, $y \in x^r H$. Case 2: There is some largest $n \in \mathbb{N}$ such that $y \in x^n H$: There exists some $z \in H$ such that $y = x^n z$. Obviously, $z \notin \sqrt{I}$ and thus $x^n \in I$. It follows that $r \leq n$, hence $y \in x^n H \subseteq x^r H$.
**2.** "$\Rightarrow$": Let $P \in \mathcal{P}(I)$. Then $P_P \in \mathcal{P}(I_P)$. Since $P_P \in s(H_P)\text{-max}(H_P)$ we obtain that $P_P$ is the only prime $s(H_P)$-ideal of $H_P$ that contains $I_P$. This implies that $P_P = \sqrt[H_P]{I_P} = (\sqrt{I})_P$. "$\Leftarrow$": Let $P_P = (\sqrt{I})_P$ and $Q \in s\text{-spec}(H)$ be such that $I \subseteq Q \subseteq P$. Then $\sqrt{I} \subseteq Q$, hence $P_P = (\sqrt{I})_P \subseteq Q_P \subseteq P_P$. Therefore, $Q_P = P_P$ and thus $Q = P$. □

**Lemma 2.3.** *Let $H$ be a monoid.*

1. *If $P \in \mathfrak{X}(H)$ such that $P \cap \mathcal{B}(H) \neq \emptyset$, then $H_P$ is a discrete valuation monoid and $P \not\subseteq \bigcup_{Q \in \mathfrak{X}(H)} Q_Q^2$.*
2. *Every prime divisor of a non-zero radical principal ideal of $H$ is minimal with respect to inclusion among the prime $s$-ideals of $H$ containing a radical element of $H$.*

*Proof.* **1.** Let $P \in \mathfrak{X}(H)$ and $x \in P \cap \mathcal{B}(H)$. Since $P \in \mathcal{P}(xH)$ it follows by Lemma 2.2.2. that $P_P = (xH)_P = xH_P$, hence $P_P$ contains a prime element of $H_P$. Since $H_P$ is primary by [8, Corollary 15.4.] we have $H_P$ is a discrete valuation monoid by [8, Theorem 16.4.]. Assume that there is some $Q \in \mathfrak{X}(H)$ such that $x \in Q_Q^2$. Then $Q \in \mathcal{P}(xH)$ and thus $Q_Q = (xH)_Q = xH_Q$ by Lemma 2.2.2.. Consequently, $x \in Q_Q^2 = x^2 H_Q$, hence $x \in H_Q^\times$, a contradiction.
**2.** Let $u \in \mathcal{B}(H)$, $P \in \mathcal{P}(uH)$ and $Q \in s\text{-spec}(H)$ be such that $Q \cap \mathcal{B}(H) \neq \emptyset$ and $Q \subseteq P$. There is some $v \in Q \cap \mathcal{B}(H)$. Lemma 2.2.2. implies that $P_P = (uH)_P = uH_P$, hence there is some $w \in H_P$ such that $v = uw$. Assume that $w \notin H_P^\times$. Then $w \in P_P = uH_P$ and thus $w^2 \in vH_P$. By Lemma 2.1.1. we have $w \in vH_P$, hence there is some $z \in H_P$ such that $w = vz$. It follows that $v = uvz$ which implies that $u \in H_P^\times$, a contradiction. Consequently, $w \in H_P^\times$ and thus $P_P = vH_P \subseteq Q_P \subseteq P_P$, hence $Q = P$. □

In the next result we collect some important facts about radical factorial monoids.

**Proposition 2.4.** *Let $H$ be a radical factorial monoid.*

1. *For every $x \in H$ it follows that $\sqrt{xH}$ is a finite intersection of radical principal ideals of $H$.*

**2.** *If $I \in \mathcal{I}_v(H)$ is primary such that $(I^l)_v$ is principal for some $l \in \mathbb{N}$, then there are some $k \in \mathbb{N}$ and $p \in H$ a prime element of $H$ such that $I = p^k H$.*
**3.** *Every $Q \in s$-spec$(H)^\bullet$ contains a radical element of $H$ and $P \nsubseteq \bigcup_{Q \in \mathfrak{X}(H)} Q_Q^2$ for all $P \in \mathfrak{X}(H)$.*
**4.** *$H_P$ is a discrete valuation monoid for all $P \in \mathfrak{V}(H)$ (especially $H$ satisfies the Principal Ideal Theorem).*
**5.** *$\bigcap_{P \in \mathfrak{X}(H)} H_P = H$ and $H$ is completely integrally closed.*

*Proof.* **1.** Without restriction let $x \in H^\bullet$. There exist some $n \in \mathbb{N}$ and some sequence $(x_i)_{i=1}^n$ of radical elements of $H$ such that $x = \prod_{i=1}^n x_i$. It follows that $\sqrt{xH} = \sqrt{\prod_{i=1}^n x_i H} = \bigcap_{i=1}^n \sqrt{x_i H} = \bigcap_{i=1}^n x_i H$.
**2.** Let $I \in \mathcal{I}_v(H)$ be primary, $l \in \mathbb{N}$ and $x \in H^\bullet$ such that $(I^l)_v = xH$. Since $I^l \subseteq (I^l)_v \subseteq I$ we have $\sqrt{I} = \sqrt{I^l} \subseteq \sqrt{(I^l)_v} \subseteq \sqrt{I}$, hence $\sqrt{I} = \sqrt{xH}$. By 1. there exist some $n \in \mathbb{N}$ and some sequence $(x_i)_{i=1}^n$ of elements of $H$ such that $\sqrt{I} = \bigcap_{i=1}^n x_i H$. Since $\sqrt{I} \in s$-spec$(H)$ there is some $i \in [1, n]$ such that $\sqrt{I} = x_i H$. Set $p = x_i$. Observe that $p$ is a prime element of $H$. By Lemma 2.2.1. there exists some $k \in \mathbb{N}$ such that $I = p^k H$.
**3.** Let $Q \in s$-spec$(H)^\bullet$. Then there are some $x \in Q^\bullet$, $n \in \mathbb{N}$ and some sequence $(x_i)_{i=1}^n$ of radical elements of $H$ such that $x = \prod_{i=1}^n x_i$. Consequently, there is some $i \in [1, n]$ such that $x_i \in Q$, hence $Q \cap \mathcal{B}(H) \neq \emptyset$. The second statement follows from of Lemma 2.3.1..
**4.** Let $P \in \mathfrak{V}(H)$. Then there is some $x \in H^\bullet$ such that $P \in \mathcal{P}(xH)$. Let $Q \in s$-spec$(H)^\bullet$ be such that $Q \subseteq P$. Then $Q \cap \mathcal{B}(H) \neq \emptyset$ by 3. and there is some $u \in \mathcal{B}(H)$ such that $x \in uH \subseteq P$. Therefore, $P \in \mathcal{P}(uH)$ and hence Lemma 2.3.2. implies that $P = Q$. Consequently, $P \in \mathfrak{X}(H)$ and thus $H_P$ is a discrete valuation monoid by 3. and Lemma 2.3.1..
**5.** Claim: For all $x \in H$, $n \in \mathbb{N}_0$ and $(u_i)_{i=1}^n \in \mathcal{B}(H)^{[1,n]}$ such that $\frac{x}{\prod_{i=1}^n u_i} \in \bigcap_{P \in \mathfrak{X}(H)} H_P$ it follows that $\frac{x}{\prod_{i=1}^n u_i} \in H$. Let $x \in H$. We use induction on $n$. The assertion is clear for $n = 0$. Now let $n \in \mathbb{N}_0$ and $(u_i)_{i=1}^{n+1} \in \mathcal{B}(H)^{[1,n+1]}$ be such that $\frac{x}{\prod_{i=1}^{n+1} u_i} \in \bigcap_{P \in \mathfrak{X}(H)} H_P$. Set $z = \frac{x}{\prod_{i=2}^{n+1} u_i}$. Then $z \in \bigcap_{P \in \mathfrak{X}(H)} H_P$ and thus $z \in H$ by the induction hypothesis. We have $\frac{z}{u_1} \in \bigcap_{P \in \mathfrak{X}(H)} H_P$. If $P \in \mathfrak{X}(H)$ such that $u_1 \in P$, then there is some $t \in H \backslash P$ such that $tz \in u_1 H \subseteq P$, hence $z \in P$. By 4. we have $z \in \bigcap_{P \in \mathfrak{X}(H), u_1 \in P} P = \bigcap_{P \in \mathcal{P}(u_1 H)} P = \sqrt{u_1 H} = u_1 H$ and thus $\frac{x}{\prod_{i=1}^{n+1} u_i} = \frac{z}{u_1} \in H$.
It is an easy consequence of the claim that $\bigcap_{P \in \mathfrak{X}(H)} H_P = H$. By 4. it follows that $H_P$ is completely integrally closed for all $P \in \mathfrak{X}(H)$. Therefore, $H$ is completely integrally closed. □

Note that if $H$ is a radical factorial monoid, then $\mathcal{C}_t(H)$ satisfies a weak form of being torsion-free by Proposition 2.4.2.. The next result is a generalization of [1] and [8, Proposition 6.6.].

**Lemma 2.5.** *Let $H$ be a monoid, $r$ a finitary ideal system on $H$ and $I \in \mathcal{I}_r(H)$ such that for each $P \in \mathcal{P}(I)$ there is some $J \in \mathcal{I}_{r,f}(H)$ such that $P = \sqrt{J}$. Then $\mathcal{P}(I)$ is finite.*

*Proof.* Without restriction let $I \neq H$. Set $\Sigma = \{(\prod_{i=1}^n P_i)_r \mid n \in \mathbb{N}, (P_i)_{i=1}^n \in \mathcal{P}(I)^{[1,n]}\}$ and $\Omega = \{J \in \mathcal{I}_{\sqrt{r}}(H) \mid I \subseteq J \text{ and } Q \nsubseteq J \text{ for all } Q \in \Sigma\}$. Assume that $\sqrt{I} \in \Omega$. Let $\emptyset \neq \mathcal{M} \subseteq \Omega$ be a chain and $J = \bigcup_{A \in \mathcal{M}} A$. Since $r$ is finitary it follows that $J \in \mathcal{I}_{\sqrt{r}}(H)$. Obviously, $I \subseteq J$. Assume that there is some $Q \in \Sigma$ such that $Q \subseteq J$. There are some $m \in \mathbb{N}$ and some sequence $(B_i)_{i=1}^m$ of $r$-finitely generated $r$-ideals of $H$ such that $Q = (\prod_{i=1}^m \sqrt{B_i})_r$. Let $B = (\prod_{i=1}^m B_i)_r$. Then $B \in \mathcal{I}_{r,f}(H)$ and $\sqrt{Q} = \sqrt{B}$. Since $B \subseteq \sqrt{Q} \subseteq J$, there is some $A \in \mathcal{M}$ such that $B \subseteq A$, hence $Q \subseteq \sqrt{B} \subseteq A$, a contradiction. Therefore, $J \in \Omega$. This implies that $\Omega$ is ordered inductively by inclusion, hence there is some maximal $M \in \Omega$. Clearly, $M \neq H$. Assume that there are some $x, y \in H \backslash M$ such that $xy \in M$. Then $\sqrt{(M \cup \{x\})_r}, \sqrt{(M \cup \{y\})_r} \notin \Omega$ and thus there are some $Q_1, Q_2 \in \Sigma$ such that $Q_1 \subseteq \sqrt{(M \cup \{x\})_r}$ and $Q_2 \subseteq \sqrt{(M \cup \{y\})_r}$. This implies that $(Q_1 Q_2)_r \subseteq (\sqrt{(M \cup \{x\})_r}\sqrt{(M \cup \{y\})_r})_r \subseteq \sqrt{(M \cup \{x\})_r} \cap \sqrt{(M \cup \{y\})_r} = \sqrt{(M \cup \{x\})_r (M \cup \{y\})_r} \subseteq M$. Since $(Q_1 Q_2)_r \in \Sigma$ we have $M \notin \Omega$, a contradiction. Therefore, $M \in r$-spec$(H)$, hence there is some $P \in \mathcal{P}(I) \subseteq \Sigma$ such that $P \subseteq M$, a contradiction.

Consequently, $\sqrt{I} \notin \Omega$ and thus there are some $n \in \mathbb{N}$ and some sequence $(P_i)_{i=1}^n$ of prime divisors of $I$ such that $(\prod_{i=1}^n P_i)_r \subseteq \sqrt{I}$. This implies that $\mathcal{P}(I) \subseteq \{P_i \mid i \in [1,n]\}$, hence $\mathcal{P}(I)$ is finite. $\square$

**Proposition 2.6.** *Let $H$ be a radical factorial monoid. The following conditions are equivalent:*

1. *$H$ is a Krull monoid.*
2. *$H$ is a Mori monoid.*
3. *For every $P \in \mathfrak{X}(H)$ there is some $J \in \mathcal{I}_{t,f}(H)$ such that $P = \sqrt{J}$.*
4. *$\{P \in \mathfrak{X}(H) \mid x \in P\}$ is finite for all $x \in H^\bullet$.*

*Proof.* **1.** $\Rightarrow$ **2.**: Clear. **2.** $\Rightarrow$ **3.**: Trivial. **3.** $\Rightarrow$ **4.**: Let $x \in H^\bullet$. By Proposition 2.4.4. we have $\mathcal{P}(xH) = \{P \in \mathfrak{X}(H) \mid x \in P\}$ and thus every $P \in \mathcal{P}(xH)$ is the radical of an $t$-finitely generated $t$-ideal of $H$. Therefore, Lemma 2.5. implies that $\{P \in \mathfrak{X}(H) \mid x \in P\}$ is finite. **4.** $\Rightarrow$ **1.**: It is well known that an intersection of finite character of a family of Krull monoids is again a Krull monoid. Consequently, Proposition 2.4.4. and Proposition 2.4.5. imply that $H$ is a Krull monoid. $\square$

It is well known that every completely integrally closed monoid $H$ where every $M \in t\text{-max}(H)$ is divisorial is already a Krull monoid. Therefore, the question arises whether $H$ is a Krull monoid if $H$ is radical factorial and every $P \in \mathfrak{X}(H)$ is divisorial. Note that this problem is closely connected with the question: Does every radical factorial monoid $H$ (with $H^\bullet \neq H^\times$) satisfy $t\text{-max}(H) = \mathfrak{X}(H)$? In the following we characterize radical factorial monoids under several additional conditions.

**Lemma 2.7.** *Let $H$ be a monoid such that $\bigcap_{P \in \mathfrak{X}(H)} H_P = H$ and $H_P$ is a discrete valuation monoid for all $P \in \mathfrak{V}(H)$.*

1. *$H^\bullet \setminus \bigcup_{L \in \mathfrak{X}(H)} L_L^2 \subseteq \mathcal{B}(H)$.*
2. *If $x \in H^\bullet$ and $y \in H$ such that $\sqrt{xH} \subseteq yH$, then $y \in \mathcal{B}(H)$.*
3. *If $P \nsubseteq \bigcup_{Q \in \mathfrak{X}(H)} Q_Q^2$ for all $P \in \mathfrak{X}(H)$, then $Q \cap \mathcal{B}(H) \neq \emptyset$ for all $Q \in s\text{-spec}(H)^\bullet$.*
4. *If $\sqrt{\{u,w\}_t}$ is principal for all $u \in \mathcal{B}(H)$ and $w \in H$, then $\Omega(H)$ is divisor-closed.*

*Proof.* **1.** Let $x \in H^\bullet \setminus \bigcup_{L \in \mathfrak{X}(H)} L_L^2$ and $Q \in \mathfrak{X}(H)$. If $x \in Q$, then $xH_Q = Q_Q$ and if $x \notin Q$, then $xH_Q = H_Q$. We have $\sqrt{xH} = \bigcap_{P \in \mathcal{P}(xH)} P = \bigcap_{P \in \mathfrak{X}(H), x \in P} P = (\bigcap_{P \in \mathfrak{X}(H), x \in P} P_P) \cap H = (\bigcap_{P \in \mathfrak{X}(H), x \in P} xH_P) \cap H = (\bigcap_{P \in \mathfrak{X}(H), x \in P} xH_P) \cap \bigcap_{P \in \mathfrak{X}(H)} H_P = \bigcap_{P \in \mathfrak{X}(H)} xH_P = xH$. Therefore, $x \in \mathcal{B}(H)$.
**2.** Let $x \in H^\bullet$ and $y \in H$ be such that $\sqrt{xH} \subseteq yH$ and $Q \in \mathfrak{X}(H)$. If $y \in Q_Q^2$, then $Q_Q \subseteq \sqrt[H_Q]{xH_Q} = (\sqrt{xH})_Q \subseteq yH_Q \subseteq Q_Q^2$, a contradiction. Therefore, $y \notin \bigcup_{L \in \mathfrak{X}(H)} L_L^2$ and thus $y \in \mathcal{B}(H)$ by 1..
**3.** Let $P \nsubseteq \bigcup_{Q \in \mathfrak{X}(H)} Q_Q^2$ for all $P \in \mathfrak{X}(H)$. It follows by 1. that $P \cap \mathcal{B}(H) \neq \emptyset$ for all $P \in \mathfrak{X}(H)$. Let $Q \in s\text{-spec}(H)^\bullet$. There is some $x \in Q^\bullet$ and thus there is some $L \in \mathcal{P}(xH) \subseteq \mathfrak{X}(H)$ such that $L \subseteq Q$. Consequently, $Q \cap \mathcal{B}(H) \neq \emptyset$.
**4.** Let $\sqrt{\{u,w\}_t}$ be principal for all $u \in \mathcal{B}(H)$ and $w \in H$. It is sufficient to show by induction that for all $n \in \mathbb{N}_0$, $(u_i)_{i=1}^n \in \mathcal{B}(H)^{[1,n]}$ and $y \in H$ such that $\prod_{i=1}^n u_i \in yH$ it follows that $y \in \Omega(H)$. The assertion is clear for $n = 0$. Now let $n \in \mathbb{N}_0$, $(u_i)_{i=1}^{n+1} \in \mathcal{B}(H)^{[1,n+1]}$ and $y \in H$ be such that $\prod_{i=1}^{n+1} u_i \in yH$. There exists some $x \in \mathcal{B}(H) \subseteq \Omega(H)$ such that $\sqrt{\{u_{n+1}, y\}_t} = xH$, hence there is some $z \in H$ such that $y = xz$. Next we prove that $\prod_{i=1}^n u_i H_Q \subseteq zH_Q$ for all $Q \in \mathfrak{X}(H)$. Let $Q \in \mathfrak{X}(H)$. Then $\prod_{i=1}^{n+1} u_i H_Q \subseteq yH_Q = xzH_Q$ by assumption. If $yH_Q = H_Q$, then $zH_Q = H_Q$, hence $\prod_{i=1}^n u_i H_Q \subseteq H_Q = zH_Q$. If $u_{n+1}H_Q = H_Q$, then $xH_Q = H_Q$ and the assertion follows. If $yH_Q \neq H_Q$ and $u_{n+1}H_Q \neq H_Q$, then $u_{n+1}H_Q = Q_Q = xH_Q$ and the assertion follows. Finally we have $(\prod_{i=1}^n u_i)H = \bigcap_{P \in \mathfrak{X}(H)} (\prod_{i=1}^n u_i)H_P \subseteq \bigcap_{P \in \mathfrak{X}(H)} zH_P = zH$, hence $z \in \Omega(H)$ by the induction hypothesis and thus $y = xz \in \Omega(H)$. $\square$

**Proposition 2.8.** *Let $H$ be a monoid such that $\bigcap_{P \in \mathfrak{X}(H)} H_P = H$ and $H_P$ is a discrete valuation monoid for all $P \in \mathfrak{V}(H)$. Let for all $x \in H$, $\sqrt{xH}$ be a finite intersection of principal ideals of $H$.*

1. *For all $x \in H^\bullet$ there is some $k \in \mathbb{N}$ such that $x \notin P_P^k$ for all $P \in \mathfrak{X}(H)$.*

**2.** *$P \cap \mathcal{B}(H) \neq \emptyset$ for all $P \in s\text{-spec}(H)^\bullet$.*
**3.** *For all $x \in H^\bullet$ and $P \in \mathfrak{X}(H)$ there is some $y \in \Omega(H)$ such that $\frac{x}{y} \in H \backslash P$.*

*Proof.* By Lemma 2.7.2., $\sqrt{xH}$ is a finite intersection of radical principal ideals of $H$ for all $x \in H$.
**1.** Let $x \in H^\bullet$. There are some $n \in \mathbb{N}$ and some sequence $(x_i)_{i=1}^n$ of radical elements of $H$ such that $\sqrt{xH} = \bigcap_{i=1}^n x_i H$. We have $\sqrt{\prod_{i=1}^n x_i H} = \bigcap_{i=1}^n \sqrt{x_i H} = \bigcap_{i=1}^n x_i H = \sqrt{xH}$ and thus there is some $l \in \mathbb{N}$ such that $(\prod_{i=1}^n x_i)^l H \subseteq xH$. Let $k = ln + 1$ and $P \in \mathfrak{X}(H)$. Observe that $P_P \subseteq x_i H_P$ for all $i \in [1, n]$. Assume that $x \in P_P^k$. Then $P_P^{ln} \subseteq (\prod_{i=1}^n x_i H_P)^l = (\prod_{i=1}^n x_i)^l H_P \subseteq P_P^{ln+1}$, a contradiction. Consequently, $x \notin P_P^k$ for all $P \in \mathfrak{X}(H)$.
**2.** Let $P \in s\text{-spec}(H)^\bullet$. There are some $x \in P^\bullet$, $n \in \mathbb{N}$ and some sequence $(x_i)_{i=1}^n$ of radical elements of $H$ such that $\sqrt{xH} = \bigcap_{i=1}^n x_i H$, hence $\bigcap_{i=1}^n x_i H \subseteq P$. Consequently, there exists some $i \in [1, n]$ such that $x_i \in P$ and thus $P \cap \mathcal{B}(H) \neq \emptyset$.
**3.** Let $x \in H^\bullet$ and $P \in \mathfrak{X}(H)$. There is some $k \in \mathbb{N}_0$ such that $xH_P = P_P^k$. It is sufficient to show by induction that for every $l \in [0, k]$ there is some $y \in \Omega(H)$ such that $\frac{x}{y} \in H \backslash P^{k+1-l}$. If $l = 0$, then set $y = 1$. Now let $l \in [0, k-1]$ and $z \in \Omega(H)$ be such that $\frac{x}{z} \in H \backslash P^{k+1-l}$. There are some $n \in \mathbb{N}$ and some sequence $(x_i)_{i=1}^n$ of radical elements of $H$ such that $\sqrt{\frac{x}{z} H} = \bigcap_{i=1}^n x_i H$. If $\frac{x}{z} \notin P$, then set $y = z$. Now let $\frac{x}{z} \in P$. There is some $i \in [1, n]$ such that $x_i \in P$. Obviously, $x_i z \in \Omega(H)$ and $\frac{x}{x_i z} \in H \backslash P^{k-l}$. ☐

**Corollary 2.9.** *Let $H$ be a monoid such that $\{\sqrt{xH} \mid x \in H^\bullet \backslash H^\times\}$ satisfies the ACC. The following statements are equivalent:*

**1.** *$H$ is radical factorial.*
**2.** *$\bigcap_{P \in \mathfrak{X}(H)} H_P = H$, $H_P$ is a discrete valuation monoid for all $P \in \mathfrak{V}(H)$ and for all $x \in H$ it follows that $\sqrt{xH}$ is a finite intersection of principal ideals of $H$.*

*Proof.* **1.** $\Rightarrow$ **2.**: This follows from Proposition 2.4.1., Proposition 2.4.4. and Proposition 2.4.5.. **2.** $\Rightarrow$ **1.**: Assume that $H$ is not radical factorial. Then $H^\bullet \backslash \Omega(H) \neq \emptyset$, hence there is some $x \in H^\bullet \backslash \Omega(H)$ such that $\sqrt{xH}$ is maximal in $\{\sqrt{vH} \mid v \in H^\bullet \backslash \Omega(H)\}$. Since $x \notin H^\times$ there is some $P \in \mathfrak{X}(H)$ such that $x \in P$. By Proposition 2.8.3. there is some $y \in \Omega(H)$ such that $\frac{x}{y} \in H \backslash P$. We have $\frac{x}{y} \in H^\bullet \backslash \Omega(H)$ and $\sqrt{xH} \subsetneq \sqrt{\frac{x}{y} H}$, a contradiction. ☐

**Proposition 2.10.** *Let $H$ be a monoid such that $\sqrt{xH}$ is principal for every $x \in H^\bullet$. Then $H$ is radical factorial.*

*Proof.* Claim 1: $\mathcal{P}(xH) \subseteq \mathfrak{X}(H)$ for all $x \in H^\bullet$. Let $x \in H^\bullet$, $P \in \mathcal{P}(xH)$ and $Q \in s\text{-spec}(H)^\bullet$ be such that $Q \subseteq P$. There is some $u \in \mathcal{B}(H)$ such that $\sqrt{xH} = uH$, hence $P \in \mathcal{P}(uH)$. There are some $y \in Q^\bullet$ and $v \in \mathcal{B}(H)$ such that $\sqrt{yH} = vH$. Therefore, $Q \cap \mathcal{B}(H) \neq \emptyset$ and thus $Q = P$ by Lemma 2.3.2..
Claim 2: For all $x \in H^\bullet$ there is some $k \in \mathbb{N}$ such that $x \notin P^{k+1}$ for all $P \in \mathcal{P}(xH)$. Let $x \in H^\bullet$. There is some $y \in H$ such that $\sqrt{xH} = yH$, hence there exists some $k \in \mathbb{N}$ such that $y^k \in xH$. Assume that there is some $P \in \mathcal{P}(xH)$ such that $x \in P^{k+1}$. We have $P \in \mathcal{P}(yH)$, hence $P_P = (yH)_P = yH_P$ by Lemma 2.2.2.. It follows that $y^k \in xH_P \subseteq P_P^{k+1} = y^{k+1} H_P$ and thus $y \in H_P^\times$, a contradiction.
By claim 2 it suffices to show (by induction) that for all $k \in \mathbb{N}_0$ and $x \in H^\bullet$ such that $x \notin P^{k+1}$ for all $P \in \mathcal{P}(xH)$ it follows that $x \in \Omega(H)$. If $x \in H^\bullet$ such that $x \notin P$ for all $P \in \mathcal{P}(xH)$, then $x \in H^\times \subseteq \Omega(H)$. Now let $k \in \mathbb{N}_0$ and $x \in H^\bullet$ be such that $x \notin P^{k+2}$ for all $P \in \mathcal{P}(xH)$. There are some $y \in \mathcal{B}(H) \subseteq \Omega(H)$ and $z \in H^\bullet$ such that $\sqrt{xH} = yH$ and $x = yz$. Assume that there is some $Q \in \mathcal{P}(zH)$ such that $z \in Q^{k+1}$. Then $x \in Q$, hence $y \in Q$. It follows that $x = yz \in Q^{k+2}$ and $Q \in \mathcal{P}(xH)$ by claim 1, a contradiction. Therefore, $z \notin P^{k+1}$ for all $P \in \mathcal{P}(zH)$. By the induction hypothesis we have $z \in \Omega(H)$. Consequently, $x = yz \in \Omega(H)$. ☐

**Lemma 2.11.** *Let $H$ be a GCD-monoid and $x, y \in H$. Then $\{x, y\}_t$ and $xH \cap yH$ are principal.*

*Proof.* See [8, Corollary 11.5.]. □

**Proposition 2.12.** *Let $H$ be a GCD-monoid such that $H^\bullet \neq H^\times$. The following are equivalent:*

1. *$H$ is radical factorial.*
2. *$t$-$\max(H) = \mathfrak{X}(H)$ and $Q \cap \mathcal{B}(H) \neq \emptyset$ for all $Q \in \mathfrak{X}(H)$.*
3. *$\bigcap_{P\in\mathfrak{X}(H)} H_P = H$, $H_Q$ is a discrete valuation monoid for all $Q \in \mathfrak{V}(H)$ and $P \nsubseteq \bigcup_{Q\in\mathfrak{X}(H)} Q_Q^2$ for all $P \in \mathfrak{X}(H)$.*
4. *$\sqrt{xH}$ is principal for all $x \in H$ (equivalently: $\sqrt{I}$ is principal for all $I \in \mathcal{I}_{t,f}(H)$).*

*Proof.* **1.** $\Rightarrow$ **2.**: Let $P \in t$-$\max(H)$. By Lemma 2.1.2. and Lemma 2.11. it follows that $H_P$ is a radical factorial valuation monoid. Therefore, every radical element of $H_P$ is a prime element of $H_P$, hence $H_P$ is factorial and thus $H_P$ is a discrete valuation monoid by [8, Theorem 16.4.]. This implies that $P \in \mathfrak{X}(H)$. Consequently, $t$-$\max(H) = \mathfrak{X}(H)$. It follows by Proposition 2.4.3. that $Q \cap \mathcal{B}(H) \neq \emptyset$ for all $Q \in \mathfrak{X}(H)$.
**2.** $\Rightarrow$ **3.**: This is an easy consequence of Lemma 2.3.1..
**3.** $\Rightarrow$ **1.**: It follows by Lemma 2.7.2. and Lemma 2.11. that $\sqrt{\{u,w\}_t}$ is principal for all $u \in \mathcal{B}(H)$ and $w \in H$. Therefore, Lemma 2.7.3. and Lemma 2.7.4. imply that $\Omega(H)$ is divisor-closed and $P \cap \mathcal{B}(H) \neq \emptyset$ for all $P \in s$-spec$(H)^\bullet$. It is sufficient to show that $\Omega(H) = H^\bullet$. Assume that $\Omega(H) \subsetneq H^\bullet$. Then there is some $x \in H^\bullet \backslash \Omega(H)$. Since $\Omega(H)$ is divisor-closed it follows that $xH \cap \Omega(H) = \emptyset$. Since $\Omega(H)$ is multiplicatively closed, there is some $P \in s$-spec$(H)$ such that $xH \subseteq P$ and $P \cap \Omega(H) = \emptyset$. Consequently, $P \in s$-spec$(H)^\bullet$ and thus $P \cap \Omega(H) \supseteq P \cap \mathcal{B}(H) \neq \emptyset$, a contradiction.
**1.** $\Rightarrow$ **4.**: Let $x \in H$. By Proposition 2.4.1. there exist some $n \in \mathbb{N}$ and some sequence $(x_i)_{i=1}^n$ of elements of $H$ such that $\sqrt{xH} = \bigcap_{i=1}^n x_iH$, hence $\sqrt{xH}$ is principal by Lemma 2.11..
**4.** $\Rightarrow$ **1.**: This follows from Proposition 2.10.. □

**Lemma 2.13.** *Let $H$ be a monoid such that $\sqrt{I}$ is principal for all $I \in \mathcal{I}_{t,f}(H)$.*

1. *Every atom of $H$ is a prime element of $H$.*
2. *If $\{uH \mid u \in \mathcal{B}(H)\}$ satisfies the ACC, then $H$ is factorial.*

*Proof.* **1.** Let $u \in \mathcal{A}(H)$ and $x, y \in H$ be such that $xy \in uH$. There are some $v, w \in H$ such that $\sqrt{\{x,u\}_t} = vH$ and $\sqrt{\{y,u\}_t} = wH$. Since $u \in \mathcal{A}(H)$, $\sqrt{uH}$ is principal and $uH \subseteq \sqrt{uH} \subsetneq H$ we obtain that $\sqrt{uH} = uH$. It follows that $vH \cap wH = \sqrt{\{x,u\}_t} \cap \sqrt{\{y,u\}_t} = \sqrt{\{x,u\}_t\{y,u\}_t} \subseteq \sqrt{\{xy, xu, uy, u^2\}_t} \subseteq \sqrt{uH} = uH$ and thus $uH = vH \cap wH$. This implies that $uH = vH$ or $uH = wH$, hence $x \in uH$ or $y \in uH$.
**2.** Let $\{uH \mid u \in \mathcal{B}(H)\}$ satisfy the ACC. It is sufficient to show that every $Q \in s$-spec$(H)^\bullet$ contains a prime element of $H$. Let $Q \in s$-spec$(H)^\bullet$. Then there is some $x \in Q^\bullet$, hence there is some $P \in \mathcal{P}(xH) \subseteq t$-spec$(H)^\bullet$ such that $P \subseteq Q$. There is some $y \in P \cap \mathcal{B}(H)$ such that $\sqrt{xH} = yH$. Therefore, $\Sigma = \{zH \mid z \in P \cap \mathcal{B}(H)\} \neq \emptyset$ and thus there is some $w \in P \cap \mathcal{B}(H)$ such that $wH$ is a maximal element of $\Sigma$. Assume that $wH \subsetneq P$. Then there is some $v \in P$ such that $v \in P \backslash wH$. Since $\sqrt{\{w,v\}_t} \subseteq P$, there is some $s \in P \cap \mathcal{B}(H)$ such that $\sqrt{\{w,v\}_t} = sH$. On the other hand $wH \subsetneq sH$, a contradiction. Consequently, $P = wH$ and thus $w$ is a prime element of $H$ and $w \in Q$. □

Lemma 2.13.2. shows that the monoids occurring in Corollary 2.9. and Proposition 2.12. are different types of monoids. Next we present the main result in this section.

**Theorem 2.14.** *Let $H$ be a monoid. The following conditions are equivalent:*

1. *$H$ is factorial.*
2. *$H$ is radical factorial, $\{P \in \mathfrak{X}(H) \mid x \in P\}$ is finite for all $x \in H^\bullet$ and $\mathcal{C}_t(H)$ is a torsion group.*
3. *$H$ is atomic and $\sqrt{I}$ is principal for all $I \in \mathcal{I}_{t,f}(H)$.*
4. *$\{uH \mid u \in \mathcal{B}(H)\}$ satisfies the ACC and $\sqrt{I}$ is principal for all $I \in \mathcal{I}_{t,f}(H)$.*
5. *$\{P \in \mathfrak{X}(H) \mid x \in P\}$ is finite and $\sqrt{xH}$ is principal for all $x \in H^\bullet$.*
6. *$\mathcal{P}(xH) \subseteq \mathfrak{X}(H)$ for all $x \in H^\bullet$ and every $P \in \mathfrak{X}(H)$ is principal.*

**7.** *$H$ is radical factorial and weakly factorial.*

*Proof.* By [7, Corollary 2.3.13.] we have $H$ is factorial if and only if $H$ is a Krull monoid and $|\mathcal{C}_v(H)| = 1$.
**1.** $\Rightarrow$ **2.**, **1.** $\Rightarrow$ **7.**: Clear, since every prime element of $H$ is radical and primary.
**1.** $\Leftrightarrow$ **3.**: This follows from Proposition 2.12. and Lemma 2.13.1..
**1.** $\Leftrightarrow$ **4.**: This is an easy consequence of Proposition 2.12. and Lemma 2.13.2..
**2.** $\Rightarrow$ **6.**: By Proposition 2.6. we have $H$ is a Krull monoid and thus $\mathcal{P}(xH) \subseteq \mathfrak{X}(H)$ for all $x \in H^\bullet$ and $v = t$. Let $P \in \mathfrak{X}(H)$. Then $P$ is $v$-invertible and since $\mathcal{C}_v(H)$ is a torsion group it follows that some $v$-power of $P$ is principal. Therefore, Proposition 2.4.2. implies that $P$ is principal.
**1.** $\Rightarrow$ **5.**: This is an easy consequence of Proposition 2.6. and Proposition 2.12..
**5.** $\Rightarrow$ **6.**: It follows by Proposition 2.10. that $H$ is radical factorial. Therefore, $H$ is a Krull monoid by Proposition 2.6. and thus $\mathcal{P}(xH) \subseteq \mathfrak{X}(H)$ for all $x \in H^\bullet$. Let $P \in \mathfrak{X}(H)$. There is some $x \in P \cap \mathcal{B}(H)$ by Proposition 2.4.3., hence there is some finite $\Sigma \subseteq \mathfrak{X}(H)$ such that $P \in \Sigma$ and $xH = (\prod_{Q \in \Sigma} Q)_t$. There are some $z \in \bigcap_{Q \in \Sigma \backslash \{P\}} Q \backslash P$ and $y \in \mathcal{B}(H)$ such that $\sqrt{zH} = yH$. This implies that $y \in \bigcap_{Q \in \Sigma \backslash \{P\}} Q \backslash P$. There is some finite $\Phi \subseteq \mathfrak{X}(H)$ such that $\Sigma \backslash \{P\} \subseteq \Phi$ and $yH = (\prod_{Q \in \Phi} Q)_t$. We have $\sqrt{xyH} = xH \cap yH = (\prod_{Q \in \Sigma} Q)_t \cap (\prod_{Q \in \Phi} Q)_t = \bigcap_{Q \in \Sigma} Q \cap \bigcap_{Q \in \Phi} Q = P \cap \bigcap_{Q \in \Phi} Q = (P \prod_{Q \in \Phi} Q)_t = yP$ is principal, hence $P$ is principal.
**6.** $\Rightarrow$ **1.**: It is sufficient to show that every $P \in s\text{-spec}(H)^\bullet$ contains a prime element of $H$. Let $P \in s\text{-spec}(H)^\bullet$. Then there are some $x \in P^\bullet$ and $Q \in \mathcal{P}(xH) \subseteq \mathfrak{X}(H)$ such that $Q \subseteq P$. There is some prime element $p \in H$ such that $Q = pH$. Consequently, $p \in P$.
**7.** $\Rightarrow$ **1.**: By Proposition 2.4.2. we have every primary element of $H$ is a finite product of prime elements of $H$. Therefore, every $x \in H^\bullet \backslash H^\times$ is a finite product of prime elements of $H$, hence $H$ is factorial. $\square$

**Lemma 2.15.** *Let $(I, \leq)$ be a directed set, $K$ a monoid and $(H_i)_{i \in I}$ a family of radical factorial submonoids of $K$ such that $H_i \subseteq H_j$ and $\mathcal{B}(H_i) \subseteq \mathcal{B}(H_j)$ for all $i, j \in I$ such that $i \leq j$. Then $\bigcup_{i \in I} H_i$ is a radical factorial monoid.*

*Proof.* Set $H = \bigcup_{i \in I} H_i$. First we show that $\mathcal{B}(H_i) \subseteq \mathcal{B}(H)$ for all $i \in I$. Let $i \in I$, $y \in \mathcal{B}(H_i)$ and $z \in \sqrt[H]{yH}$. Then there are some $k \in \mathbb{N}$ and $w \in H$ such that $z^k = yw$. There is some $j \in I$ such that $i \leq j$, $H_i \subseteq H_j$ and $z, w \in H_j$. We have $y \in \mathcal{B}(H_j)$. Therefore, $z \in \sqrt[H_j]{yH_j} = yH_j \subseteq yH$. This implies that $\sqrt{yH} = yH$ and thus $y \in \mathcal{B}(H)$. Now let $x \in H^\bullet$. Then there is some $l \in I$ such that $x \in H_l^\bullet$. Since $x$ is a finite product of radical elements of $H_l$ we have $x$ is a finite product of radical elements of $H$. $\square$

If $R$ is an integral domain, $X$ is an indeterminate over $R$, $k \in \mathbb{N}_0$ and $f \in R[X]$ (or $f \in R[\![X]\!]$), then let $f_k$ denote the $k$-th coefficient (the coefficient belonging to $X^k$) of $f$. The next Lemma is probably well known, but we were not able to find a reference.

**Lemma 2.16.** *Let $R$ be an integrally closed domain, $X$ an indeterminate over $R$ and $I \in \mathcal{I}_t(R[X])$ such that $I \cap R \neq \{0\}$. Then $I \cap R \in \mathcal{I}_t(R)$ and $I = (I \cap R)[X]$.*

*Proof.* It follows by [12, Lemme 2.] that $J = (J \cap R)[X]$ for all $J \in \mathcal{I}_v(R[X])$ such that $J \cap R \neq \{0\}$. First we show that $I = (I \cap R)[X]$. "$\subseteq$": Let $f \in I$. There is some finite $E \subseteq I$ such that $E \cap R \neq \{0\}$ and $f \in E_{v_{R[X]}}$. Set $J = E_{v_{R[X]}}$. Then $f \in J = (J \cap R)[X] \subseteq (I \cap R)[X]$. "$\supseteq$": Trivial. It is well known that $A_t[X] = (A[X])_{t_{R[X]}}$ for every ideal $A$ of $R$. Therefore, $(I \cap R)[X] = I = I_{t_{R[X]}} = ((I \cap R)[X])_{t_{R[X]}} = (I \cap R)_t[X]$, hence $(I \cap R)_t = I \cap R$ and thus $I \cap R \in \mathcal{I}_t(R)$. $\square$

**Proposition 2.17.** *Let $R$ be an integral domain, $K$ a field of quotients of $R$ and $\mathbb{X}$ a set of independent indeterminates over $K$.*

**1.** $\mathcal{B}(R[\mathbb{X}]) \cap R = \mathcal{B}(R)$.
**2.** *If $R[\mathbb{X}]$ is radical factorial, then $R$ is radical factorial.*
**3.** *If $R$ is a radical factorial GCD-domain, then $R[\mathbb{X}]$ is a radical factorial GCD-domain.*

*Proof.* **1.** Let $X \in \mathbb{X}$. Claim: $\mathcal{B}(R) \subseteq \mathcal{B}(R[X])$. Let $y \in \mathcal{B}(R)$, $f \in R[X]$ and $k \in \mathbb{N}$ be such that $f^k \in yR[X]$. It is sufficient to show by induction that $f_j \in yR$ for all $j \in \mathbb{N}_0$. Let $j \in \mathbb{N}_0$ be such that $f_i \in yR$ for all $i \in \mathbb{N}_0$ such that $i < j$. Set $g = \sum_{i=0}^{j-1} f_i X^i$. Then $g \in yR[X]$, hence $(f-g)^k = \sum_{i=0}^{k} \binom{k}{i} f^i(-g)^{k-i} \in yR[X]$. There is some $h \in R[X]$ such that $(f-g)^k = yh$ and thus $f_j^k = yh_{jk} \in yR$. Consequently, $f_j \in yR$.
"$\subseteq$": Let $y \in \mathcal{B}(R[\mathbb{X}]) \cap R$. Let $x \in R$ and $k \in \mathbb{N}$ be such that $x^k \in yR$. Then $x^k \in yR[\mathbb{X}]$, hence $x \in yR[\mathbb{X}]$ and thus there is some $f \in R[\mathbb{X}]$ such that $x = yf$. Observe that $f \in R$, hence $x = yf \in yR$. "$\supseteq$": Let $y \in \mathcal{B}(R)$, $f \in R[\mathbb{X}]$ and $k \in \mathbb{N}$ be such that $f^k \in yR[\mathbb{X}]$. There is some finite $E \subseteq \mathbb{X}$ such that $f \in R[E]$ and $f^k \in yR[E]$. Using induction it follows by the claim that $y \in \mathcal{B}(R[E])$, hence $f \in yR[E] \subseteq yR[\mathbb{X}]$.
**2.** Let $x \in R^\bullet$. Then there are some $n \in \mathbb{N}$ and some sequence $(x_i)_{i=1}^n$ of radical elements of $R[\mathbb{X}]$ such that $x = \prod_{i=1}^n x_i$. Obviously, $x_i \in R$ for every $i \in [1,n]$. Therefore, 1. implies that $x_i \in \mathcal{B}(R[\mathbb{X}]) \cap R = \mathcal{B}(R)$ for all $i \in [1,n]$. Consequently, $R$ is radical factorial.
**3.** Let $R$ be a radical factorial GCD-domain and $X \in \mathbb{X}$. Claim: $R[X]$ is a radical factorial GCD-domain. By Proposition 2.4.5. it follows that $R$ is completely integrally closed. Therefore, $I \cap R \in \mathcal{I}_t(R)$ and $I = (I \cap R)[X]$ for all $I \in \mathcal{I}_t(R[X])$ such that $I \cap R \neq \{0\}$ by Lemma 2.16.. It is well known that $R[X]$ is a GCD-domain. By Proposition 2.12. it suffices to show that $P \in \mathfrak{X}(R[X])$ and $P \cap \mathcal{B}(R[X]) \neq \emptyset$ for all $P \in t\text{-spec}(R[X])^\bullet$. Let $P \in t\text{-spec}(R[X])^\bullet$.
Case 1: $P \cap R = \{0\}$: Set $T = R^\bullet$. Observe that $T^{-1}R[X] = K[X]$, hence $T^{-1}P \in \text{spec}(K[X])^\bullet$. Therefore, there is some prime element $h \in K[X]$ such that $T^{-1}P = hK[X]$. Since $R$ is a GCD-domain there is some $f \in R[X]$ such that $f$ is a prime element of $K[X]$, $1 \in \text{GCD}(\{f_i \mid i \in \mathbb{N}_0\})$ and $T^{-1}P = fK[X]$. Observe that $f$ is a prime element of $R[X]$, hence $f \in \mathcal{B}(R[X])$. Since $f \in T^{-1}P \cap R[X] = P$ we have $P \cap \mathcal{B}(R[X]) \neq \emptyset$. Let $Q \in \text{spec}(R[X])^\bullet$ be such that $Q \subseteq P$. Then $Q \cap R = \{0\}$, $T^{-1}Q \in \text{spec}(K[X])^\bullet$ and $T^{-1}Q \subseteq T^{-1}P$. Therefore, $T^{-1}Q = T^{-1}P$, hence $Q = P$. This implies that $P \in \mathfrak{X}(R[X])$.
Case 2: $P \cap R \neq \{0\}$: We have $P \cap R \in t\text{-spec}(R)^\bullet$. Consequently, there exists some $u \in \mathcal{B}(R)$ such that $u \in P$. By 1. it follows that $u \in \mathcal{B}(R[X])$, hence $P \cap \mathcal{B}(R[X]) \neq \emptyset$. Together with case 1 we obtain that $Q \cap \mathcal{B}(R[X]) \neq \emptyset$ for all $Q \in t\text{-spec}(R[X])^\bullet$. To prove that $P \in \mathfrak{X}(R[X])$ it suffices to show that for all $Q \in t\text{-spec}(R[X])^\bullet$ such that $Q \subseteq P$ it follows that $P = Q$. Let $Q \in t\text{-spec}(R[X])^\bullet$ be such that $Q \subseteq P$. There is some $A \in \mathcal{P}(uR[X])$ such that $A \subseteq P$. Since $A \in t\text{-spec}(R[X])^\bullet$ and $A \cap R \neq \{0\}$ we have $A \cap R \in t\text{-spec}(R)^\bullet$ and $A = (A \cap R)[X]$. We have $A \cap R \subseteq P \cap R$ and thus $A \cap R = P \cap R$ by Proposition 2.12.. This implies that $P = (P \cap R)[X] = (A \cap R)[X] = A \in \mathcal{P}(uR[X])$. Since $Q \cap \mathcal{B}(R[X]) \neq \emptyset$ it follows that $Q = P$ by Lemma 2.3.2..
We have $R[E]$ is a radical factorial GCD-domain for every finite $E \subseteq \mathbb{X}$ by the claim. It is straightforward to prove that $R[\mathbb{X}]$ is a GCD-domain. As in 1. it follows that $\mathcal{B}(R[E]) \subseteq \mathcal{B}(R[F])$ for all finite $E, F \subseteq \mathbb{X}$ such that $E \subseteq F$. Therefore, $R[\mathbb{X}] = \bigcup_{E \subseteq \mathbb{X}, |E| < \infty} R[E]$ is radical factorial by Lemma 2.15.. ☐

## 3. $r$-SP-monoids

Let $H$ be a monoid, $r$ a finitary ideal system on $H$, $Q \in r\text{-max}(H)$, $\Sigma \subseteq r\text{-spec}(H)$ and $(I_i)_{i \in \mathbb{N}}$ a sequence of $r$-ideals of $H$.

- $H$ is called an $r$-SP-monoid if every $r$-ideal of $H$ is a finite $r$-product of radical $r$-ideals of $H$.
- $H$ is called an almost $r$-Dedekind monoid if $H_M$ is a discrete valuation monoid for all $M \in r\text{-max}(H)$.
- $H$ is called an $r$-Prüfer monoid if $H_M$ is a valuation monoid for all $M \in r\text{-max}(H)$.
- $Q$ is called $r$-critical if for all $I \in \mathcal{I}_{r,f}(H)$ such that $I \subseteq Q$ there is some $M \in r\text{-max}(H)$ such that $I \subseteq (M^2)_r$.
- $\Sigma$ is called $r$-closed if there is some $I \in \mathcal{I}_r(H)$ such that $\Sigma = \{P \in r\text{-spec}(H) \mid I \subseteq P\}$.
- $(I_i)_{i \in \mathbb{N}}$ is called formally infinite if there is some $k \in \mathbb{N}$ such that $I_l = H$ for all $l \in \mathbb{N}_{\geq k}$.

Observe that $H$ is an $r$-Prüfer monoid if and only if every $I \in \mathcal{I}_{r,f}^\bullet(H)$ is $r$-invertible. Obviously, every valuation monoid and every almost $r$-Dedekind monoid is an $r$-Prüfer monoid. Note that if $H$ is an almost $r$-Dedekind monoid, then $\mathcal{P}(I) = \{P \in r\text{-spec}(H) \mid I \subseteq P\}$ for each $I \in \mathcal{I}_r(H)^\bullet$. Observe that every intersection of $r$-closed subsets of $r$-spec$(H)$ is $r$-closed. Recall that $I\in\mathcal{I}_r(H)$ is $r$-cancellative if for all $A,B\in\mathcal{I}_r(H)$ such that $(IA)_r=(IB)_r$ it follows that $A=B$. It is easy to prove that if $H$ is an almost $r$-Dedekind monoid, then each $I\in\mathcal{I}_r(H)^\bullet$ is $r$-cancellative.

**Lemma 3.1.** *Let $H$ be a monoid and $r$ a finitary ideal system on $H$ such that $H$ is an $r$-Prüfer monoid.*

1. *If $P\in r\text{-spec}(H)$ and $I,J\in\mathcal{I}_r(H)$ are $P$-primary, then $(IJ)_r$ is $P$-primary.*
2. *If $I,J\in\mathcal{I}_{r,f}(H)$, then $I\cap J\in\mathcal{I}_{r,f}(H)$.*

*Proof.* See [8, Theorem 17.4.] and [8, Chapter 17, Exercise 8.]. □

**Lemma 3.2.** *Let $H$ be a monoid, $r$ a finitary ideal system on $H$ such that $H$ is an almost $r$-Dedekind monoid and $I\in\mathcal{I}_r(H)^\bullet$. Then $\sqrt{I}=I$ if and only if $I\nsubseteq(Q^2)_r$ for all $Q\in r\text{-max}(H)$.*

*Proof.* Let $M\in r\text{-max}(H)$. It follows by Lemma 3.1.1. that $M_M^2\cap H=((M^2)_r)_M\cap H=(M^2)_r$, hence $(\sqrt{I})_M=I_M$ if and only if ${}^{H_M}\!\sqrt{I_M}=I_M$ if and only if $I_M\in\{M_M,H_M\}$ if and only if $I_M\nsubseteq M_M^2$ if and only if $I\nsubseteq(M^2)_r$. Therefore, $\sqrt{I}=I$ if and only if $(\sqrt{I})_Q=I_Q$ for all $Q\in r\text{-max}(H)$ if and only if $I\nsubseteq(Q^2)_r$ for all $Q\in r\text{-max}(H)$. □

The next Theorem is the first main result in this section. It generalizes [11, Theorem 2.1.] (together with the following Corollary).

**Theorem 3.3.** *Let $H$ be a monoid, $r$ a finitary ideal system on $H$ such that $H$ is an almost $r$-Dedekind monoid and $I\in\mathcal{I}_r(H)^\bullet$. For $i\in\mathbb{N}$ set $V_i(I)=\{M\in r\text{-max}(H)\mid I\subseteq(M^i)_r\}$ and $I_i=\bigcap_{M\in V_i(I)}M$. Let $(J_j)_{j\in\mathbb{N}}$ be such that $J_1=I$ and $J_{j+1}=(J_j:_H\sqrt{J_j})$ for all $j\in\mathbb{N}$.*

1. *Let $I\in\mathcal{I}_{r,f}(H)^\bullet$. The following statements are equivalent:*
   - **a.** *$I$ is a finite $r$-product of radical $r$-ideals of $H$.*
   - **b.** *For all $M\in r\text{-max}(H)$ such that $I\subseteq M$ there is some $J\in\mathcal{I}_{\sqrt{r},f}(H)^\bullet$ such that $J\subseteq M$.*
   - **c.** *Every $M\in r\text{-max}(H)$ such that $I\subseteq M$ is not $r$-critical.*
   - **d.** *$\sqrt{J}\in\mathcal{I}_{r,f}(H)$ for all $J\in\mathcal{I}_{r,f}(H)$ such that $I\subseteq J$.*
   - **e.** *For every $J\in\mathcal{I}_{r,f}(H)$ such that $I\subseteq J$ we have $J$ is a finite $r$-product of radical $r$-ideals of $H$.*
2. *The following conditions are equivalent:*
   - **a.** *$I$ is a finite $r$-product of radical $r$-ideals of $H$.*
   - **b.** *$V_j(I)$ is $r$-closed for all $j\in\mathbb{N}$ and there is some $n\in\mathbb{N}_0$ such that $V_l(I)=\emptyset$ for all $l\in\mathbb{N}_{>n}$.*

   *If these equivalent conditions are satisfied, then $(I_i)_{i\in\mathbb{N}}=(\sqrt{J_i})_{i\in\mathbb{N}}$ is the unique formally infinite sequence $(A_i)_{i\in\mathbb{N}}$ of radical $r$-ideals of $H$ such that $A_j\subseteq A_{j+1}$ for all $j\in\mathbb{N}$ and $I=(\prod_{i\in\mathbb{N}}A_i)_r$.*

*Proof.* **1.a. ⇒ 1.b.**: Let $M\in r\text{-max}(H)$ be such that $I\subseteq M$. There exist some $n\in\mathbb{N}$ and some sequence $(I_i)_{i=1}^n$ of radical $r$-ideals of $H$ such that $I=(\prod_{i=1}^n I_i)_r$. Since $(\prod_{i=1}^n I_i)_r\subseteq M$ there is some $j\in[1,n]$ such that $I_j\subseteq M$. Set $J=I_j$. Since $I$ is $r$-invertible and $I=(\prod_{i=1}^n I_j)_r$ it follows that $J$ is $r$-invertible, hence $J\in\mathcal{I}_{r,f}(H)^\bullet$.
**1.b. ⇒ 1.c.**: This follows from Lemma 3.2..
**1.c. ⇒ 1.d.**: Let $J\in\mathcal{I}_{r,f}(H)$ be such that $I\subseteq J$, $K$ a quotient monoid of $H$ and $M\in r\text{-max}(H)$. There is some $n\in\mathbb{N}_0$ such that $J_M=M_M^n$. Therefore, $J\nsubseteq(M^{n+1})_r$. Claim: For every $i\in[0,n]$ there exist some $A,B\in\mathcal{I}_{r,f}(H)$ such that $A\nsubseteq(M^{n-i+1})_r$, $\sqrt{B}\in\mathcal{I}_{r,f}(H)$ and $J=(AB)_r$. We prove the claim by induction on $i$. If $i=0$, then set $A=J$ and $B=H$. Now let $i\in[0,n-1]$ and $C,D\in\mathcal{I}_{r,f}(H)$ be such that $C\nsubseteq(M^{n-i+1})_r$, $\sqrt{D}\in\mathcal{I}_{r,f}(H)$ and $J=(CD)_r$. If $C\nsubseteq M$, then set $A=C$ and $B=D$. Since $A\nsubseteq(M^{n-i})_r$ we are done. Now let $C\subseteq M$. There is some $N\in\mathcal{I}_{r,f}(H)$ such that $N\subseteq M$ and $N\nsubseteq(Q^2)_r$ for all $Q\in r\text{-max}(H)$. Set $L=(C\cup N)_r$. Then $L\in\mathcal{I}_{r,f}(H)^\bullet$. It follows by Lemma 3.2. that $\sqrt{L}=L$. Set $A=(CL^{-1})_r$ and $B=(LD)_r$. Since $L$ is $r$-invertible it follows that $A,B\in\mathcal{I}_{r,f}(H)$ and $J=(AB)_r$. We have $\sqrt{B}=L\cap\sqrt{D}\in\mathcal{I}_{r,f}(H)$ by Lemma 3.1.2.. If $A\subseteq(M^{n-i})_r$, then $C=(AL)_r\subseteq(M^{n-i+1})_r$, a contradiction. Therefore, $A\nsubseteq(M^{n-i})_r$.
By the claim there exist some $A,B\in\mathcal{I}_{r,f}(H)$ such that $A\nsubseteq M$, $\sqrt{B}\in\mathcal{I}_{r,f}(H)$ and $J=(AB)_r$. This

implies that $\sqrt{J} = \sqrt{A} \cap \sqrt{B}$ and $(\sqrt{A})_M = H_M$, hence $(\sqrt{J})_M = (\sqrt{A})_M \cap (\sqrt{B})_M = (\sqrt{B})_M$. Since $\sqrt{B} \in \mathcal{I}_{r,f}(H)$ this implies that $(H_M :_K (\sqrt{J})_M) = (H_M :_K (\sqrt{B})_M) = ((\sqrt{B})^{-1})_M \subseteq ((\sqrt{J})^{-1})_M \subseteq (H_M :_K (\sqrt{J})_M)$. Consequently, $((\sqrt{J}(\sqrt{J})^{-1})_r)_M = ((\sqrt{J})_M(H_M :_K (\sqrt{J})_M))_{r_M} = H_M$. Since $M \in r\text{-max}(H)$ was arbitrary we have $(\sqrt{J}(\sqrt{J})^{-1})_r = H$, hence $\sqrt{J}$ is $r$-invertible and thus $\sqrt{J} \in \mathcal{I}_{r,f}(H)$.
**1.d.** $\Rightarrow$ **1.e.**: Claim: For every $J \in \mathcal{I}_{r,f}(H)$ such that $I \subseteq J$ there is some $k \in \mathbb{N}_0$ such that $V_{k+1}(J) = \emptyset$. Let $J \in \mathcal{I}_{r,f}(H)$ be such that $I \subseteq J$. Since $\sqrt{J} \in \mathcal{I}_{r,f}(H)$ there is some $k \in \mathbb{N}_0$ such that $(\sqrt{J})^k \subseteq J$. Assume that $V_{k+1}(J) \neq \emptyset$. Then there is some $M \in r\text{-max}(H)$ such that $(\sqrt{J})^k \subseteq (M^{k+1})_r$ and thus $M_M^k \subseteq (\sqrt{J})_M^k \subseteq ((M^{k+1})_r)_M = M_M^{k+1}$, a contradiction. Consequently, $V_{k+1}(J) = \emptyset$.
It is sufficient to show by induction that for all $k \in \mathbb{N}_0$ and $J \in \mathcal{I}_{r,f}(H)^\bullet$ such that $I \subseteq J$ and $V_{k+1}(J) = \emptyset$ it follows that $J$ is a finite $r$-product of radical $r$-ideals of $H$. Let $k \in \mathbb{N}_0$. If $k = 0$ and $J \in \mathcal{I}_{r,f}(H)^\bullet$ such that $I \subseteq J$ and $V_{k+1}(J) = \emptyset$, then $J = H$. Now let $J \in \mathcal{I}_{r,f}(H)^\bullet$ be such that $I \subseteq J$ and $V_{k+2}(J) = \emptyset$. Since $\sqrt{J} \in \mathcal{I}_{r,f}(H)^\bullet$ it follows that $\sqrt{J}$ is $r$-invertible. Let $N = (J(\sqrt{J})^{-1})_r$. Then $N \in \mathcal{I}_{r,f}(H)^\bullet$ and $J = (\sqrt{J}N)_r$. Assume that $V_{k+1}(N) \neq \emptyset$. There is some $M \in r\text{-max}(H)$ such that $N \subseteq (M^{k+1})_r$. Then $J = (\sqrt{J}N)_r \subseteq (\sqrt{J}M^{k+1})_r$, hence $J \subseteq M$. Therefore, $\sqrt{J} \subseteq M$ and thus $J \subseteq (M^{k+2})_r$, a contradiction. Consequently, $V_{k+1}(N) = \emptyset$. It follows by the induction hypothesis that $N$ is a finite $r$-product of radical $r$-ideals of $H$. Since $J = (\sqrt{J}N)_r$ we have $J$ is a finite $r$-product of radical $r$-ideals of $H$.
**1.e.** $\Rightarrow$ **1.a.**: Trivial.
**2.a.** $\Rightarrow$ **2.b.**: Let $n \in \mathbb{N}$ and $(A_i)_{i=1}^n$ a finite sequence of radical $r$-ideals of $H$ be such that $I = (\prod_{i=1}^n A_i)_r$. Set $B_j = \bigcap_{\Sigma \subseteq [1,n], |\Sigma| \geq j} (\bigcup_{i \in \Sigma} A_i)_r$ for $j \in \mathbb{N}$. It is sufficient to show that $V_j(I) = \mathcal{P}(B_j)$ for all $j \in \mathbb{N}$, since then $V_l(I) = \mathcal{P}(B_l) = \mathcal{P}(H) = \emptyset$ for all $l \in \mathbb{N}_{>n}$. Let $j \in \mathbb{N}$. "$\subseteq$": Let $M \in V_j(I)$ and $\Sigma = \{i \in [1,n] \mid A_i \subseteq M\}$. Then $M_M^{|\Sigma|} = \prod_{i \in \Sigma} (A_i)_M = \prod_{i=1}^n (A_i)_M = ((\prod_{i=1}^n A_i)_r)_M = I_M \subseteq ((M^j)_r)_M = M_M^j$, hence $|\Sigma| \geq j$. Therefore, $B_j \subseteq (\bigcup_{i \in \Sigma} A_i)_r \subseteq M$, hence $M \in \mathcal{P}(B_j)$. "$\supseteq$": Let $M \in \mathcal{P}(B_j)$. There is some $\Sigma \subseteq [1,n]$ such that $|\Sigma| \geq j$ and $(\bigcup_{i \in \Sigma} A_i)_r \subseteq M$. It follows that $I \subseteq (\prod_{i \in \Sigma} A_i)_r \subseteq (M^{|\Sigma|})_r \subseteq (M^j)_r$. Consequently, $M \in V_j(I)$.
**2.b.** $\Rightarrow$ **2.a.**: There is some $n \in \mathbb{N}_0$ such that $V_l(I) = \emptyset$ for all $l \in \mathbb{N}_{>n}$, hence $I_l = H$ for all $l \in \mathbb{N}_{>n}$. Therefore, $(I_i)_{i \in \mathbb{N}}$ is a formally infinite sequence of radical $r$-ideals of $H$. Obviously, $I_i \subseteq I_{i+1}$ for all $i \in \mathbb{N}$. Let $M \in r\text{-max}(H)$. There is some $k \in \mathbb{N}_0$ such that $I_M = M_M^k = ((M^k)_r)_M$. By Lemma 3.1.1. we have $I \subseteq (M^k)_r$ and $I \nsubseteq (M^{k+1})_r$. Therefore, for all $i \in [1,n]$ it follows that $I_i \subseteq M$ if and only if $i \leq k$. Consequently, $((\prod_{i=1}^n I_i)_r)_M = \prod_{i=1}^n (I_i)_M = \prod_{i=1}^k (I_i)_M = M_M^k = I_M$. Since $M \in r\text{-max}(H)$ was arbitrary this implies that $(\prod_{i \in \mathbb{N}} I_i)_r = (\prod_{i=1}^n I_i)_r = I$.
Now let $(A_i)_{i \in \mathbb{N}}$ be a formally infinite sequence of radical $r$-ideals of $H$ such that $A_j \subseteq A_{j+1}$ for all $j \in \mathbb{N}$ and $I = (\prod_{i \in \mathbb{N}} A_i)_r$. It suffices to show by induction on $l$ that $A_l = I_l = \sqrt{J_l}$ and $J_l = (\prod_{i \in \mathbb{N}_{\geq l}} I_i)_r = (\prod_{i \in \mathbb{N}_{\geq l}} A_i)_r$ for all $l \in \mathbb{N}$. Clearly, $J_1 = I = (\prod_{i \in \mathbb{N}} I_i)_r = (\prod_{i \in \mathbb{N}} A_i)_r$. Since $(I_i)_{i \in \mathbb{N}}$ and $(A_i)_{i \in \mathbb{N}}$ are ascending sequences we obtain that $\mathcal{P}(A_1) = \mathcal{P}(I_1) = \mathcal{P}(J_1)$ and thus $A_1 = I_1 = \sqrt{J_1}$. Now let $l \in \mathbb{N}$ and let the assertion be true for $l$. We have $J_l = (\sqrt{J_l} \prod_{i \in \mathbb{N}_{\geq l+1}} I_i)_r = (\sqrt{J_l} \prod_{i \in \mathbb{N}_{\geq l+1}} A_i)_r$. Therefore, $(\prod_{i \in \mathbb{N}_{\geq l+1}} I_i)_r \subseteq (J_l :_H \sqrt{J_l}) = J_{l+1}$ and thus $J_l = (\sqrt{J_l} J_{l+1})_r$. Since $H$ is an almost $r$-Dedekind monoid we have $\sqrt{J_l}$ is $r$-cancellative, hence $J_{l+1} = (\prod_{i \in \mathbb{N}_{\geq l+1}} I_i)_r = (\prod_{i \in \mathbb{N}_{\geq l+1}} A_i)_r$. Since $(I_i)_{i \in \mathbb{N}}$ and $(A_i)_{i \in \mathbb{N}}$ are ascending sequences it follows that $\mathcal{P}(A_{l+1}) = \mathcal{P}(I_{l+1}) = \mathcal{P}(J_{l+1})$ and thus $A_{l+1} = I_{l+1} = \sqrt{J_{l+1}}$. $\square$

**Corollary 3.4.** *Let $H$ be a monoid and $r$ a finitary ideal system on $H$ such that $H$ is an almost $r$-Dedekind monoid. The following conditions are equivalent:*

1. *$H$ is an $r$-SP-monoid.*
2. *$I$ is a finite $r$-product of radical $r$-ideals of $H$ for all $I \in \mathcal{I}_{r,f}(H)$.*
3. *$xH$ is a finite $r$-product of radical $r$-ideals of $H$ for all $x \in H$.*
4. *$\sqrt{I} \in \mathcal{I}_{r,f}(H)$ for all $I \in \mathcal{I}_{r,f}(H)$.*
5. *$\sqrt{xH} \in \mathcal{I}_{r,f}(H)$ for all $x \in H$.*
6. *For every $M \in r\text{-max}(H)$ there exists some $I \in \mathcal{I}_{\sqrt{r},f}(H)^\bullet$ such that $I \subseteq M$.*

**7.** *Every $M \in r$-max$(H)$ is not $r$-critical.*

*Proof.* **1.** $\Rightarrow$ **2.**, **2.** $\Rightarrow$ **3.**, **4.** $\Rightarrow$ **5.**, **5.** $\Rightarrow$ **6.**: Trivial. **2.** $\Rightarrow$ **4.**, **3.** $\Rightarrow$ **5.**, **6.** $\Rightarrow$ **7.**, **7.** $\Rightarrow$ **2.**: Are an immediate consequence of Theorem 3.3.1.. **2.** $\Rightarrow$ **1.**: For $i \in \mathbb{N}$ and $J \in \mathcal{I}_r(H)$ set $V_i(J) = \{M \in r\text{-max}(H) \mid J \subseteq (M^i)_r\}$. Let $I \in \mathcal{I}_r(H)^\bullet$. There is some $A \in \mathcal{I}_{r,f}(H)^\bullet$ such that $A \subseteq I$. By Theorem 3.3.2. there is some $n \in \mathbb{N}_0$ such that $V_l(A) = \emptyset$ for all $l \in \mathbb{N}_{>n}$. Obviously, $V_l(I) \subseteq V_l(A)$ for all $l \in \mathbb{N}_{>n}$ and thus $V_l(I) = \emptyset$ for all $l \in \mathbb{N}_{>n}$. Let $j \in \mathbb{N}$. Since $I = \bigcup_{J \in \mathcal{I}_{r,f}(H)^\bullet, J \subseteq I} J$ it follows by Theorem 3.3.2. that $V_j(I) = \bigcap_{J \in \mathcal{I}_{r,f}(H)^\bullet, J \subseteq I} V_j(J)$ is $r$-closed. Consequently, $I$ is a finite $r$-product of radical $r$-ideals of $H$ by Theorem 3.3.2.. □

Let $H$ be a monoid and $r$ a finitary ideal system on $H$.

- We say that $H$ is primary $r$-ideal inclusive if for all $P, Q \in r$-spec$(H)$ such that $P \subsetneq Q$, there is some primary $I \in \mathcal{I}_r(H)$ such that $P \subseteq I \subsetneq \sqrt{I} \subseteq Q$.
- We say that $H$ satisfies the $r$-prime power condition if for every primary $Q \in \mathcal{I}_r(H)$ there is some $k \in \mathbb{N}$ such that $Q = ((\sqrt{Q})^k)_r$.

Note that every almost $r$-Dedekind monoid satisfies the $r$-prime power condition.

**Lemma 3.5.** *Let $H$ be a monoid, $r$ a finitary ideal system on $H$, $I \in \mathcal{I}_r(H)$, $P \in \mathcal{P}(I)$ and $J = I_P \cap H$. Then $J \in \mathcal{I}_r(H)$ and $J$ is $P$-primary.*

*Proof.* This follows from [8, Theorem 7.3.]. □

**Lemma 3.6.** *Let $R$ be an integral domain. Then $R$ is primary $d$-ideal inclusive.*

*Proof.* Let $P, Q \in$ spec$(R)$ be such that $P \subsetneq Q$. Then there is some $y \in Q \backslash P$. There exists some $M \in \mathcal{P}(P + y^2R)$ such that $M \subseteq Q$. Set $I = (P + y^2R)_M \cap R$. By Lemma 3.5. it follows that $I \in \mathcal{I}_d(R)$, $I$ is $M$-primary and $P \subseteq I \subseteq Q$. Assume that $I = M$. Then $y \in (P + y^2R)_M$, hence there is some $t \in R \backslash M$ such that $yt \in P + y^2R$. Consequently, there exists some $z \in R$ such that $y(t + yz) \in P$, hence $t + yz \in P \subseteq M$ and thus $t \in M$, a contradiction. This implies that $I \subsetneq M$. □

**Lemma 3.7.** *Let $H$ be a monoid. Then $H$ is primary $s$-ideal inclusive.*

*Proof.* Let $P, Q \in s$-spec$(H)$ be such that $P \subsetneq Q$. Then there is some $y \in Q \backslash P$. There exists some $M \in \mathcal{P}(P \cup y^2H)$ such that $M \subseteq Q$. Set $I = (P \cup y^2H)_M \cap H$. By Lemma 3.5. it follows that $I \in \mathcal{I}_s(H)$, $I$ is $M$-primary and $P \subseteq I \subseteq Q$. Assume that $I = M$. Then $y \in (P \cup y^2H)_M$, hence there is some $t \in H \backslash M$ such that $yt \in P \cup y^2H$. Since $yt \notin P$ it follows that $yt \in y^2H$ and thus $t \in yH \subseteq M$, a contradiction. This implies that $I \subsetneq M$. □

**Lemma 3.8.** *Let $H$ be a monoid, $r$ a finitary ideal system on $H$ and $T \subseteq H^\bullet$ multiplicatively closed.*

1. *If $H$ is primary $r$-ideal inclusive, then $T^{-1}H$ is primary $T^{-1}r$-ideal inclusive.*
2. *If $H$ is an $r$-SP-monoid, then $T^{-1}H$ is an $T^{-1}r$-SP-monoid.*
3. *If $H$ satisfies the $r$-prime power condition, then $T^{-1}H$ satisfies the $T^{-1}r$-prime power condition.*
4. *Let $H_M$ be primary $r_M$-ideal inclusive for all $M \in r$-max$(H)$. Then $H$ is primary $r$-ideal inclusive.*

*Proof.* In the following we use some facts about primary $r$-ideals which can be found in [8, Theorem 7.2.].
**1.** Let $H$ be primary $r$-ideal inclusive. Let $A, B \in T^{-1}r$-spec$(T^{-1}H)$ be such that $A \subsetneq B$. Set $P = A \cap H$ and $Q = B \cap H$. Then $P, Q \in r$-spec$(H)$ and $P \subsetneq Q$, hence there is some primary $I \in \mathcal{I}_r(H)$ such that $P \subseteq I \subsetneq \sqrt{I} \subseteq Q$. Set $J = T^{-1}I$. Then $J \in \mathcal{I}_{T^{-1}r}(T^{-1}H)$, $J$ is primary and $A = T^{-1}P \subseteq J \subseteq T^{-1}Q = B$. Since $J \cap H = I$ and ${}^{T^{-1}H}\!\!\sqrt{J} \cap H = \sqrt{I}$ it follows that $A \subseteq J \subsetneq {}^{T^{-1}H}\!\!\sqrt{J} \subseteq B$.
**2.** Let $H$ be an $r$-SP-monoid and $J \in \mathcal{I}_{T^{-1}r}(T^{-1}H)$. There is some $I \in \mathcal{I}_r(H)$ such that $J = T^{-1}I$, hence there exist some $k \in \mathbb{N}$ and some sequence $(I_i)_{i=1}^k$ of radical $r$-ideals of $H$ such that $I = (\prod_{i=1}^k I_i)_r$. Observe that $(T^{-1}I_i)_{i=1}^k$ is a sequence of radical $T^{-1}r$-ideals of $T^{-1}H$ and $T^{-1}I = (\prod_{i=1}^k T^{-1}I_i)_{T^{-1}r}$. Consequently, $T^{-1}H$ is an $T^{-1}r$-SP-monoid.

**3.** Let $H$ satisfy the $r$-prime power condition. Let $I \in \mathcal{I}_{T^{-1}r}(T^{-1}H)$ be primary. Set $J = I \cap H$. Then $J \in \mathcal{I}_r(H)$ and $J$ is primary. There is some $k \in \mathbb{N}$ such that $J = ((\sqrt{J})^k)_r$. Consequently, $I = T^{-1}J = T^{-1}((\sqrt{J})^k)_r = ((T^{-1}\sqrt{J})^k)_{T^{-1}r} = ((\sqrt[T^{-1}H]{I})^k)_{T^{-1}r}$.
**4.** Let $P, Q \in r\text{-spec}(H)$ be such that $P \subsetneq Q$. There is some $M \in r\text{-max}(H)$ such that $Q \subseteq M$. We have $P_M, Q_M \in r_M\text{-spec}(H_M)$ and $P_M \subsetneq Q_M$. Therefore, there is some primary $J \in \mathcal{I}_{r_M}(H_M)$ such that $P_M \subseteq J \subsetneq \sqrt[H_M]{J} \subseteq Q_M$. Let $I = J \cap H$. Then $I \in \mathcal{I}_r(H)$, $I$ is primary and $P = P_M \cap H \subseteq I \subseteq Q_M \cap H = Q$. Assume that $I = \sqrt{I}$. Then $J = I_M = \sqrt[H_M]{I_M} = \sqrt[H_M]{J}$, a contradiction. Consequently, $P \subseteq I \subsetneq \sqrt{I} \subseteq Q$. □

**Proposition 3.9.** *Let $H$ be a monoid and $r$ a finitary ideal system on $H$ such that $H$ is an $r$-Prüfer monoid or $r\text{-max}(H) = \mathfrak{X}(H)$. Then $H$ is primary $r$-ideal inclusive.*

*Proof.* Case 1: $H$ is an $r$-Prüfer monoid: Let $M \in r\text{-max}(H)$. Then $H_M$ is a valuation monoid, hence $r_M = s(H_M)$. Therefore, Lemma 3.7. implies that $H_M$ is primary $r_M$-ideal inclusive. Consequently, $H$ is primary $r$-ideal inclusive by Lemma 3.8.4.. Case 2: $r\text{-max}(H) = \mathfrak{X}(H)$: Let $P, Q \in r\text{-spec}(H)$ be such that $P \subsetneq Q$. Then $P = \{0\}$. There is some $x \in Q^\bullet$. Let $I = x^2 H_Q \cap H$. Since $Q \in \mathcal{P}(x^2H)$ it follows by Lemma 3.5. that $I \in \mathcal{I}_r(H)$ and $I$ is $Q$-primary. Assume that $I = Q$. Then $x \in I$, hence $x \in H_Q^\times$, a contradiction. Therefore, $P \subseteq I \subsetneq \sqrt{I} = Q$. □

Recall that if $H$ is a monoid, then $\mathfrak{V}(H)$ is the set of all prime divisors of non-zero principal ideals of $H$.

**Proposition 3.10.** *Let $H$ be a monoid and $r$ a finitary ideal system on $H$ such that $H$ is an $r$-SP-monoid.*

1. *$H$ satisfies the $r$-prime power condition.*
2. *If $\mathcal{C}_r(H)$ is trivial, then $H$ is radical factorial.*
3. *$\bigcap_{P \in \mathfrak{X}(H)} H_P = H$, $H_P$ is a discrete valuation monoid for all $P \in \mathfrak{V}(H)$ and $H$ is completely integrally closed.*

*Proof.* **1.** Let $Q \in \mathcal{I}_r(H)$ be primary. There are some $n \in \mathbb{N}$ and some sequence $(I_i)_{i=1}^n$ of radical $r$-ideals of $H$ such that $Q = (\prod_{i=1}^n I_i)_r$. Let $\Sigma = \{i \in [1,n] \mid I_i \neq \sqrt{Q}\}$ and $k = |[1,n] \setminus \Sigma|$. Observe that $k \in \mathbb{N}$. It follows that $Q = ((\sqrt{Q})^k \prod_{i \in \Sigma} I_i)_r$. Assume that $Q \neq ((\sqrt{Q})^k)_r$. Then $(\sqrt{Q})^k \nsubseteq Q$ and $(\sqrt{Q})^k \prod_{i \in \Sigma} I_i \subseteq Q$. Since $Q$ is primary this implies that $\prod_{i \in \Sigma} I_i \subseteq \sqrt{Q}$. Consequently, there exists some $j \in \Sigma$ such that $I_j \subseteq \sqrt{Q}$, hence $I_j = \sqrt{Q}$, a contradiction. Therefore, $Q = ((\sqrt{Q})^k)_r$.
**2.** Let $\mathcal{C}_r(H)$ be trivial and $x \in H^\bullet$. Then there are some $n \in \mathbb{N}$ and some sequence of $(I_i)_{i=1}^n$ of radical $r$-ideals of $H$ such that $xH = (\prod_{i=1}^n I_i)_r$. Observe that $I_i$ is $r$-invertible and thus $I_i$ is principal for every $i \in [1,n]$. Therefore, $xH$ is a finite product of radical principal ideals of $H$.
**3.** By Lemma 3.8.2. and 2. we have $H_M$ is radical factorial for all $M \in r\text{-max}(H)$. Moreover, it follows that $\mathfrak{X}(H_M) = \{P_M \mid P \in \mathfrak{X}(H), P \subseteq M\}$ and $\bigcap_{Q \in \mathfrak{X}(H_M)} (H_M)_Q = \bigcap_{P \in \mathfrak{X}(H), P \subseteq M} (H_M)_{P_M} = \bigcap_{P \in \mathfrak{X}(H), P \subseteq M} H_P$ for all $M \in r\text{-max}(H)$. Therefore, Proposition 2.4.5. implies that $\bigcap_{P \in \mathfrak{X}(H)} H_P = \bigcap_{M \in r\text{-max}(H)} (\bigcap_{P \in \mathfrak{X}(H), P \subseteq M} H_P) = \bigcap_{M \in r\text{-max}(H)} (\bigcap_{Q \in \mathfrak{X}(H_M)} (H_M)_Q) = \bigcap_{M \in r\text{-max}(H)} H_M = H$. Now let $P \in \mathfrak{V}(H)$. There is some $Q \in r\text{-max}(H)$ such that $P \subseteq Q$. It follows that $P_Q \in \mathfrak{V}(H_Q)$ and thus Proposition 2.4.4. implies that $H_P = (H_Q)_{P_Q}$ is a discrete valuation monoid. We have $H_M$ is completely integrally closed for all $M \in r\text{-max}(H)$ by Proposition 2.4.5., hence $H$ is completely integrally closed. □

Note that every $r$-SP-monoid fulfills the Principal Ideal Theorem by Proposition 3.10.3.. Next we provide a few conditions that enforce a radical factorial monoid to be an $r$-SP-monoid.

**Proposition 3.11.** *Let $H$ be a radical factorial monoid such that $H^\bullet \neq H^\times$ and $r$ a finitary ideal system on $H$. If $r\text{-max}(H) = \mathfrak{X}(H)$ or $H$ is an $r$-Prüfer monoid, then $H$ is an $r$-SP-monoid.*

*Proof.* Case 1: $r\text{-max}(H) = \mathfrak{X}(H)$: By Proposition 2.4.4. we have $H$ is an almost $r$-Dedekind monoid. Case 2: $H$ is an $r$-Prüfer monoid: Let $P \in r\text{-max}(H)$. It follows by Lemma 2.1.2. that $H_P$ is a radical factorial valuation monoid. Moreover, every radical element of $H_P$ is a prime element of $H_P$. Therefore,

$H_P$ is a factorial valuation monoid, hence $H_P$ is a discrete valuation monoid by [8, Theorem 16.4.]. In any case $H$ is an almost $r$-Dedekind monoid and thus $H$ is an $r$-SP-monoid by Corollary 3.4.. □

**Lemma 3.12.** *Let $H$ be a monoid, $r$ a finitary ideal system on $H$ such that $H$ is $r$-local, $H$ is primary $r$-ideal inclusive, $M = H\backslash H^\times$ and $I \in \mathcal{I}_r(H)$ such that $I \subsetneqq \sqrt{I} = M$.*

1. *If $H$ satisfies the $r$-prime power condition, then $M$ is principal.*
2. *If $H$ satisfies the $r$-prime power condition and $H$ is radical factorial, then $H$ is a discrete valuation monoid.*
3. *If $H$ is an $r$-SP-monoid, then $H$ is a discrete valuation monoid.*

*Proof.* **1.** Let $H$ satisfy the $r$-prime power condition. Since $I$ is $M$-primary there exists some $k \in \mathbb{N}_{\geq 2}$ such that $I = (M^k)_r$, hence $(M^k)_r \subsetneqq M$ and thus $(M^2)_r \subsetneqq M$. Let $A \in r\text{-spec}(H)\backslash\{M\}$. Then there is some primary $J \in \mathcal{I}_r(H)$ such that $A \subseteq J \subsetneqq \sqrt{J}$. There exists some $l \in \mathbb{N}_{\geq 2}$ such that $J = ((\sqrt{J})^l)_r$. Therefore, $A \subseteq J \subseteq ((\sqrt{J})^2)_r \subseteq (M^2)_r$. Consequently, $B \subseteq (M^2)_r$ for all $B \in r\text{-spec}(H)\backslash\{M\}$. There is some $z \in M\backslash(M^2)_r$, hence $\sqrt{zH} = M$. Since $zH$ is primary there is some $m \in \mathbb{N}$ such that $zH = (M^m)_r$. We have $H$ is $r$-local and thus $\mathcal{C}_r(H)$ is trivial. Consequently, $M$ is principal.
**2.** Let $H$ satisfy the $r$-prime power condition and let $H$ be radical factorial. By 1. there is some $x \in M$ such that $M = xH$. Therefore, Proposition 2.4.4. implies that $H = H_M$ is a discrete valuation monoid.
**3.** Let $H$ be an $r$-SP-monoid. It follows by Proposition 3.10.1. that $H$ satisfies the $r$-prime power condition. Since $H$ is $r$-local we have $\mathcal{C}_r(H)$ is trivial and thus $H$ is radical factorial by Proposition 3.10.2.. Therefore, $H$ is a discrete valuation monoid by 2.. □

Next we present the second main result in this section, which generalizes parts of [15].

**Theorem 3.13.** *Let $H$ be a monoid such that $H^\bullet \neq H^\times$ and $r$ a finitary ideal system on $H$ such that $H$ is primary $r$-ideal inclusive.*

1. *If $H$ satisfies the $r$-prime power condition and $H$ is radical factorial, then $H$ is an $r$-SP-monoid.*
2. *If $H$ is an $r$-SP-monoid, then $H$ is an almost $r$-Dedekind monoid.*

*Proof.* **1.** Let $H$ satisfy the $r$-prime power condition and let $H$ be radical factorial. By Proposition 3.11. it is sufficient to show that $r\text{-max}(H) = \mathfrak{X}(H)$. Assume that $r\text{-max}(H) \neq \mathfrak{X}(H)$. Then there are some $P, Q \in r\text{-spec}(H)^\bullet$ such that $P \subsetneqq Q$. Therefore, there is some primary $I \in \mathcal{I}_r(H)$ such that $P \subseteq I \subsetneqq \sqrt{I} \subseteq Q$. Set $M = \sqrt{I}$. It follows by Lemma 2.1.2. and Lemma 3.8. that $H_M$ is radical factorial, $H_M$ is primary $r_M$-ideal inclusive and $H_M$ satisfies the $r_M$-prime power condition. Since $I_M \in \mathcal{I}_{r_M}(H_M)$ and $I_M \subsetneqq \sqrt[H_M]{I_M} = M_M$ it follows by Lemma 3.12.2. that $H_M$ is a discrete valuation monoid. Consequently, $P_M = \{0\}$ and thus $P = \{0\}$, a contradiction.
**2.** Let $H$ be an $r$-SP-monoid. Claim: If $B \in \mathcal{I}_r(H)$ is primary and $B \subsetneqq \sqrt{B}$, then $H_{\sqrt{B}}$ is a discrete valuation monoid. Let $B \in \mathcal{I}_r(H)$ be primary and $B \subsetneqq \sqrt{B}$. Set $P = \sqrt{B}$. Then Lemma 3.8. implies that $H_P$ is an $r_P$-local $r_P$-SP-monoid that is primary $r_P$-ideal inclusive. Moreover, $B_P \in \mathcal{I}_{r_P}(H_P)$ and $B_P \subsetneqq \sqrt[H_P]{B_P} = P_P$, hence $H_P$ is a discrete valuation monoid by Lemma 3.12.3..
Let $Q \in r\text{-max}(H)$. Since $\{0\} \subsetneqq Q$, there is some primary $I \in \mathcal{I}_r(H)$ such that $I \subsetneqq \sqrt{I} \subseteq Q$. Assume that $\sqrt{I} \subsetneqq Q$. Then there is some primary $J \in \mathcal{I}_r(H)$ such that $\sqrt{I} \subseteq J \subsetneqq \sqrt{J} \subseteq Q$. We have $H_{\sqrt{J}}$ is a discrete valuation monoid by the claim. Consequently, $(\sqrt{I})_{\sqrt{J}} = \{0\}$ and thus $\sqrt{I} = \{0\}$, a contradiction. Therefore, $Q = \sqrt{I}$, hence $H_Q$ is a discrete valuation monoid by the claim. □

**Corollary 3.14.** *Let $H$ be a monoid such that $H^\bullet \neq H^\times$ and $r$ a finitary ideal system on $H$. The following statements are equivalent:*

1. *$H$ is an $r$-SP-monoid that is primary $r$-ideal inclusive.*
2. *$H$ is an $r$-Prüfer monoid, $r\text{-max}(H) = \mathfrak{X}(H)$ and every $M \in r\text{-max}(H)$ is not $r$-critical.*

*Proof.* **1.** $\Rightarrow$ **2.**: It follows by Theorem 3.13.2. that $H$ is an almost $r$-Dedekind monoid. Therefore, $H$ is an $r$-Prüfer monoid and $r$-max$(H) = \mathfrak{X}(H)$. It follows by Corollary 3.4. that every $M \in r$-max$(H)$ is not $r$-critical. **2.** $\Rightarrow$ **1.**: Let $M \in r$-max$(H)$. Assume that $M_M^2 = M_M$. Then $((M^2)_r)_M = (M_M^2)_{r_M} = (M_M)_{r_M} = M_M$ and thus $(M^2)_r = ((M^2)_r)_M \cap H = M_M \cap H = M$ by Lemma 3.1.1.. This implies that $M$ is $r$-critical, a contradiction. Therefore, $M_M^2 \neq M_M$, hence $M_M$ is principal and thus $H_M$ is a discrete valuation monoid by [8, Corollary 5.4.] and [8, Theorem 6.4.]. Consequently, $H$ is an almost $r$-Dedekind monoid. By Corollary 3.4. we have $H$ is an $r$-SP-monoid. Proposition 3.9. implies that $H$ is primary $r$-ideal inclusive. $\square$

By Lemma 3.6. we obtain that Corollary 3.14. improves [11, Corollary 2.2.].

**Corollary 3.15.** *Let $H$ be an $s$-SP-monoid such that $H^\bullet \neq H^\times$. Then $H$ is a discrete valuation monoid.*

*Proof.* By Theorem 3.13.2. and Lemma 3.7. we have $H$ is an almost $s$-Dedekind monoid. Since $H$ is $s$-local this implies that $H$ is a discrete valuation monoid. $\square$

## 4. Examples and Remarks

Let $H$ be a monoid, $K$ a quotient monoid of $H$, $x \in H^\bullet$ and $n \in \mathbb{N}$. Set $\mathcal{D}_n(x) = \{u \in \mathcal{A}(H) \mid u|x^n\}$ and $\mathcal{D}_\infty(x) = \bigcup_{k\in\mathbb{N}} \mathcal{D}_k(x)$. Observe that $\mathcal{D}_\infty(a) = \mathcal{D}_1(a)$ for all $a \in H^\bullet$ if and only if $\mathcal{A}(H) \subseteq \mathcal{B}(H)$. Especially if $H$ is atomic, then $H$ is radical factorial if and only if $\mathcal{D}_\infty(a) = \mathcal{D}_1(a)$ for all $a \in H^\bullet$. Recall that $H$ is seminormal if for all $x \in K$ such that $x^2, x^3 \in H$ it follows that $x \in H$ (see also [14]). By $\widehat{H}$ we denote the complete integral closure of $H$.

**Example 4.1.** *Let $L$ be a field, $K \subsetneq L$ a subfield, $X$ an indeterminate over $L$ and $R = K + XL[\![X]\!]$. Then $R$ is a primary Mori domain that is not completely integrally closed and $\mathcal{D}_\infty(f) = \mathcal{D}_2(f)$ for all $f \in R^\bullet$. Especially $R$ is weakly factorial, $\mathcal{C}_t(R)$ is trivial and $R$ is not factorial.*

*Proof.* Note that $R$ and $L[\![X]\!]$ have the same field of quotients. Clearly, $\widehat{R} = L[\![X]\!]$ is a discrete valuation domain, hence $\widehat{R}$ is factorial. Observe that $R$ is seminormal and not completely integrally closed. We have $\widehat{R}\backslash\widehat{R}^\times = XL[\![X]\!]$. Since $\widehat{R}^\times \cap R = (L^\times + XL[\![X]\!]) \cap (K + XL[\![X]\!]) = (L^\times \cap K) + XL[\![X]\!] = K^\times + XL[\![X]\!] = R^\times$ it follows that $R\backslash R^\times = (\widehat{R}\backslash\widehat{R}^\times) \cap R = XL[\![X]\!]$. Therefore, $R$ is local. Let $P \in \mathrm{spec}(R)^\bullet$. Then $P \subseteq XL[\![X]\!]$, hence $\{0\} \neq P\widehat{R} \subseteq X\widehat{R}$ and thus $\sqrt[\widehat{R}]{P\widehat{R}} = X\widehat{R}$ (since $\widehat{R}$ is local and 1-dimensional). There exists some $k \in \mathbb{N}$ such that $X^k \in P\widehat{R}$, hence $X^{k+1} \in XP\widehat{R} \subseteq PR = P$ and thus $X \in P$. This implies that $(XL[\![X]\!])^2 = X^2\widehat{R} \subseteq XR \subseteq P$. Consequently, $XL[\![X]\!] \subseteq P$, hence $P = XL[\![X]\!]$. This implies that $R$ is 1-dimensional and since $R$ is local we have $R$ is primary, hence $R$ is weakly factorial and $\mathcal{C}_t(R)$ is trivial. It follows by [14, Lemma 2.6.] that $R$ is a Mori domain.

Next we show that $\mathcal{A}(R) \subseteq \mathcal{A}(\widehat{R})$. Let $f \in \mathcal{A}(R)$. Obviously, $f \in \widehat{R}^\bullet\backslash\widehat{R}^\times$. Let $g, h \in \widehat{R}$ be such that $f = gh$. Case 1: $g_0 = h_0 = 0$: We have $g, h \in R$, hence $g \in R^\times \subseteq \widehat{R}^\times$ or $h \in R^\times \subseteq \widehat{R}^\times$. Case 2: $g_0 \neq 0$ or $h_0 \neq 0$: Without restriction let $g_0 \neq 0$. Then $f = gh = g_0^{-1}gg_0h$ and $g_0^{-1}g, g_0h \in R$, hence $g_0^{-1}g \in R^\times$ or $g_0h \in R^\times$. Since $g_0 \in L^\times \subseteq \widehat{R}^\times$ it follows that $g \in \widehat{R}^\times$ or $h \in \widehat{R}^\times$.

It remains to show that $\mathcal{D}_\infty(f) \subseteq \mathcal{D}_2(f)$ for all $f \in R^\bullet\backslash R^\times$. Let $f \in R^\bullet\backslash R^\times$ and $u \in \mathcal{D}_\infty(f)$. Then there exists some $k \in \mathbb{N}$ such that $u|_R f^k$. This implies that $u|_{\widehat{R}} f^k$. Since $u \in \mathcal{A}(R) \subseteq \mathcal{A}(\widehat{R})$ and $\widehat{R}$ is factorial it follows that $u|_{\widehat{R}} f$. Therefore, there exists some $g \in \widehat{R}$ such that $f = ug$. Case 1: $f_0 \neq 0$: We have $u_0 \neq 0$, hence $g_0 = f_0u_0^{-1} \in K$. This implies that $g \in R$ and thus $u \in \mathcal{D}_1(f) \subseteq \mathcal{D}_2(f)$. Case 2: $f_0 = 0$: It follows that $fg \in R$. Since $f^2 = ufg$ we have $u|_R f^2$, hence $u \in \mathcal{D}_2(f)$. $\square$

**Example 4.2.** *There exists some radical factorial GCD-domain that is not factorial.*

*Proof.* By [15, Theorem 3.4.] there exists some Bézout domain $R$ that is an SP-domain but not a Dedekind domain. Therefore, $R$ is a 1-dimensional Bézout domain and thus $R$ is a GCD-domain, hence $Pic(R) = \mathcal{C}_t(R)$ is trivial. Assume that $R$ is factorial. Then $R$ is an atomic Bézout domain, hence $R$

is a principal ideal domain and thus $R$ is a Dedekind domain, a contradiction. Consequently, $R$ is not factorial. By Proposition 3.10.2. we have $R$ is radical factorial. $\square$

Together with Proposition 2.17. it follows by Example 4.2. that there are plenty examples of radical factorial GCD-domains that fail to be factorial. The following example is a generalization of [5, Example 3-2.] and [5, Example 3-3.].

**Example 4.3.** *Each of the following properties is satisfied by some Dedekind domain $R$ that is not factorial such that* $\max(R)$ *is countable, $M$ is not principal and $|R/M| < \infty$ for all $M \in \max(R)$ and $\mathcal{C}_t(R)$ is torsion-free*:

1. *$R$ is radical factorial.*
2. *$\mathcal{B}(R) = R^\times$.*
3. *$M \cap \mathcal{B}(R) \neq \emptyset$ for all $M \in \max(R)$ and $R$ is not radical factorial.*

*Proof.* Let $G$ be a countably generated (additive) free abelian group with basis $(x_i)_{i\in\mathbb{N}}$, $(l_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, $H = \langle\{x_i + l_i x_{i+1} \mid i \in \mathbb{N}\}\rangle$ and $I = \{z \in H \mid z = \sum_{i\in\mathbb{N}} r_i x_i$ for some $(r_i)_{i\in\mathbb{N}} \in \mathbb{N}_0^{(\mathbb{N})}\}$. Set $l_0 = 0$. First we show that for every finite $E \subseteq \mathbb{N}$ and all $(n_e)_{e\in E} \in \mathbb{N}_0^E$ there is some $(m_e)_{e\in\mathbb{N}} \in \mathbb{N}_0^{(\mathbb{N})}$ such that $m_e = n_e$ for all $e \in E$ and $\sum_{e\in\mathbb{N}} m_e x_e \in I$. Let $E \subseteq \mathbb{N}$ be finite and $(n_e)_{e\in E} \in \mathbb{N}_0^E$. There is some finite $F \subseteq \mathbb{N}$ and some bijection $h : E \to F$ such that $E \cap F = \emptyset$ and $h(e) - e$ is an odd natural number for all $e \in E$. Set $m_e = n_e$ if $e \in E$, $m_e = (\prod_{i=h^{-1}(e)}^{e-1} l_i) n_{h^{-1}(e)}$ if $e \in F$ and $m_e = 0$ if $e \in \mathbb{N}\backslash(E\cup F)$. Then $(m_e)_{e\in\mathbb{N}} \in \mathbb{N}_0^{(\mathbb{N})}$. Note that $x_e + (\prod_{j=e}^{h(e)-1} l_j) x_{h(e)} = \sum_{i=e}^{h(e)-1}(\prod_{j=e}^{i-1}(-l_j))(x_i + l_i x_{i+1}) \in I$ for all $e \in E$. Therefore, $\sum_{e\in\mathbb{N}} m_e x_e = \sum_{e\in E} n_e x_e + \sum_{e\in F}(\prod_{i=h^{-1}(e)}^{e-1} l_i) n_{h^{-1}(e)} x_e = \sum_{e\in E} n_e(x_e + (\prod_{j=e}^{h(e)-1} l_j) x_{h(e)}) \in I$.
By [5, Theorem 2.1.] there exists some Dedekind domain $R$ and some bijection $f : \{x_i \mid i \in \mathbb{N}\} \to \max(R)$ such that $R/M$ is finite for all $M \in \max(R)$ and such that the canonical group isomorphism $g : G \to \mathcal{F}(R)^\bullet$ that extends $f$ satisfies $g(I) = \{yR \mid y \in R^\bullet\}$. Obviously, $\max(R)$ is countable. Next we show that $H \cap \{kx_j \mid k \in \mathbb{Z}, j \in \mathbb{N}\} = \{0\}$. Let $v \in H$, $k \in \mathbb{Z}$, $j \in \mathbb{N}$ and $(z_i)_{i\in\mathbb{N}} \in \mathbb{Z}^{(\mathbb{N})}$ be such that $v = kx_j = \sum_{i\in\mathbb{N}} z_i(x_i + l_i x_{i+1})$. Set $z_0 = 0$. Then $z_i + l_{i-1} z_{i-1} = 0$ for all $i \in \mathbb{N}\backslash\{j\}$ and $k = z_j + l_{j-1} z_{j-1}$. Assume that $v \neq 0$. Then there exist some smallest $a \in \mathbb{N}$ and some largest $b \in \mathbb{N}$ such that $z_a \neq 0$ and $z_b \neq 0$. This implies that $a = j = b + 1$, hence $z_a = z_{b+1} = 0$, a contradiction. Consequently, every $M \in \max(R)$ is not principal. Therefore, $R$ is not a principal ideal domain, hence $R$ is not factorial. Now we prove that $Pic(R) = \mathcal{C}_t(R)$ is torsion-free. Let $J$ be a non-zero ideal of $R$ and $c \in \mathbb{N}$ such that $J^c$ is a principal ideal of $R$. There is some $(v_i)_{i\in\mathbb{N}} \in \mathbb{N}_0^{(\mathbb{N})}$ such that $g(\sum_{i\in\mathbb{N}} v_i x_i) = J$. It follows that $\sum_{i\in\mathbb{N}} cv_i x_i \in I$, hence there is some $(d_i)_{i\in\mathbb{N}} \in \mathbb{Z}^{(\mathbb{N})}$ such that $\sum_{i\in\mathbb{N}} cv_i x_i = \sum_{i\in\mathbb{N}} d_i(x_i + l_i x_{i+1})$. Set $d_0 = 0$. Then $cv_i = d_i + l_{i-1} d_{i-1}$ for all $i \in \mathbb{N}$. By induction we obtain that $c|d_i$ for all $i \in \mathbb{N}$, hence there is some $(e_i)_{i\in\mathbb{N}} \in \mathbb{Z}^{(\mathbb{N})}$ such that $d_i = ce_i$ for all $i \in \mathbb{N}$. This implies that $\sum_{i\in\mathbb{N}} v_i x_i = \sum_{i\in\mathbb{N}} e_i(x_i + l_i x_{i+1}) \in I$ and thus $J$ is principal.
**1.** Set $l_i = 1$ for all $i \in \mathbb{N}$. We show that $R$ is radical factorial. Since $R$ is atomic it remains to show that $\mathcal{A}(R) \subseteq \mathcal{B}(R)$. Let $u \in \mathcal{A}(R)$. Then there is some $z \in I$ such that $g(z) = uR$. There exist some $(r_i)_{i\in\mathbb{N}} \in \mathbb{N}_0^{(\mathbb{N})}$ and $(t_i)_{i\in\mathbb{N}} \in \mathbb{Z}^{(\mathbb{N})}$ such that $z = \sum_{i\in\mathbb{N}} r_i x_i = \sum_{i\in\mathbb{N}} t_i(x_i + x_{i+1})$. Set $t_0 = 0$. Then $r_i = t_i + t_{i-1}$ for all $i \in \mathbb{N}$. Since $uR \neq R$ we have $z \neq 0$ and thus there is some least $n \in \mathbb{N}$ such that $t_n \neq 0$. Assume that $r_{n+2i+1} = 0$ for all $i \in \mathbb{N}_0$. If $i \in \mathbb{N}_0$, then $t_{n+2i+1} = -t_{n+2i}$ and $t_{n+2i+2} + t_{n+2i+1} \geq 0$, hence $t_{n+2i+2} \geq t_{n+2i}$. Since $t_n = r_n > 0$ we have $t_{n+2i} > 0$ for all $i \in \mathbb{N}_0$, a contradiction. Consequently, there is some $j \in \mathbb{N}_0$ such that $r_{n+2j+1} \neq 0$. Set $m = n + 2j + 1$. We have $x_n + x_m = \sum_{i=0}^{m-n-1}(-1)^i(x_{i+n} + x_{i+n+1}) \in I$. Therefore, there is some $v \in R^\bullet$ such that $g(x_n + x_m) = vR$. This implies that $uR = \prod_{i\in\mathbb{N}} g(x_i)^{r_i} \subseteq g(x_n)g(x_m) = vR \subsetneq R$, hence $uR = g(x_n)g(x_m)$. Since $g(x_n) \neq g(x_m)$ it follows that $\sqrt{uR} = g(x_n) \cap g(x_m) = g(x_n)g(x_m) = uR$ and thus $u \in \mathcal{B}(R)$.
**2.** Let $l_i \in \mathbb{N}_{\geq 2}$ for all $i \in \mathbb{N}$. Assume that $\mathcal{B}(R) \neq R^\times$. Then there are some $u \in \mathcal{B}(R)\backslash R^\times$ and $w \in I$ such that $g(w) = uR$. Consequently, there exist some $(s_i)_{i\in\mathbb{N}} \in \{0,1\}^{(\mathbb{N})}$ and $(u_i)_{i\in\mathbb{N}} \in \mathbb{Z}^{(\mathbb{N})}$ such that $w = \sum_{i\in\mathbb{N}} s_i x_i = \sum_{i\in\mathbb{N}} u_i(x_i + l_i x_{i+1})$. Set $u_0 = 0$. Then $s_i = u_i + l_{i-1} u_{i-1}$ for all $i \in \mathbb{N}$. Since

$w \neq 0$, there are some largest $m, n \in \mathbb{N}$ such that $s_m \neq 0$ and $u_n \neq 0$. Set $p = \max(m, n)$. Then $s_{p+1} = u_{p+1} = 0$, hence $u_p = 0$ and thus $1 = s_p = l_{p-1}u_{p-1}$, a contradiction.
**3.** Set $l_2 = 2$ and $l_i = 1$ for all $i \in \mathbb{N}\backslash\{2\}$. Let $N \in \max(R)$. There is some $b \in \mathbb{N}$ such that $g(x_b) = N$. If $b \neq 2$, then there is some $w \in \mathcal{B}(R)$ such that $g(x_b + x_{b+1}) = wR$ and thus $w \in N \cap \mathcal{B}(R)$. If $b = 2$, then there exists some $w \in \mathcal{B}(R)$ such that $g(x_1 + x_2) = wR$, hence $w \in N \cap \mathcal{B}(R)$. There is some $v \in R^\bullet$ such that $g(x_2 + 2x_3) = vR$. Since $v \in g(x_3)^2$ it follows that $v \notin \mathcal{B}(R)$. On the other hand we have $x_2 + x_3, x_2, x_3, 2x_3 \notin I$. Therefore, $v \in \mathcal{A}(R)$. This implies that $\mathcal{A}(R) \nsubseteq \mathcal{B}(R)$ and thus $R$ is not radical factorial. □

Note that $R$ in Example 4.3.2. is an integral domain that fulfills the conditions in Lemma 2.7.4., hence $\Omega(R)$ is divisor-closed and yet $R$ is not radical factorial. Observe that there exists a valuation domain $S$ that is not a field and such that $\mathfrak{X}(S) = \emptyset$, hence every $P \in \mathfrak{X}(S)$ is principal and yet $S$ is not factorial. By this remark and Example 4.1., Example 4.2. and Example 4.3. it follows that none of the conditions in Theorem 2.14. can be omitted.

**Example 4.4.** *There exists some local, 2-dimensional, noetherian domain $R$ such that $R_P$ is a discrete valuation domain for all $P \in \mathfrak{X}(R)$ and $R$ is not primary $t$-ideal inclusive.*

*Proof.* Let $S$ be a discrete valuation domain, $p$ a prime element of $S$, $K$ a quotient field of $S$, $X$ an indeterminate over $K$, $R = \{f \in S[\![X]\!] \mid p|f_1\}$ and $Q = (p, pX, X^2, X^3)_R$. Clearly, $S[\![X]\!]$ is a local, 2-dimensional noetherian domain. By [13, Lemma 5.2.1.] we have $\overline{R} = S[\![X]\!] = (1, X)_R$ and thus $R$ is local and 2-dimensional. Moreover, $R$ is noetherian by the Theorem of Eakin-Nagata. By [13, Lemma 5.2.5.] we have $R_P$ is a discrete valuation domain for all $P \in \mathfrak{X}(R)$. It follows that $Q = Q_v = \{f \in S[\![X]\!] \mid p|f_0, p|f_1\}$ by [13, Lemma 5.2.2.]. Obviously, $Q \in \max(R)$ and thus $\max(R) = t\text{-}\max(R) = \{Q\}$. Let $I \in \mathcal{I}_t(R)$ be such that $\sqrt{I} = Q$. Then there is some $k \in \mathbb{N}$ such that $p^k, X^k \in I$. We have $Q^{-1} \subseteq I^{-1} \subseteq ((p^k, X^k)_R)^{-1} \subseteq S[\![X]\!] = Q^{-1}$, hence $I = I_v = Q$. Consequently, $\{J \in \mathcal{I}_t(R) \mid P \subseteq J \subseteq Q\} = \{P, Q\}$ for all $P \in \mathfrak{X}(R) = t\text{-spec}(R)^\bullet\backslash\{Q\}$, hence $R$ is not primary $t$-ideal inclusive. □

**Example 4.5.** *There is some Dedekind domain $R$ such that $R$ is not radical factorial, $\mathcal{C}_t(R)$ is a torsion group and yet $P \cap \mathcal{B}(R) \neq \emptyset$ for all $P \in \max(R)$.*

*Proof.* Let $R = \mathbb{Z}[\sqrt{-5}]$. It is well known that $R$ is a Dedekind domain that is not factorial and $|\mathcal{C}_t(R)| = 2$. Therefore, Theorem 2.14. implies that $R$ is not radical factorial. Let $P \in \max(R)$. There is some $p \in \mathbb{P}$ such that $P \cap \mathbb{Z} = p\mathbb{Z}$. If $p$ is not ramified, then $p \in P \cap \mathcal{B}(R)$. If $p$ is ramified, then $p = 2$ or $p = 5$. Case 1: $p = 2$: We have $P = (2, 1+\sqrt{-5})_R$, $(3, 1+\sqrt{-5})_R \in \max(R)$, $P \neq (3, 1+\sqrt{-5})_R$ and $(1+\sqrt{-5})R = P(3, 1+\sqrt{-5})_R$. Therefore, $1+\sqrt{-5} \in P \cap \mathcal{B}(R)$. Case 2: $p = 5$: It follows that $P = \sqrt{-5}R$ and thus $\sqrt{-5} \in P \cap \mathcal{B}(R)$. □

By Proposition 2.12. we have if $H$ is a GCD-monoid such that $t\text{-}\max(H) = \mathfrak{X}(H)$ and every height-one prime $s$-ideal of $H$ contains a radical element, then $H$ is radical factorial. By Example 4.5. we know that this is far away from being true if we replace the term "GCD-monoid" by "Dedekind domain". Observe that the last example fails to be radical factorial, since the property in Proposition 2.4.2. is not satisfied. By Example 4.3. we obtain that even if the $t$-class group of a Dedekind domain $R$ is torsion-free (and thus the property in Proposition 2.4.2. is satisfied) and all maximal ideals of $R$ contain a radical element, then $R$ needs not be radical factorial.
Up to now we could not decide whether every radical factorial monoid $H$ such that $H^\bullet \neq H^\times$ satisfies $t\text{-}\max(H) = \mathfrak{X}(H)$. The following remark shows (together with Proposition 2.4.5.) that at least the divisorial prime ideals of $H$ behave nicely.

**Remark 4.6.** *Let $H$ be a monoid such that $H^\bullet \neq H^\times$ and $\bigcap_{P \in \mathfrak{X}(H)} H_P = H$. Then for each $I \in \mathcal{I}_v(H)\backslash\{H\}$ there is some $P \in \mathfrak{X}(H)$ such that $I \subseteq P$. Especially $v\text{-spec}(H)^\bullet \subseteq \mathfrak{X}(H)$.*

*Proof.* Let $K$ be a quotient monoid of $H$, $I \in \mathcal{I}_v(H)\backslash\{H\}$ and $x \in \bigcap_{Q\in\mathfrak{X}(H)} I_Q$. It follows that $xI^{-1} \subseteq I_Q(H_Q :_K I_Q) \subseteq H_Q$ for all $Q \in \mathfrak{X}(H)$, hence $xI^{-1} \subseteq H$ and thus $x \in I_v = I$. Therefore, $\bigcap_{Q\in\mathfrak{X}(H)} I_Q = I$. There is some $P \in \mathfrak{X}(H)$ such that $I_P \neq H_P$, hence $I \subseteq P$. $\square$

## 5. Appendix: Strongly $r$-discrete $r$-Prüfer monoids

In the following we refine some of the statements of [4] and [10]. The first result in this section extends [10, Lemma 2.2.] (in the local case), but its proof is very similar. Let $H$ be a monoid, $r$ a finitary ideal system on $H$, $X \subseteq H$ and $P \in r\text{-spec}(H)$. Set $\mathcal{O}_r(X) = \{Q \in r\text{-spec}(H) \mid X \subsetneq Q\}$. Next, $H$ is called strongly $r$-discrete if for all $Q \in r\text{-spec}(H)$ such that $(Q^2)_r = Q$ it follows that $Q = \{0\}$. Observe that if $\emptyset \neq K \subseteq r\text{-spec}(H)$ is a chain, then $\bigcap_{Q\in K} Q, \bigcup_{Q\in K} Q \in r\text{-spec}(H)$. We call $P$ $r$-branched if there exists some $P$-primary $I \in \mathcal{I}_r(H)$ such that $I \neq P$. Let $r\text{-}\mathfrak{B}(H)$ denote the set of all $r$-branched prime $r$-ideals of $H$.

**Proposition 5.1.** *Let $H$ be a monoid, $r$ a finitary ideal system on $H$ and $M = H\backslash H^\times$.*

1. *If $M$ is principal, then $\bigcap_{n\in\mathbb{N}} M^n \in s\text{-spec}(H)$ and $Q \subseteq \bigcap_{n\in\mathbb{N}} M^n$ for all $Q \in s\text{-spec}(H)\backslash\{M\}$.*
2. *The following conditions are equivalent:*
   - **a.** *$H$ is a strongly $s$-discrete valuation monoid.*
   - **b.** *$H$ is a strongly $r$-discrete valuation monoid.*
   - **c.** *$P_P$ is principal in $H_P$ for all $P \in s\text{-spec}(H)$.*
   - **d.** *$H$ is $r$-local and $P_P$ is principal in $H_P$ for all $P \in r\text{-spec}(H)$.*

   *If these conditions are satisfied, then $s\text{-spec}(H)$ satisfies the ACC.*

*Proof.* **1.** Let $M$ be principal and $N = \bigcap_{n\in\mathbb{N}} M^n$. There is some $s \in M$ such that $M = sH$. Without restriction we can assume that $s \neq 0$. Let $x, y \in H$ be such that $xy \in N$. Assume that $x, y \notin N$. Then there are some least $n, m \in \mathbb{N}$ such that $x \notin M^n$ and $y \notin M^m$. It follows that $x \in M^{n-1}$, $y \in M^{m-1}$ and $xy \in M^{n+m-1}$. There exist some $u, v \in H^\times$ such that $x = s^{n-1}u$ and $y = s^{m-1}v$. We have $s^{n+m-2}uv = xy \in s^{n+m-1}H$, hence $s \in H^\times$, a contradiction. Let $Q \in s\text{-spec}(H)$ be such that $Q \neq M$. We show by induction on $k$ that $Q \subseteq M^k$, for all $k \in \mathbb{N}$. Clearly, $Q \subseteq M$. Now let $k \in \mathbb{N}$ be such that $Q \subseteq M^k$. Then $s^{-k}Q \subseteq H$. Assume that $s^{-k}Q = H$. Then $s^k \in Q$, hence $s \in Q$ and thus $Q = M$, a contradiction. Therefore, $s^{-k}Q \subseteq M = sH$, hence $Q \subseteq s^{k+1}H = M^{k+1}$.
**2. a. $\Leftrightarrow$ b.**: In any case $H$ is a valuation monoid. Hence $s = r$ and thus $H$ is strongly $s$-discrete if and only if $H$ is strongly $r$-discrete.
**a. $\Rightarrow$ c.**: Let $P \in s\text{-spec}(H)$. Obviously, $H_P$ is a valuation monoid and thus $P_P = P_P^2$ or $P_P$ is principal. Assume that $P_P$ is not principal. Then $P_P = P_P^2$. Since $P^2$ is $P$-primary by Lemma 3.1.1. it follows that $P = P_P \cap H = P_P^2 \cap H = (P^2)_P \cap H = P^2$. Therefore, $P = \{0\}$ and thus $P_P = \{0\}$, a contradiction.
**c. $\Rightarrow$ d.**: Since $M_M = M$ is principal in $H_M = H$ it follows that $M_r = M$, hence $H$ is $r$-local. Clearly, $P_P$ is principal in $H_P$ for all $P \in r\text{-spec}(H)$.
**d. $\Rightarrow$ a.**: Claim 1: $r\text{-spec}(H)$ is a chain. Let $P, Q \in r\text{-spec}(H)$. Then there is some $A \in \mathcal{P}((P \cup Q)_r)$. It follows that $P_A, Q_A \subseteq A_A$. Assume that $P_A, Q_A \subsetneq A_A$. Since $A \in r\text{-spec}(H)$ we have $A_A$ is principal, hence it follows by 1. that $P_A, Q_A \subseteq \bigcap_{k\in\mathbb{N}} A_A^k$. By 1. we have $\bigcap_{k\in\mathbb{N}} A_A^k \in s\text{-spec}(H_A)$. Set $B = \bigcap_{k\in\mathbb{N}} A_A^k \cap H$. Then $B \in r\text{-spec}(H)$, $B \subseteq A$ and $B_A = \bigcap_{k\in\mathbb{N}} A_A^k$. Consequently, $P_A, Q_A \subseteq B_A$, hence $(P \cup Q)_r \subseteq B \subseteq A$ and thus $B = A$. We have $A_A = \bigcap_{k\in\mathbb{N}} A_A^k$. Therefore, $A_A = A_A^2$, and thus $A_A = \{0\}$, a contradiction. It follows that $P_A = A_A$ or $Q_A = A_A$, hence $Q \subseteq P$ or $P \subseteq Q$.
Claim 2: $r\text{-spec}(H)$ satisfies the ACC. Let $(P_i)_{i\in\mathbb{N}}$ be an ascending sequence of prime $r$-ideals of $H$. Set $Q = \bigcup_{i\in\mathbb{N}} P_i$. Then $Q \in r\text{-spec}(H)$. There is some $q \in Q$ such that $Q_Q = qH_Q$, hence there is some $k \in \mathbb{N}$ such that $q \in P_k$. Consequently, $Q_Q = qH_Q \subseteq (P_k)_Q \subseteq Q_Q$, hence $Q_Q = (P_k)_Q$ and thus $Q = P_k$. Therefore, $P_j = P_k$ for all $j \in \mathbb{N}_{\geq k}$.
Claim 3: If $\Omega \subseteq r\text{-spec}(H)$ and if for every $P \in r\text{-spec}(H)$ satisfying $\mathcal{O}_r(P) \subseteq \Omega$ it follows that $P \in \Omega$, then $\Omega = r\text{-spec}(H)$. Let $\Omega \subseteq r\text{-spec}(H)$ be such that for every $P \in r\text{-spec}(H)$ satisfying $\mathcal{O}_r(P) \subseteq \Omega$ it follows that $P \in \Omega$. Assume that $\Omega \neq r\text{-spec}(H)$. By claim 2 there is some $P \in r\text{-spec}(H)\backslash\Omega$ that is

maximal. Therefore, $\mathcal{O}_r(P) \subseteq \Omega$ and thus $P \in \Omega$, a contradiction.
Let $\Omega = \{P \in r\text{-spec}(H) \mid \text{for all } x, y \in H\backslash P \text{ we have } P \subseteq xH \text{ and it follows that } x \in yH \text{ or } y \in xH\}$. Let $P \in r\text{-spec}(H)$ be such that $\mathcal{O}_r(P) \subseteq \Omega$. We show that $P \in \Omega$. Since $M \in \Omega$ we can assume that $P \subsetneqq M$. Set $L = \bigcap_{Q \in \mathcal{O}_r(P)} Q$. Note that $M \in \mathcal{O}_r(P)$, hence $L \in r\text{-spec}(H)$ by claim 1. There exists some $t \in L$ such that $L_L = tH_L$. First we show that if $P \subsetneqq L$, then $P_L = \bigcap_{k \in \mathbb{N}} t^k H_L$. Let $P \subsetneqq L$. Since $P_L \subsetneqq L_L$ and $L_L$ is principal it follows by 1. that $P_L \subseteq \bigcap_{k \in \mathbb{N}} L_L^k \in s\text{-spec}(H_L)$. Set $B = \bigcap_{k \in \mathbb{N}} L_L^k \cap H$. Then $B \in r\text{-spec}(H)$, $B \subseteq L$ and $\bigcap_{k \in \mathbb{N}} L_L^k = B_L$. If $B = L$, then $L_L^2 = L_L$, hence $L_L = \{0\}$, a contradiction. Therefore, $P \subseteq B \subsetneqq L$, hence $P = B$ and $P_L = \bigcap_{k \in \mathbb{N}} t^k H_L$.
Let $x, y \in H\backslash P$. Case 1: $x \notin L$: There is some $Q \in \mathcal{O}_r(P)$ such that $x \notin Q$. Therefore, $P \subseteq Q \subseteq xH$. Case 1a: $y \notin L$: There is some $Q' \in \mathcal{O}_r(P)$ such that $y \notin Q'$. By claim 1 we have $Q \cap Q' \in \mathcal{O}_r(P)$ and $x, y \notin Q \cap Q'$. Therefore, $x \in yH$ or $y \in xH$. Case 1b: $y \in L$: We have $y \in L \subseteq Q \subseteq xH$. Case 2: $x \in L$: It follows that $P \subsetneqq L$ and thus $L \in \Omega$. Therefore, $L \subseteq zH$ for all $z \in H\backslash L$, hence $L_L \subseteq H$. Since $x \notin P_L$ there is some $i \in \mathbb{N}$ such that $x \in t^i H_L$ and $x \notin t^{i+1} H_L$, hence there is some $u \in H_L^\times$ such that $x = t^i u$. It follows that $P \subseteq P_L \subseteq t^{i+1} H_L = xtH_L = xL_L \subseteq xH$. Case 2a: $y \notin L$: We have $x \in L \subseteq yH$. Case 2b: $y \in L$: Since $y \notin P_L$ there exists some $j \in \mathbb{N}$ such that $y \in t^j H_L$ and $y \notin t^{j+1} H_L$ and thus there is some $v \in H_L^\times$ such that $y = t^j v$. If $i < j$, then $y = xt^{j-i} vu^{-1} \in xL_L \subseteq xH$. If $j < i$, then $x = yt^{i-j} uv^{-1} \in yL_L \subseteq yH$. Now let $i = j$. There exists some $c \in H\backslash L$ such that $cu, cv \in H\backslash L$. Consequently, $cu \in cvH$ or $cv \in cuH$ and hence $x = t^i c^{-1} cu \in t^i c^{-1} cvH = yH$ or $y = t^i c^{-1} cv \in t^i c^{-1} cuH = xH$. Finally $P \in \Omega$.
By claim 3 we have $\Omega = r\text{-spec}(H)$. Therefore, $\{0\} \in \Omega$ and thus $H$ is a valuation monoid. Let $I \in s\text{-spec}(H)$ be such that $I^2 = I$. Then $I \in r\text{-spec}(H)$. Since $I_I = (I^2)_I = I_I^2$ and $I_I$ is principal we have $I_I = \{0\}$, hence $I = \{0\}$.
Set $r = s$. Then it follows by claim 2 that $s\text{-spec}(H)$ satisfies the ACC. $\square$

**Proposition 5.2.** *Let $H$ be a monoid and $r$ a finitary ideal system on $H$ such that $H$ satisfies the $r$-prime power condition and $H$ is primary $r$-ideal inclusive.*

1. *$P_P$ is principal in $H_P$ for all $P \in r\text{-}\mathfrak{B}(H)$ and $(Q^k)_r$ is $Q$-primary for all $k \in \mathbb{N}$ and $Q \in r\text{-spec}(H)$.*
2. *If $P \in r\text{-spec}(H)$ and $Q \in s\text{-spec}(H)$ such that $Q \subsetneqq P$, then $Q \subseteq \bigcap_{n \in \mathbb{N}} (P^n)_r \in r\text{-spec}(H)$.*

*Proof.* **1.** Let $P \in r\text{-}\mathfrak{B}(H)$. There is some primary $I \in \mathcal{I}_r(H)$ such that $I \subsetneqq \sqrt{I} = P$. This implies that $I_P \in \mathcal{I}_{r_P}(H_P)$ and $I_P \subsetneqq \sqrt[H_P]{I_P} = P_P$. By Lemma 3.8. we have $H_P$ satisfies the $r_P$-prime power condition and $H_P$ is primary $r_P$-ideal inclusive. Since $H_P$ is $r_P$-local it follows by Lemma 3.12.1. that $P_P$ is principal in $H_P$. Let $Q \in r\text{-spec}(H)$. Without restriction let $Q \neq (Q^2)_r$. Set $\Omega = \{A \in r\text{-spec}(H) \mid A \subsetneqq Q\}$. Then $\{0\} \in \Omega$. Let $B \in \Omega$. There are some primary $I \in \mathcal{I}_r(H)$ and some $l \in \mathbb{N}_{\geq 2}$ such that $B \subseteq I \subsetneqq \sqrt{I} \subseteq Q$ and $I = ((\sqrt{I})^l)_r$. This implies that $B \subseteq I \subseteq ((\sqrt{I})^2)_r \subseteq (Q^2)_r$. Therefore, $C \subseteq (Q^2)_r$ for all $C \in \Omega$. Let $\emptyset \neq \Sigma \subseteq \Omega$ be a chain. Then $\bigcup_{A \in \Sigma} A \in r\text{-spec}(H)$ and $\bigcup_{A \in \Sigma} A \subseteq (Q^2)_r \subsetneqq Q$, hence $\bigcup_{A \in \Sigma} A \in \Omega$. Therefore, $\Omega$ is ordered inductively by inclusion. Consequently, there exists some maximal $A \in \Omega$. There is some primary $J \in \mathcal{I}_r(H)$ such that $A \subseteq J \subsetneqq \sqrt{J} \subseteq Q$ and thus $Q = \sqrt{J}$. It follows that $Q \in r\text{-}\mathfrak{B}(H)$, hence $Q_Q$ is principal in $H_Q$. We have $Q_Q^k \cap H$ is $Q$-primary for all $k \in \mathbb{N}$. It suffices to show by induction on $k$ that $Q_Q^k \cap H = (Q^k)_r$ for all $k \in \mathbb{N}$. The assertion is clear for $k = 1$. Now let $k \in \mathbb{N}$ be such that $Q_Q^k \cap H = (Q^k)_r$. Since $Q_Q$ is principal we have $Q_Q^{k+1} \cap H \subsetneqq Q_Q^k \cap H$ and $Q_Q^{k+1} \cap H \in \mathcal{I}_r(H)$. Therefore, there is some $l \in \mathbb{N}$ such that $Q_Q^{k+1} \cap H = (Q^l)_r$. Since $(Q^l)_r = Q_Q^{k+1} \cap H \subsetneqq Q_Q^k \cap H = (Q^k)_r$ it follows that $l \geq k+1$. Consequently, $Q_Q^{k+1} \cap H = (Q^l)_r \subseteq (Q^{k+1})_r \subseteq ((Q^{k+1})_r)_Q \cap H = Q_Q^{k+1} \cap H$, hence $Q_Q^{k+1} \cap H = (Q^{k+1})_r$.
**2.** Let $P \in r\text{-spec}(H)$ and $Q \in s\text{-spec}(H)$ be such that $Q \subsetneqq P$. Without restriction let $P \neq (P^2)_r$. By 1. it follows that $(P^2)_r$ is $P$-primary, hence $P \in r\text{-}\mathfrak{B}(H)$ and thus $P_P$ is principal in $H_P$ by 1.. Since $Q_P \subsetneqq P_P$ we have $Q_P \subseteq \bigcap_{n \in \mathbb{N}} P_P^n \in s(H_P)\text{-spec}(H_P)$ by Proposition 5.1.1.. Observe that $(P^k)_r = ((P^k)_r)_P \cap H = P_P^k \cap H$ for all $k \in \mathbb{N}$. Therefore, $Q = Q_P \cap H \subseteq (\bigcap_{n \in \mathbb{N}} P_P^n) \cap H = \bigcap_{n \in \mathbb{N}} (P^n)_r \in r\text{-spec}(H)$. $\square$

**Corollary 5.3.** *Let $H$ be a monoid and $r$ a finitary ideal system on $H$ such that $H$ is primary $r$-ideal inclusive. The following conditions are equivalent:*

1. *$H$ satisfies the $r$-prime power condition.*
2. *$P_P$ is principal in $H_P$ for all $P \in r\text{-}\mathfrak{B}(H)$ and $(Q^k)_r$ is $Q$-primary for all $k \in \mathbb{N}$ and $Q \in r\text{-spec}(H)$.*
3. *$P_P$ is principal in $H_P$ and $(P^k)_r$ is $P$-primary for all $k \in \mathbb{N}$ and $P \in r\text{-}\mathfrak{B}(H)$.*

*Proof.* **1.** $\Rightarrow$ **2.**: This follows from Proposition 5.2.1.. **2.** $\Rightarrow$ **3.**: Trivial. **3.** $\Rightarrow$ **1.**: Let $I \in \mathcal{I}_r(H)$ be primary and $P = \sqrt{I}$. Without restriction let $I \neq P$. Then $P \in r\text{-}\mathfrak{B}(H)$. Since $I_P$ is $P_P$-primary, it follows by Lemma 2.2.1. that there is some $k \in \mathbb{N}$ such that $I_P = P_P^k = (P_P^k)_{r_P} = ((P^k)_r)_P$. Consequently, $I = I_P \cap H = ((P^k)_r)_P \cap H = (P^k)_r$. □

The following Theorem is the main result in this section.

**Theorem 5.4.** *Let $H$ be a monoid and $r$ a finitary ideal system on $H$. The following are equivalent:*

1. *$H$ is a strongly $r$-discrete $r$-Prüfer monoid.*
2. *$H$ satisfies the $r$-prime power condition, $H$ is primary $r$-ideal inclusive and $r\text{-spec}(H)$ satisfies the ACC.*
3. *$H$ satisfies the $r$-prime power condition, $H$ is primary $r$-ideal inclusive and $H$ is strongly $r$-discrete.*
4. *$P_P$ is principal in $H_P$ for all $P \in r\text{-spec}(H)$.*

*Proof.* **1.** $\Rightarrow$ **2.**, **1.** $\Rightarrow$ **3.**: First we show that $L_L$ is principal in $H_L$ for all $L \in r\text{-spec}(H)$. Let $L \in r\text{-spec}(H)$. Assume that $L_L$ is not principal in $H_L$. Since $H_L$ is a valuation monoid it follows that $L_L^2 = L_L$, hence $((L^2)_r)_L = (L_L^2)_{r_L} = L_L^2 = L_L$. Because of Lemma 3.1.1. we have $(L^2)_r = ((L^2)_r)_L \cap H = L_L \cap H = L$. Therefore, $L = \{0\}$, a contradiction. By Proposition 3.9. it follows that $H$ is primary $r$-ideal inclusive. By Lemma 3.1.1. and Corollary 5.3. we have $H$ satisfies the $r$-prime power condition. Let $M \in r\text{-max}(H)$ and $J \in r_M\text{-spec}(H_M)^\bullet$. Then there is some $N \in r\text{-spec}(H)^\bullet$ such that $N \subseteq M$ and $J = N_M$. We have $N_N$ is principal in $H_N$. This implies that $J_J = (N_M)_{N_M} = N_N$ is principal in $(H_M)_J = (H_M)_{N_M} = H_N$. Since $H_M$ is $r_M$-local it follows by Proposition 5.1.2. that $r_M\text{-spec}(H_M) = s(H_M)\text{-spec}(H_M)$ satisfies the ACC. Therefore, $\{P \in r\text{-spec}(H) \mid P \subseteq B\}$ satisfies the ACC for all $B \in r\text{-max}(H)$ and thus $r\text{-spec}(H)$ satisfies the ACC.
**2.** $\Rightarrow$ **4.**: Let $P \in r\text{-spec}(H)^\bullet$. There is some $Q \in r\text{-spec}(H)$ such that $Q$ is maximal in $\{A \in r\text{-spec}(H) \mid A \subsetneq P\}$. There is some primary $I \in \mathcal{I}_r(H)$ such that $Q \subseteq I \subsetneq \sqrt{I} \subseteq P$ and thus $\sqrt{I} = P$. Therefore, $P \in r\text{-}\mathfrak{B}(H)$, hence $P_P$ is principal in $H_P$ by Proposition 5.2.1..
**3.** $\Rightarrow$ **4.**: Let $P \in r\text{-spec}(H)^\bullet$. By Proposition 5.2.1. it follows that $(P^2)_r$ is $P$-primary. Since $(P^2)_r \subsetneq P$ we have $P \in r\text{-}\mathfrak{B}(H)$ and thus $P_P$ is principal in $H_P$ by Proposition 5.2.1..
**4.** $\Rightarrow$ **1.**: Let $M \in r\text{-max}(H)$. Let $\overline{P} \in r_M\text{-spec}(H_M)$ and $P = \overline{P} \cap H$. Then $P \in r\text{-spec}(H)$, $P \subseteq M$ and $\overline{P} = P_M$. It follows that $\overline{P}_{\overline{P}} = (P_M)_{P_M} = P_P$ is principal in $(H_M)_{\overline{P}} = (H_M)_{P_M} = H_P$. Since $H_M$ is $r_M$-local it follows by Proposition 5.1.2. that $H_M$ is a strongly $r_M$-discrete valuation monoid. Consequently, $H$ is an $r$-Prüfer monoid. Let $Q \in r\text{-spec}(H)$ be such that $(Q^2)_r = Q$. There is some $N \in r\text{-max}(H)$ such that $Q \subseteq N$. We have $(Q_N^2)_{r_N} = ((Q^2)_r)_N = Q_N$, hence $Q_N = \{0\}$ and thus $Q = \{0\}$. Therefore, $H$ is strongly $r$-discrete. □

**Corollary 5.5.** *Let $R$ be an integral domain. The following statements are equivalent:*

1. *$R$ is a strongly $d$-discrete Prüfer domain.*
2. *$R$ satisfies the $d$-prime power condition and $\text{spec}(R)$ satisfies the ACC.*
3. *$R$ satisfies the $d$-prime power condition and $R$ is strongly $d$-discrete.*
4. *$P_P$ is principal in $R_P$ for all $P \in \text{spec}(R)$.*

*Proof.* This is an easy consequence of Lemma 3.6. and Theorem 5.4.. □

Therefore, Theorem 5.4. is a common generalization of [10, Lemma 2.2.] and [4, Theorem 7.].

**Corollary 5.6.** *Let $H$ be a monoid. The following conditions are equivalent:*

**1.** *$H$ is a strongly $s$-discrete valuation monoid.*
**2.** *$H$ satisfies the $s$-prime power condition and $s$-spec$(H)$ satisfies the ACC.*
**3.** *$H$ satisfies the $s$-prime power condition and $H$ is strongly $s$-discrete.*

*Proof.* Since $H$ is $s$-local, this is an immediate consequence of Lemma 3.7. and Theorem 5.4.. □

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## References


[1] D.D. Anderson, *A note on minimal prime ideals*, Proc. Amer. Math. Soc. **122** (1994), 13-14.
[2] D.D. Anderson, L.A. Mahaney, *Commutative rings in which every ideal is a product of primary ideals*, J. Algebra **106** (1987), 528-535.
[3] D.D. Anderson, L.A. Mahaney, *On primary factorizations*, J. Pure Appl. Algebra **54** (1988), 141-154.
[4] H.S. Butts, R. Gilmer, *Primary ideals and prime power ideals*, Canad. J. Math. **18** (1966), 1183-1195.
[5] L. Claborn, *Specified relations in the ideal group*, Michigan Math. J. **15** (1968), 249-255.
[6] M. Fontana, E. Houston, T. Lucas, *Factoring Ideals in Integral domains*, Lecture Notes of the Unione Mathematica Italiana, Springer-Verlag Berlin Heidelberg, 2013.
[7] A. Geroldinger, F. Halter-Koch, *Non-unique factorizations: Algebraic, combinatorial and analytic theory*, Pure Appl. Math., Chapman and Hall/CRC, Boca Raton, FL, 2006.
[8] F. Halter-Koch, *Ideal Systems. An Introduction to Multiplicative Ideal Theory*, Marcel Dekker, New York, 1998.
[9] W.W. McGovern, *Bézout SP-domains*, Comm. Algebra **35** (2007), 1777-1781.
[10] B. Olberding, *Factorization into prime and invertible ideals*, J. London Math. Soc. **62** (2000), 336-344.
[11] B. Olberding, *Factorization into radical ideals*, Arithmetical properties of commutative rings and monoids, 363-377, Lect. Notes Pure Appl. Math., 241, Chapman and Hall/CRC, Boca Raton, FL, 2005.
[12] J. Querré, *Idéaux divisoriels d'un anneau de polynômes*, J. Algebra **64** (1980), 270-284.
[13] A. Reinhart, *Structure of general ideal semigroups of monoids and domains*, J. Commut. Algebra **4** (2012), 413-444.
[14] A. Reinhart, *On integral domains that are C-monoids*, Houston J. Math. **39** (2013), 1095-1116.
[15] N.H. Vaughan, R.W. Yeagy, *Factoring ideals into semiprime ideals*, Canad. J. Math. **30** (1978), 1313-1318.
[16] R.W. Yeagy, *Semiprime factorizations in unions of Dedekind domains*, J. Reine Angew. Math. **310** (1979), 182-186.


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