ON A COUNTEREXAMPLE TO MORDELL'S PELLIAN EQUATION CONJECTURE: A NON-COMPUTER BASED APPROACH

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Abstract. In this note, we investigate a recently discovered counterexample to Mordell's Pellian equation conjecture. We provide a verification of this counterexample that can be checked without computer assistance.

1. Introduction and main result

Let $\mathbb{P}$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$ denote the sets of prime numbers, positive integers, nonnegative integers, integers and rational numbers, respectively. For each $p \in \mathbb{P}$ and each $a \in \mathbb{Z} \setminus \{0\}$, let $\mathrm{v}_p(a)$ be the p-adic exponent of a (i.e., the largest $k \in \mathbb{N}_0$ with $p^k \mid a$). Let $d \in \mathbb{N}_{\geq 2}$ be squarefree, let $K = \mathbb{Q}\left(\sqrt{d}\right)$, let $\mathcal{O}_K$ be the ring of algebraic integers of K, let $\mathcal{O}_d = \mathbb{Z} + d\mathcal{O}_K$ and let $\varepsilon > 1$ be the fundamental unit of $\mathcal{O}_K$. By $\mathcal{O}_K^\times$ (respectively $\mathcal{O}_d^\times$) we denote the unit group of $\mathcal{O}_K$ (respectively of $\mathcal{O}_d$). Let $\mathrm{N} : K \to \mathbb{Q}$ defined by $\mathrm{N}\left(a + b\sqrt{d}\right) = a^2 - db^2$ for each $a, b \in \mathbb{Q}$ be the norm map. Moreover, for each nonzero ideal I of $\mathcal{O}_K$, let $\mathrm{N}(I) = |\mathcal{O}_K/I|$ be the ideal norm of I. We set

$$\omega = \begin{cases} \sqrt{d} & \text{if } d \equiv 2, 3 \mod 4, \\ \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \mod 4. \end{cases}$$

Observe that $\mathcal{O}_K = \mathbb{Z}[\omega] = \mathbb{Z} \oplus \omega\mathbb{Z}$. Let $p \in \mathbb{P}$. Then p is called *ramified* in $\mathcal{O}_K$ if $p\mathcal{O}_K$ is the square of a maximal ideal of $\mathcal{O}_K$. Moreover, we say that p *splits* in $\mathcal{O}_K$ if $p\mathcal{O}_K$ is the product of two distinct maximal ideals of $\mathcal{O}_K$. After introducing the most important terminology above, we are now prepared to formulate Mordell's Pellian equation conjecture.

Conjecture 1.1. (Mordell's Pellian equation conjecture) Let $x, y \in \mathbb{N}_0$ be such that $\varepsilon = x + y\omega$. If $d \in \mathbb{P}$ with $d \equiv 3 \mod 4$, then $d \nmid y$.

This conjecture was stated by L. J. Mordell in 1961 ([5, page 283]) and independently by A. A. Kiselev and I. Š. Slavutskiĭ in 1959 ([4]). It was investigated in various papers and by many authors (for instance, see [1, 3, 8, 9]). Next, we present the main result of this note.

Theorem 1.2. *The number* $d = 39028039587479$ *is a counterexample to Mordell's Pellian equation conjecture.*

The counterexample in Theorem 1.2 was first announced in [7] and it was discovered and verified with computer assistance. For the methods, algorithms and programs that were used to find (and check) this counterexample, we refer to [7, Section 3].

The purpose of this note is to provide a detailed proof of Theorem 1.2 that can be checked without relying on computers. That said, the proof in Section 3 was created with computer assistance (and with the help of the programs Mathematica 12.0.0 and Pari/GP 2.15.2). We want to outline why (we think that)

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this proof can be verified without computers (within a manageable amount of time). Note that the vast majority of the verification process involves doing basic arithmetic. Therefore, we focus entirely on the arithmetic in the estimate below. We define an arithmetical operation as *hard* if it is a multiplication of integers a and b with $|a| \geq 10^3$, $|b| \geq 10^3$ and $|ab| \geq 10^{10}$. All other arithmetical operations in this note (additions, subtractions and multiplications of integers) are defined to be *easy*. We also define additions and subtractions where (at least) one of the members is a single digit integer as *trivial*. A rough estimate yields that there are approximately 550 hard operations and about 2200 (nontrivial) easy operations in the proof of Theorem 1.2 below. If we assume that (the verification of) a hard operation takes about 15 minutes (on average) and (the verification of) an easy (but nontrivial) operation takes about 3 minutes (on average), then a verification of the proof below is doable in about 250 hours. (That said, we think that skilled arithmeticians can perform the verification of all details below in substantially less than 100 hours.)

2. Preliminaries

We recall a variant of a result of H. C. Pocklington (cf. [2, Theorem 4] and [6]) that will be used to show that d is prime. For the readers' convenience we include a proof. This proof goes along the lines of the proof of [2, Theorem 4 and Corollary 1].

Proposition 2.1. *Let $a, b, c \in \mathbb{N}$ be such that $a^2 > c > 1$, $a \mid c-1$, $b^{c-1} \equiv 1 \mod c$ and for each $p \in \mathbb{P}$ with $p \mid a$, $\gcd\left(b^{\frac{c-1}{p}} - 1, c\right) = 1$. Then $c \in \mathbb{P}$.*

Proof. Claim: For each $q \in \mathbb{P}$ with $q \mid c$, it follows that $q > a$.
Let $q \in \mathbb{P}$ be such that $q \mid c$. Then $q \nmid b$ (since $b^{c-1} \equiv 1 \mod c$). Set $e = \min\{k \in \mathbb{N} \mid b^k \equiv 1 \mod q\}$. We obtain that $e \mid q-1$. It remains to show that $\mathrm{v}_p(a) \leq \mathrm{v}_p(e)$ for each $p \in \mathbb{P}$ with $p \mid a$. (Then $a \mid e \mid q-1$, and hence $q = 1 + a\ell$ for some $\ell \in \mathbb{N}$, which implies that $q > a$.) Let $p \in \mathbb{P}$ be such that $p \mid a$. Since $b^{c-1} \equiv 1 \mod c$, we have that $b^{c-1} \equiv 1 \mod q$, and thus $e \mid c-1$. Moreover, since $\gcd\left(b^{\frac{c-1}{p}} - 1, c\right) = 1$, we infer that $q \nmid b^{\frac{c-1}{p}} - 1$. Therefore, $b^{\frac{c-1}{p}} \not\equiv 1 \mod q$, and hence $e \nmid \frac{c-1}{p}$. In particular, we obtain that $p \nmid \frac{c-1}{e}$, and thus $\mathrm{v}_p(c-1) = \mathrm{v}_p\left(\frac{c-1}{e}\right) + \mathrm{v}_p(e) = \mathrm{v}_p(e)$. Since $a \mid c-1$, we have that $\mathrm{v}_p(a) \leq \mathrm{v}_p(e)$. □(Claim)
Assume to the contrary that $c \notin \mathbb{P}$. Then there are some $r, s \in \mathbb{P}$ such that $rs \mid c$. We infer by the claim that $c \geq rs > a^2 > c$, a contradiction. □

Next we mention a simple (well-known) lemma that will be used to compute integer powers modulo d.

Lemma 2.2. *Let $a, n \in \mathbb{N}$, let $m \in \mathbb{N}_0$ and let $(g_i)_{i=0}^m$ be the sequence of integers with $g_i \in \{0,1\}$ for each $i \in [0,m]$, $g_m = 1$ and $n = \sum_{j=0}^m g_j 2^j$ (i.e., $(g_i)_{i=0}^m$ is the binary representation of n). Furthermore, let $(h_i)_{i=0}^{m+1}$ be defined recursively by $h_0 = 1$ and $h_{r+1} = a^{g_{m-r}} h_r^2$ for each $r \in [0,m]$. Then $h_{m+1} = a^n$.*

Proof. It is straightforward to show by induction that $h_{r+1} = a^{\sum_{j=0}^r g_{m-j} 2^{r-j}}$ for each $r \in [0,m]$. We obtain that $h_{m+1} = a^{\sum_{j=0}^m g_{m-j} 2^{m-j}} = a^{\sum_{j=0}^m g_j 2^j} = a^n$. □

3. Proof of Theorem 1.2

In what follows, we will use the sieve of Eratosthenes and well-known facts about $\mathcal{O}_K$ (e.g. that $\mathcal{O}_K$ is a Dedekind domain) without further mention. Besides that, we will indicate the binary representation of a positive integer by using the subscript "$_2$".
Let $d = 39028039587479$ and let $x, y \in \mathbb{N}_0$ be such that $\varepsilon = x + y\omega$. We need to show that $d \equiv 3 \mod 4$, $d \in \mathbb{P}$ and $d \mid y$. Since $79 = 19 \cdot 4 + 3$, we have that $d \equiv 79 \equiv 3 \mod 4$. Observe that $\omega = \sqrt{d}$ and $\varepsilon = x + y\sqrt{d}$.

CLAIM 1: $d \in \mathbb{P}$.

Set $p = 3617$, $q = 4021$ and $a = pq$. By Proposition 2.1, it is sufficient to show that $p, q \in \mathbb{P}$, $a^2 > d$, $a \mid d-1$, $2^{d-1} \equiv 1 \mod d$ and $\gcd\left(2^{\frac{d-1}{p}} - 1, d\right) = \gcd\left(2^{\frac{d-1}{q}} - 1, d\right) = 1$.

Note that $\{t \in \mathbb{P} \mid t^2 \leq p\} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59\}$ and

$$\begin{aligned} p =& 1808 \cdot 2 + 1 = 1205 \cdot 3 + 2 = 723 \cdot 5 + 2 = 516 \cdot 7 + 5 = 328 \cdot 11 + 9 = 278 \cdot 13 + 3 = 212 \cdot 17 + 13 \\ =& 190 \cdot 19 + 7 = 157 \cdot 23 + 6 = 124 \cdot 29 + 21 = 116 \cdot 31 + 21 = 97 \cdot 37 + 28 = 88 \cdot 41 + 9 = 84 \cdot 43 + 5 \\ =& 76 \cdot 47 + 45 = 68 \cdot 53 + 13 = 61 \cdot 59 + 18. \end{aligned}$$

This implies that $p \in \mathbb{P}$.

Moreover, $\{t \in \mathbb{P} \mid t^2 \leq q\} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61\}$ and

$$\begin{aligned} q =& 2010 \cdot 2 + 1 = 1340 \cdot 3 + 1 = 804 \cdot 5 + 1 = 574 \cdot 7 + 3 = 365 \cdot 11 + 6 = 309 \cdot 13 + 4 = 236 \cdot 17 + 9 \\ =& 211 \cdot 19 + 12 = 174 \cdot 23 + 19 = 138 \cdot 29 + 19 = 129 \cdot 31 + 22 = 108 \cdot 37 + 25 = 98 \cdot 41 + 3 \\ =& 93 \cdot 43 + 22 = 85 \cdot 47 + 26 = 75 \cdot 53 + 46 = 68 \cdot 59 + 9 = 65 \cdot 61 + 56. \end{aligned}$$

Consequently, $q \in \mathbb{P}$. Observe that $a^2 > (3000 \cdot 4000)^2 > 10^{14} > d$ and $d = 1 + 10790168534p = 1 + 9706053118q = 1 + 2683454a$. In particular, we obtain that $a \mid d-1$.

Next, we show that $2^{d-1} \equiv 1 \mod d$ by applying Lemma 2.2. Each item in the list of equations below consists of two parts. The first part is used to derive the binary representation of $d-1$, while the second part is the outcome of determining the representatives of the integers h_i (in Lemma 2.2) modulo d. Also note that the second part contains an additional factor 2 if and only if the remainder (modulo 2) in the first part is 1. (Keep in mind that the computation of the binary representation of $d-1$ is technically done from the bottom to the top.)

- $2 \cdot 0 + 1 = 1$, $2 \cdot 1^2 = 2$.
- $2 \cdot 1 = 2$, $2^2 = 4$.
- $2 \cdot 2 = 4$, $4^2 = 16$.
- $2 \cdot 4 = 8$, $16^2 = 256$.
- $2 \cdot 8 + 1 = 17$, $2 \cdot 256^2 = 131072$.
- $2 \cdot 17 + 1 = 35$, $2 \cdot 131072^2 = 34359738368$.
- $2 \cdot 35 = 70$, $34359738368^2 = 30249831d + 18934861837375$.
- $2 \cdot 70 + 1 = 141$, $2 \cdot 18934861837375^2 = 18372892750447d + 8878320928137$.
- $2 \cdot 141 + 1 = 283$, $2 \cdot 8878320928137^2 = 4039382112766d + 4872136924624$.
- $2 \cdot 283 + 1 = 567$, $2 \cdot 4872136924624^2 = 1216444303284d + 37954390101716$.
- $2 \cdot 567 + 1 = 1135$, $2 \cdot 37954390101716^2 = 73820552772801d + 14864000930633$.
- $2 \cdot 1135 + 1 = 2271$, $2 \cdot 14864000930633^2 = 11322040563715d + 5306939836893$.
- $2 \cdot 2271 + 1 = 4543$, $2 \cdot 5306939836893^2 = 1443250069954d + 7254835280932$.
- $2 \cdot 4543 = 9086$, $7254835280932^2 = 1348585158511d + 1741096904855$.
- $2 \cdot 9086 + 1 = 18173$, $2 \cdot 1741096904855^2 = 155345667583d + 18644662148793$.
- $2 \cdot 18173 + 1 = 36347$, $2 \cdot 18644662148793^2 = 17814034746144d + 34365896782722$.
- $2 \cdot 36347 + 1 = 72695$, $2 \cdot 34365896782722^2 = 60521352041448d + 19084525628976$.
- $2 \cdot 72695 = 145390$, $19084525628976^2 = 9332242211824d + 32132914656880$.
- $2 \cdot 145390 + 1 = 290781$, $2 \cdot 32132914656880^2 = 52911917444994d + 20520829038674$.
- $2 \cdot 290781 + 1 = 581563$, $2 \cdot 20520829038674^2 = 21579583749811d + 29933591140083$.
- $2 \cdot 581563 = 1163126$, $29933591140083^2 = 22958362449471d + 9944907473280$.
- $2 \cdot 1163126 = 2326252$, $9944907473280^2 = 2534105881245d + 7176351027045$.

- $2 \cdot 2326252 = 4652504$, $7176351027045^2 = 1319564462056d + 17827533235201$.
- $2 \cdot 4652504 + 1 = 9305009$, $2 \cdot 17827533235201^2 = 16286800188352d + 30177590176194$.
- $2 \cdot 9305009 + 1 = 18610019$, $2 \cdot 30177590176194^2 = 46668341964810d + 29055194037282$.
- $2 \cdot 18610019 = 37220038$, $29055194037282^2 = 21630712417718d + 35791355394602$.
- $2 \cdot 37220038 + 1 = 74440077$, $2 \cdot 35791355394602^2 = 65646193584044d + 32203984891732$.
- $2 \cdot 74440077 + 1 = 148880155$, $2 \cdot 32203984891732^2 = 53146233009337d + 18209288628225$.
- $2 \cdot 148880155 + 1 = 297760311$, $2 \cdot 18209288628225^2 = 16991793379874d + 22534432303604$.
- $2 \cdot 297760311 + 1 = 595520623$, $2 \cdot 22534432303604^2 = 26022349296203d + 5977961735395$.
- $2 \cdot 595520623 + 1 = 1191041247$, $2 \cdot 5977961735395^2 = 1831300105645d + 26065072393095$.
- $2 \cdot 1191041247 = 2382082494$, $26065072393095^2 = 17407689600562d + 2873437115827$.
- $2 \cdot 2382082494 = 4764164988$, $2873437115827^2 = 211556638403d + 25924124537892$.
- $2 \cdot 4764164988 + 1 = 9528329977$, $2 \cdot 25924124537892^2 = 34439866319688d + 247631620776$.
- $2 \cdot 9528329977 = 19056659954$, $247631620776^2 = 1571214446d + 9241789320542$.
- $2 \cdot 19056659954 + 1 = 38113319909$, $2 \cdot 9241789320542^2 = 4376887527432d + 9150470123600$.
- $2 \cdot 38113319909 + 1 = 76226639819$, $2 \cdot 9150470123600^2 = 4290817800121d + 10455437635041$.
- $2 \cdot 76226639819 = 152453279638$, $10455437635041^2 = 2800965082942d + 24962411388463$.
- $2 \cdot 152453279638 + 1 = 304906559277$, $2 \cdot 24962411388463^2 = 31932015490051d + 31490910333309$.
- $2 \cdot 304906559277 = 609813118554$, $31490910333309^2 = 25409358094908d + 8143824432549$.
- $2 \cdot 609813118554 = 1219626237108$, $8143824432549^2 = 1699339169714d + 30354920226395$.
- $2 \cdot 1219626237108 + 1 = 2439252474217$, $2 \cdot 30354920226395^2 = 47218419971389d + 29633566753719$.
- $2 \cdot 2439252474217 = 4878504948434$, $29633566753719^2 = 22500445521451d + 15832914818932$.
- $2 \cdot 4878504948434 + 1 = 9757009896869$, $2 \cdot 15832914818932^2 = 12846209766784d + 7728534743712$.
- $2 \cdot 9757009896869 + 1 = 19514019793739$, $2 \cdot 7728534743712^2 = 3060889038553d + 1$.
- $2 \cdot 19514019793739 = 39028039587478$, $1^2 = 1$.

This shows that $d - 1 = 39028039587478 = 100011011111101110110001101111100101101010010110_2$. By Lemma 2.2, we obtain that $2^{d-1} \equiv 1 \mod d$. Now we show that $2^{\frac{d-1}{p}} \equiv 10285064380914 \mod d$ by applying Lemma 2.2 again. This is done along the lines of the proof of $2^{d-1} \equiv 1 \mod d$ above.

- $2 \cdot 0 + 1 = 1$, $2 \cdot 1^2 = 2$.
- $2 \cdot 1 = 2$, $2^2 = 4$.
- $2 \cdot 2 + 1 = 5$, $2 \cdot 4^2 = 32$.
- $2 \cdot 5 = 10$, $32^2 = 1024$.
- $2 \cdot 10 = 20$, $1024^2 = 1048576$.
- $2 \cdot 20 = 40$, $1048576^2 = 1099511627776$.
- $2 \cdot 40 = 80$, $1099511627776^2 = 30975827440d + 31390886082416$.
- $2 \cdot 80 = 160$, $31390886082416^2 = 25248199485668d + 897251646084$.
- $2 \cdot 160 + 1 = 321$, $2 \cdot 897251646084^2 = 41255493481d + 23445371345713$.
- $2 \cdot 321 + 1 = 643$, $2 \cdot 23445371345713^2 = 28168744489781d + 6603555904639$.
- $2 \cdot 643 = 1286$, $6603555904639^2 = 1117323622877d + 23588814563238$.
- $2 \cdot 1286 = 2572$, $23588814563238^2 = 14257241162513d + 36673782069917$.
- $2 \cdot 2572 + 1 = 5145$, $2 \cdot 36673782069917^2 = 68923077127515d + 7453272389093$.
- $2 \cdot 5145 = 10290$, $7453272389093^2 = 1423368170504d + 28404889843233$.
- $2 \cdot 10290 = 20580$, $28404889843233^2 = 20673284529132d + 20389277954061$.
- $2 \cdot 20580 + 1 = 41161$, $2 \cdot 20389277954061^2 = 21303793881634d + 11174538322756$.
- $2 \cdot 41161 = 82322$, $11174538322756^2 = 3199502410231d + 12534158337887$.
- $2 \cdot 82322 = 164644$, $12534158337887^2 = 4025442397307d + 8237359105716$.
- $2 \cdot 164644 + 1 = 329289$, $2 \cdot 8237359105716^2 = 3477196690058d + 10027987161530$.
- $2 \cdot 329289 + 1 = 658579$, $2 \cdot 10027987161530^2 = 5153245080958d + 10764345756918$.
- $2 \cdot 658579 + 1 = 1317159$, $2 \cdot 10764345756918^2 = 5937840629415d + 7258376622663$.
- $2 \cdot 1317159 = 2634318$, $7258376622663^2 = 1349902064087d + 13688184444896$.

- $2 \cdot 2634318 = 5268636$, $13688184444896^2 = 4800814885347d + 11643098680603$.
- $2 \cdot 5268636 + 1 = 10537273$, $2 \cdot 11643098680603^2 = 6946889893478d + 1481444325256$.
- $2 \cdot 10537273 + 1 = 21074547$, $2 \cdot 1481444325256^2 = 112466693794d + 1299469525746$.
- $2 \cdot 21074547 + 1 = 42149095$, $2 \cdot 1299469525746^2 = 86533736574d + 22516898956086$.
- $2 \cdot 42149095 + 1 = 84298191$, $2 \cdot 22516898956086^2 = 25981870673373d + 12486243382125$.
- $2 \cdot 84298191 + 1 = 168596383$, $2 \cdot 12486243382125^2 = 7989449403329d + 36029689713659$.
- $2 \cdot 168596383 = 337192766$, $36029689713659^2 = 33261689661681d + 18099097476082$.
- $2 \cdot 337192766 + 1 = 674385533$, $2 \cdot 18099097476082^2 = 16786768329189d + 8643489516917$.
- $2 \cdot 674385533 = 1348771066$, $8643489516917^2 = 1914262458958d + 16355540998007$.
- $2 \cdot 1348771066 + 1 = 2697542133$, $2 \cdot 16355540998007^2 = 13708283796212d + 16782573114550$.
- $2 \cdot 2697542133 + 1 = 5395084267$, $2 \cdot 16782573114550^2 = 14433456731225d + 18430065073225$.
- $2 \cdot 5395084267 = 10790168534$, $18430065073225^2 = 8703160655609d + 10285064380914$.

Observe that $\frac{d-1}{p} = 10790168534 = 1010000011001001001110011111010110_2$. We infer by Lemma 2.2 that $2^{\frac{d-1}{p}} - 1 \equiv 10285064380913 \mod d$. Since $7336389398826 \cdot 10285064380913 = 1933359658541d - 1$, we have that $\gcd\left(2^{\frac{d-1}{p}} - 1, d\right) = 1$. Finally, we prove that $2^{\frac{d-1}{q}} \equiv 15901499388071 \mod d$ by applying Lemma 2.2. We proceed along the lines of the proof of $2^{d-1} \equiv 1 \mod d$ above.

- $2 \cdot 0 + 1 = 1$, $2 \cdot 1^2 = 2$.
- $2 \cdot 1 = 2$, $2^2 = 4$.
- $2 \cdot 2 = 4$, $4^2 = 16$.
- $2 \cdot 4 + 1 = 9$, $2 \cdot 16^2 = 512$.
- $2 \cdot 9 = 18$, $512^2 = 262144$.
- $2 \cdot 18 = 36$, $262144^2 = 68719476736$.
- $2 \cdot 36 = 72$, $68719476736^2 = 120999325d + 36711407762021$.
- $2 \cdot 72 = 144$, $36711407762021^2 = 34532286892056d + 31998527837617$.
- $2 \cdot 144 + 1 = 289$, $2 \cdot 31998527837617^2 = 52470264691605d + 38620486063583$.
- $2 \cdot 289 = 578$, $38620486063583^2 = 38217188450990d + 25764109643679$.
- $2 \cdot 578 + 1 = 1157$, $2 \cdot 25764109643679^2 = 34016022979769d + 6589566597731$.
- $2 \cdot 1157 = 2314$, $6589566597731^2 = 1112594647461d + 6956899607542$.
- $2 \cdot 2314 = 4628$, $6956899607542^2 = 1240094369611d + 30110629581095$.
- $2 \cdot 4628 = 9256$, $30110629581095^2 = 23230734194007d + 5999347360672$.
- $2 \cdot 9256 = 18512$, $5999347360672^2 = 922213084091d + 7444474594995$.
- $2 \cdot 18512 + 1 = 37025$, $2 \cdot 7444474594995^2 = 2840019769443d + 7552148495853$.
- $2 \cdot 37025 + 1 = 74051$, $2 \cdot 7552148495853^2 = 2922767707846d + 10516484734984$.
- $2 \cdot 74051 = 148102$, $10516484734984^2 = 2833769063220d + 28051642057876$.
- $2 \cdot 148102 + 1 = 296205$, $2 \cdot 28051642057876^2 = 40324578454903d + 38284942303215$.
- $2 \cdot 296205 = 592410$, $38284942303215^2 = 37555993656179d + 37268879953484$.
- $2 \cdot 592410 = 1184820$, $37268879953484^2 = 35589013121550d + 38577918665806$.
- $2 \cdot 1184820 + 1 = 2369641$, $2 \cdot 38577918665806^2 = 76265978220592d + 10176084091704$.
- $2 \cdot 2369641 + 1 = 4739283$, $2 \cdot 10176084091704^2 = 5306578989668d + 15426040080260$.
- $2 \cdot 4739283 + 1 = 9478567$, $2 \cdot 15426040080260^2 = 12194448661681d + 9515608643001$.
- $2 \cdot 9478567 = 18957134$, $9515608643001^2 = 2320044993389d + 36985824109670$.
- $2 \cdot 18957134 + 1 = 37914269$, $2 \cdot 36985824109670^2 = 70100942785264d + 9512534908344$.
- $2 \cdot 37914269 + 1 = 75828539$, $2 \cdot 9512534908344^2 = 4637092784516d + 22708882969508$.
- $2 \cdot 75828539 + 1 = 151657079$, $2 \cdot 22708882969508^2 = 264268137048896d + 7292632926944$.
- $2 \cdot 151657079 + 1 = 303314159$, $2 \cdot 7292632926944^2 = 2725348009752d + 37465133263064$.
- $2 \cdot 303314159 + 1 = 606628319$, $2 \cdot 37465133263064^2 = 71929629325755d + 32577343114547$.
- $2 \cdot 606628319 + 1 = 1213256639$, $2 \cdot 32577343114547^2 = 54385682479598d + 31854972276976$.
- $2 \cdot 1213256639 + 1 = 2426513279$, $2 \cdot 31854972276976^2 = 52000524212466d + 35754130095938$.

- $2 \cdot 2426513279 + 1 = 4853026559$, $2 \cdot 35754130095938^2 = 65509712116177d + 10262965651905$.
- $2 \cdot 4853026559 = 9706053118$, $10262965651905^2 = 2698789513526d + 15901499388071$.

It is now clear that $\frac{d-1}{q} = 9706053118 = 1001000001010000110100111011111111110_2$. Moreover, $2^{\frac{d-1}{q}} - 1 \equiv 15901499388070 \mod d$ by Lemma 2.2. Since $11826010015564 \cdot 15901499388070 = 4818363746001d + 1$, it follows that $\gcd\left(2^{\frac{d-1}{q}} - 1, d\right) = 1$. This shows that $d \in \mathbb{P}$. □(Claim 1)

Finally, we show that $d \mid y$. Set $r = 57$ and $s = 56$, and let $(a_i)_{i=1}^{r}, (b_i)_{i=1}^{r}, (c_i)_{i=1}^{s}, (d_i)_{i=1}^{s}, (e_i)_{i=1}^{s}, (f_i)_{i=1}^{s}$ be defined as follows.

i	a_i	b_i	c_i	d_i	e_i	f_i
1	−3033477	3410579	2	84702667	1	19514019793739
2	−2271496	3206358	5	146634276	2	7805607917495
3	−1787484	82862	23	112959051	7	1696871286410
4	−1758518	4049619	29	57435327	14	1345794468527
5	−1699348	2129966	37	54701496	5	1054811880742
6	−1427442	8014362	47	60589478	3	830383821010
7	−832653	717212	53	37470722	22	736378105415
8	−748505	9346906	61	9465381	6	639803927663
9	−674823	15717595	79	40989357	9	494025817562
10	−554788	3962534	83	26886549	11	470217344426
11	−521509	17200418	89	16628268	8	438517298735
12	−482142	13587629	97	33227623	5	402350923582
13	−445057	8567013	107	27266087	25	364748033522
14	−425248	2262519	109	16545406	22	358055409055
15	−322867	3536609	113	31075882	50	345380881283
16	−280973	3410579	137	14887445	36	284876201359
17	−252892	4558274	139	18716004	49	280777263202
18	−226498	2007139	157	18190318	57	248586239390
19	−215202	3644517	163	23253236	3	239435825690
20	−207973	4049619	167	4568510	23	233700835850
21	−206198	5460335	179	16995512	63	218033740690
22	−193242	2284255	193	16028724	94	202217821651
23	−126128	8014362	199	21271274	15	196120801946
24	−121640	5480460	223	1909615	52	175013630425
25	−54422	13915416	227	19110033	90	171929689777
26	−48408	8365791	229	9530079	41	170428120462
27	−40763	864323	239	4315464	40	163297236761
28	−34641	7529543	263	13737288	63	148395587770
29	−22980	14887445	269	6324658	125	145085648966
30	3843	1909615	271	16934137	49	144014906218
31	29419	6660471	277	5275486	9	140895449774
32	37368	3205421	283	6324658	96	137908267061
33	38302	2075306	307	8891039	68	127127164765
34	80167	4486315	311	8094883	49	125492088698
35	83134	7103618	337	8014362	90	115810206467
36	102633	6324658	349	21686733	103	111828193630
37	115435	9578599	353	17200418	128	110561018615
38	116792	11851993	359	6177547	132	108713202145

39	156001	4568510	383	35556425	23	101900886650
40	197807	3013477	409	11851993	182	95423079595
41	211873	4414221	421	4049619	1	92703181918
42	219482	1158859	433	1909615	54	90134040611
43	229892	6177547	439	3410579	13	88902140290
44	264088	6019577	443	13587629	158	88099412105
45	269512	147111	449	5020616	202	86922137075
46	315998	4315464	467	20082812	8	83571819245
47	339243	1212808	557	26469383	149	70068293654
48	471496	2548	563	15717595	214	69321562241
49	499231	1109106	569	5278544	205	68590579166
50	596696	1909615	593	6589566	75	65814569278
51	1124049	5020616	601	4414221	280	64938501679
52	1154432	1909615	631	13309440	139	61851092818
53	1214853	23621570	641	17283280	265	60886177094
54	1403789	16545406	647	13915416	74	60321544949
55	1947578	5128375	683	11066152	192	57142078405
56	1991126	10145918	691	9578599	38	56480520385
57	2126573	6589566	–	–	–	–

For each $j \in [1,s]$, set $P_j = c_j\mathcal{O}_K + \left(e_j + \sqrt{d}\right)\mathcal{O}_K$ and $\overline{P}_j = c_j\mathcal{O}_K + \left(e_j - \sqrt{d}\right)\mathcal{O}_K$. Then $P_1 = \overline{P}_1$, $c_1f_1 + e_1^2 = d$, $c_1 \in \mathbb{P}$ and $\gcd(c_1, e_1) = 1$. Observe that $c_jf_j + e_j^2 = d$, $c_j \in \mathbb{P}$ and $\gcd(c_j, 2e_j) = 1$ for each $j \in [2,s]$. In particular, $P_j, \overline{P}_j \in \max(\mathcal{O}_K)$, $c_j\mathcal{O}_K = P_j\overline{P}_j$ and $\mathrm{N}(P_j) = \mathrm{N}\left(\overline{P}_j\right) = c_j$ for each $j \in [1,s]$. Set $z = \prod_{i=1}^{r}\left(a_i + \sqrt{d}\right)^{b_i}$, set $n = \prod_{j=1}^{s} c_j^{d_j}$ and set $\eta = \frac{z}{n}$. Clearly, $\eta \in K$ and $\eta > 0$.

Claim 2: $\eta \in \mathcal{O}_K^{\times}$.

Observe the following.

$$\begin{aligned} d =& 1561121583499c_2^2 + 2^2 = 312224316694c_2^3 + 27^2 = 12488972039c_2^5 + 1402^2 \\ =& 73777012451c_3^2 + 30^2 = 46406705350c_4^2 + 623^2 = 17667740470c_6^2 + 943^2 \\ =& 13893926015c_7^2 + 1188^2 = 5665268302c_{10}^2 + 2501^2 = 3056458675c_{15}^2 + 10898^2 \\ =& 1399382135c_{20}^2 + 25908^2 = 1218065438c_{21}^2 + 2211^2 = 1047746695c_{22}^2 + 22868^2 \\ =& 320417518c_{36}^2 + 29419^2 = 125690710c_{47}^2 + 180617^2. \end{aligned}$$

Also note that $2 = e_2$, $27 = 5c_2 + e_2$, $1402 = 280c_2 + e_2$, $30 = c_3 + e_3$, $623 = 21c_4 + e_4$, $943 = 20c_6 + e_6$, $1188 = 22c_7 + e_7$, $2501 = 30c_{10} + e_{10}$, $10898 = 96c_{15} + e_{15}$, $25908 = 155c_{20} + e_{20}$, $2211 = 12c_{21} + e_{21}$, $22868 = 118c_{22} + e_{22}$, $29419 = 84c_{36} + e_{36}$ and $180617 = 324c_{47} + e_{47}$.

Let $j \in [2,s]$, $k, m \in \mathbb{N}_0$, $\ell \in \mathbb{N}$ be such that $d = kc_j^{\ell} + (mc_j + e_j)^2$. Set $I = c_j^{\ell}\mathcal{O}_K + \left(mc_j + e_j + \sqrt{d}\right)\mathcal{O}_K$ and $\overline{I} = c_j^{\ell}\mathcal{O}_K + \left(mc_j + e_j - \sqrt{d}\right)\mathcal{O}_K$. Clearly, I and $\overline{I}$ are ideals of $\mathcal{O}_K$ such that $I \subseteq P_j$ and $\overline{I} \subseteq \overline{P}_j$. Observe that $I\overline{I} = c_j^{\ell}J$ for some ideal J of $\mathcal{O}_K$ with $c_j^{\ell}, 2(mc_j + e_j) \in J$. Since $\gcd\left(c_j^{\ell}, 2(mc_j + e_j)\right) = 1$, we obtain that $J = \mathcal{O}_K$, $I \nsubseteq \overline{P}_j$ and $\overline{I} \nsubseteq P_j$. Since $I\overline{I} = c_j^{\ell}\mathcal{O}_K = P_j^{\ell}\overline{P}_j^{\ell}$, we infer that $P_j^{\ell} = I$ and $\overline{P}_j^{\ell} = \overline{I}$. Using this, we get the following equations (and in analogy the equations for the conjugates $\overline{P}_j^{\ell}$).

Equations, part A

$P_2^2 = c_2^2\mathcal{O}_K + \left(2+\sqrt{d}\right)\mathcal{O}_K$, $P_2^3 = c_2^3\mathcal{O}_K + \left(27+\sqrt{d}\right)\mathcal{O}_K$, $P_2^5 = c_2^5\mathcal{O}_K + \left(1402+\sqrt{d}\right)\mathcal{O}_K$,
$P_3^2 = c_3^2\mathcal{O}_K + \left(30+\sqrt{d}\right)\mathcal{O}_K$, $P_4^2 = c_4^2\mathcal{O}_K + \left(623+\sqrt{d}\right)\mathcal{O}_K$, $P_6^2 = c_6^2\mathcal{O}_K + \left(943+\sqrt{d}\right)\mathcal{O}_K$,
$P_7^2 = c_7^2\mathcal{O}_K + \left(1188+\sqrt{d}\right)\mathcal{O}_K$, $P_{10}^2 = c_{10}^2\mathcal{O}_K + \left(2501+\sqrt{d}\right)\mathcal{O}_K$, $P_{15}^2 = c_{15}^2\mathcal{O}_K + \left(10898+\sqrt{d}\right)\mathcal{O}_K$,
$P_{20}^2 = c_{20}^2\mathcal{O}_K + \left(25908+\sqrt{d}\right)\mathcal{O}_K$, $P_{21}^2 = c_{21}^2\mathcal{O}_K + \left(2211+\sqrt{d}\right)\mathcal{O}_K$, $P_{22}^2 = c_{22}^2\mathcal{O}_K + \left(22868+\sqrt{d}\right)\mathcal{O}_K$,
$P_{36}^2 = c_{36}^2\mathcal{O}_K + \left(29419+\sqrt{d}\right)\mathcal{O}_K$, $P_{47}^2 = c_{47}^2\mathcal{O}_K + \left(180617+\sqrt{d}\right)\mathcal{O}_K$.

The following equations can easily be derived (by using elementary arithmetic).

Equations, part B

$a_1 = -1516739c_1 + e_1 = -121339c_2^2 - 2 = -131890c_3 - e_3 = -81986c_5 + e_5 = -64542c_6 - e_6 = -57235c_7 - e_7 = -6910c_{43} + e_{43} = -4732c_{53} - e_{53}$,
$a_2 = -4294c_3^2 + 30 = -42858c_7 - e_7 = -28753c_9 - e_9 = -16580c_{16} - e_{16} = -9504c_{27} - e_{27} = -4864c_{46} - e_{46}$,
$a_3 = -77717c_3 + e_3 = -20084c_{11} - e_{11} = -15818c_{15} - e_{15} = -4667c_{39} - e_{39} = -2833c_{52} + e_{52} = -2789c_{53} + e_{53}$,
$a_4 = -351704c_2 + e_2 = -76457c_3 - e_3 = -33180c_7 + e_7 = -18129c_{12} - e_{12} = -4982c_{37} + e_{37} = -4300c_{40} + e_{40} = -4177c_{41} - e_{41}$,
$a_5 = -13595c_2^3 + 27 = -73885c_3 + e_3 = -17519c_{12} - e_{12} = -4437c_{39} + e_{39} = -3018c_{48} - e_{48} = -2828c_{51} + e_{51}$,
$a_6 = -285488c_2 - e_2 = -62063c_3 + e_3 = -18069c_9 + e_9 = -13096c_{14} + e_{14} = -7173c_{23} - e_{23} = -4236c_{35} + e_{35} = -2563c_{47} + e_{47}$,
$a_7 = -416327c_1 + e_1 = -166531c_2 + e_2 = -36202c_3 - e_3 = -22504c_5 - e_5 = -8584c_{12} - e_{12} = -3006c_{31} + e_{31} = -2319c_{38} - e_{38} = -1783c_{46} + e_{46}$,
$a_8 = -374253c_1 + e_1 = -1415c_3^2 + 30 = -25811c_4 + e_4 = -20230c_5 + e_5 = -9018c_{10} - e_{10} = -1186c_{52} - e_{52} = -1157c_{54} + e_{54}$,
$a_9 = -337412c_1 + e_1 = -26993c_2^2 + 2 = -14358c_6 + e_6 = -4140c_{19} - e_{19} = -1762c_{39} + e_{39} = -1445c_{46} - e_{46} = -1199c_{48} + e_{48}$,
$a_{10} = -110958c_2 + e_2 = -5090c_{14} + e_{14} = -1572c_{37} + e_{37} = -1188c_{46} + e_{46} = -879c_{52} - e_{52} = -812c_{55} - e_{55}$,
$a_{11} = -260755c_1 + e_1 = -22674c_3 - e_3 = -11096c_6 + e_6 = -2297c_{25} - e_{25} = -1494c_{36} - e_{36} = -1477c_{37} - e_{37} = -814c_{53} + e_{53}$,
$a_{12} = -96428c_2 - e_2 = -20963c_3 + e_3 = -13031c_5 + e_5 = -3469c_{17} + e_{17} = -1833c_{28} - e_{28} = -1088c_{44} - e_{44} = -856c_{48} - e_{48}$,
$a_{13} = -222529c_1 + e_1 = -89011c_2 - e_2 = -19350c_3 - e_3 = -5362c_{10} - e_{10} = -3939c_{15} + e_{15} = -2486c_{21} - e_{21} = -1961c_{25} + e_{25} = -1005c_{44} + e_{44}$,
$a_{14} = -17010c_2^2 + 2 = -5383c_9 + e_9 = -2137c_{23} + e_{23} = -1569c_{30} - e_{30} = -747c_{49} - e_{49} = -663c_{53} - e_{53}$,
$a_{15} = -161434c_1 + e_1 = -64573c_2 - e_2 = -14038c_3 + e_3 = -8726c_5 - e_5 = -5293c_8 + e_8 = -1038c_{34} - e_{34} = -915c_{37} + e_{37} = -473c_{55} + e_{55}$,
$a_{16} = -140487c_1 + e_1 = -2248c_2^3 + 27 = -7594c_5 + e_5 = -1790c_{18} + e_{18} = -1412c_{23} + e_{23} = -915c_{33} - e_{33} = -640c_{43} - e_{43}$,
$a_{17} = -50578c_2 - e_2 = -478c_3^2 - 30 = -933c_{30} - e_{30} = -913c_{31} + e_{31} = -813c_{34} - e_{34} = -401c_{52} + e_{52}$,
$a_{18} = -9060c_2^2 + 2 = -7c_{21}^2 - 2211 = -738c_{33} + e_{33} = -642c_{37} + e_{37} = -504c_{45} - e_{45}$,
$a_{19} = -8608c_2^2 - 2 = -4060c_7 - e_7 = -3528c_8 + e_8 = -2011c_{13} - e_{13} = -16c_{15}^2 - 10898 = -610c_{37} + e_{37}$,
$a_{20} = -103987c_1 + e_1 = -67c_2^5 + 1402 = -9042c_3 - e_3 = -7171c_4 - e_4 = -2144c_{12} - e_{12} = -908c_{26} - e_{26} = -494c_{41} + e_{41}$,
$a_{21} = -8248c_2^2 + 2 = -1313c_{18} - e_{18} = -1265c_{19} - e_{19} = -767c_{29} + e_{29} = -574c_{38} - e_{38} = -327c_{52} + e_{52}$,

$a_{22} = -38648c_2 - e_2 = -6664c_4 + e_4 = -3168c_8 + e_8 = -6c_{20}^2 - 25908 = -735c_{28} + e_{28} = -322c_{51} + e_{51}$,
$a_{23} = -25226c_2 + e_2 = -3409c_5 + e_5 = -1179c_{13} + e_{13} = -803c_{18} - e_{18} = -4c_{22}^2 + 22868 = -374c_{35} - e_{35}$,
$a_{24} = -230c_3^2 + 30 = -4194c_4 - e_4 = -1994c_8 - e_8 = -531c_{26} - e_{26} = -396c_{33} - e_{33} = -205c_{50} - e_{50}$,
$a_{25} = -10884c_2 - e_2 = -1471c_5 + e_5 = -689c_9 + e_9 = -561c_{12} - e_{12} = -347c_{18} + e_{18} = -201c_{30} + e_{30} = -84c_{54} - e_{54}$,
$a_{26} = -9682c_2 + e_2 = -2105c_3 + e_3 = -544c_{11} + e_{11} = -499c_{12} - e_{12} = -297c_{19} + e_{19} = -139c_{36} + e_{36} = -70c_{56} - e_{56}$,
$a_{27} = -20382c_1 + e_1 = -8153c_2 + e_2 = -77c_3^2 - 30 = -260c_{18} + e_{18} = -152c_{29} + e_{29} = -133c_{33} + e_{33} = -72c_{49} + e_{49}$,
$a_{28} = -17321c_1 + e_1 = -1195c_4 + e_4 = -307c_{15} + e_{15} = -179c_{22} - e_{22} = -174c_{23} - e_{23} = -153c_{25} + e_{25} = -51c_{55} + e_{55}$,
$a_{29} = -489c_6 + e_6 = -434c_7 + e_7 = -291c_9 + e_9 = -277c_{10} + e_{10} = -215c_{13} + e_{13} = -168c_{16} + e_{16} = -141c_{19} + e_{19}$,
$a_{30} = 1921c_1 + e_1 = 769c_2 - e_2 = 133c_4 - e_4 = 104c_5 - e_5 = 28c_{17} - e_{17} = 17c_{24} + e_{24} = 14c_{30} + e_{30} = 9c_{42} - e_{42}$,
$a_{31} = 14709c_1 + e_1 = 626c_6 - e_6 = 215c_{16} - e_{16} = 212c_{17} - e_{17} = 164c_{21} + e_{21} = 29419$,
$a_{32} = 7474c_2 - e_2 = 1625c_3 - e_3 = 795c_6 + e_6 = 349c_{13} + e_{13} = 163c_{26} + e_{26} = 80c_{46} + e_{46} = 59c_{52} + e_{52}$,
$a_{33} = 1532c_2^2 + 2 = 1665c_3 + e_3 = 815c_6 - e_6 = 628c_8 - e_8 = 235c_{19} - e_{19} = 123c_{34} + e_{34} = 82c_{46} + e_{46}$,
$a_{34} = 40083c_1 + e_1 = 16033c_2 + e_2 = 1513c_7 - e_7 = 12c_{10}^2 - 2501 = 709c_{15} + e_{15} = 296c_{30} - e_{30} = 230c_{36} - e_{36}$,
$a_{35} = 2247c_5 - e_5 = 934c_{11} + e_{11} = 857c_{12} + e_{12} = 217c_{39} + e_{39} = 178c_{46} + e_{46} = 122c_{55} - e_{55}$,
$a_{36} = 51316c_1 + e_1 = 20527c_2 - e_2 = 4462c_3 + e_3 = 2774c_5 - e_5 = 448c_{26} + e_{26} = 390c_{28} + e_{28} = 382c_{29} - e_{29} = 363c_{32} - e_{32}$,
$a_{37} = 57717c_1 + e_1 = 138c_4^2 - 623 = 3120c_5 - e_5 = 2456c_6 + e_6 = 1190c_{12} + e_{12} = 580c_{23} + e_{23} = 167c_{56} + e_{56}$,
$a_{38} = 23358c_2 + e_2 = 2485c_6 - e_6 = 1407c_{10} + e_{10} = 10c_{15}^2 - 10898 = 305c_{39} - e_{39} = 286c_{40} - e_{40}$,
$a_{39} = 78000c_1 + e_1 = 2943c_7 + e_7 = 1431c_{14} + e_{14} = 994c_{18} - e_{18} = 934c_{20} + e_{20} = 784c_{23} - e_{23} = 241c_{54} + e_{54}$,
$a_{40} = 98903c_1 + e_1 = 39561c_2 + e_2 = 8600c_3 + e_3 = 5346c_5 + e_5 = 2504c_9 - e_9 = 871c_{25} + e_{25} = 441c_{45} - e_{45} = 348c_{49} - e_{49}$,
$a_{41} = 105936c_1 + e_1 = 8475c_2^2 - 2 = 4508c_6 - e_6 = 1184c_{21} - e_{21} = 782c_{30} - e_{30} = 372c_{49} + e_{49} = 353c_{51} - e_{51}$,
$a_{42} = 43896c_2 + e_2 = 9543c_3 - e_3 = 2466c_{11} + e_{11} = 2051c_{13} + e_{13} = 1103c_{23} - e_{23} = 573c_{39} + e_{39} = 470c_{46} - e_{46}$,
$a_{43} = 45978c_2 + e_2 = 9995c_3 + e_3 = 4338c_7 - e_7 = 2034c_{15} + e_{15} = 812c_{32} + e_{32} = 640c_{38} + e_{38} = 413c_{47} - e_{47}$,
$a_{44} = 52818c_2 - e_2 = 9106c_4 + e_4 = 3343c_9 - e_9 = 1475c_{21} + e_{21} = 1327c_{23} + e_{23} = 860c_{33} + e_{33} = 849c_{34} + e_{34}$,
$a_{45} = 53902c_2 + e_2 = 9294c_4 - e_4 = 3247c_{10} + e_{10} = 1717c_{18} - e_{18} = 1025c_{28} - e_{28} = 973c_{31} - e_{31} = 952c_{32} + e_{32}$,
$a_{46} = 12640c_2^2 - 2 = 375c_4^2 + 623 = 2796c_{15} + e_{15} = 1765c_{21} + e_{21} = 1322c_{27} + e_{27} = 825c_{39} + e_{39}$,
$a_{47} = 169621c_1 + e_1 = 67849c_2 - e_2 = 14750c_3 - e_3 = 154c_6^2 - 943 = 1252c_{30} - e_{30} = 829c_{40} + e_{40} = 491c_{56} - e_{56}$,
$a_{48} = 16258c_4 + e_4 = 12743c_5 + e_5 = 1793c_{28} - e_{28} = 1231c_{39} + e_{39} = 829c_{49} - e_{49} = 747c_{52} + e_{52}$,
$a_{49} = 249615c_1 + e_1 = 21706c_3 - e_3 = 10622c_6 - e_6 = 2089c_{27} - e_{27} = 1842c_{30} + e_{30} = 1069c_{46} + e_{46} = 842c_{50} - e_{50}$,
$a_{50} = 25943c_3 + e_3 = 212c_7^2 + 1188 = 9782c_8 - e_8 = 7553c_9 + e_9 = 2676c_{24} - e_{24} = 1071c_{47} + e_{47}$,
$a_{51} = 562024c_1 + e_1 = 48872c_3 - e_3 = 23916c_6 - e_6 = 8205c_{16} - e_{16} = 2537c_{44} + e_{44} = 2503c_{45} + e_{45} = 1754c_{53} - e_{53}$,
$a_{52} = 230886c_2 + e_2 = 50193c_3 - e_3 = 31201c_5 - e_5 = 5982c_{22} - e_{22} = 5086c_{25} - e_{25} = 2666c_{42} + e_{42} = 2472c_{46} + e_{46}$,
$a_{53} = 607426c_1 + e_1 = 242971c_2 - e_2 = 52820c_3 - e_3 = 41891c_4 + e_4 = 32834c_5 - e_5 = 25848c_6 - e_6 = 15378c_9 - e_9 = 11354c_{13} - e_{13} = 3172c_{39} - e_{39}$,

$a_{54} = 701894c_1 + e_1 = 61034c_3 + e_3 = 48407c_4 - e_4 = 26487c_7 - e_7 = 15773c_{11} - e_{11} = 14472c_{12} + e_{12} = 12879c_{14} - e_{14} = 2520c_{47} + e_{47}$,
$a_{55} = 389516c_2 - e_2 = 84677c_3 + e_3 = 24653c_9 - e_9 = 14011c_{17} + e_{17} = 7405c_{28} + e_{28} = 7031c_{31} - e_{31} = 5085c_{39} + e_{39}$,
$a_{56} = 3764c_3^2 - 30 = 68660c_4 - e_4 = 37568c_7 + e_7 = 14325c_{17} - e_{17} = 7c_{47}^2 - 180617$,
$a_{57} = 1063286c_1 + e_1 = 85063c_2^2 - 2 = 92460c_3 - e_3 = 11019c_{22} - e_{22} = 5199c_{40} + e_{40} = 3586c_{50} + e_{50} = 3318c_{53} - e_{53}$.

It is straightforward to prove that the equations below are satisfied. Note that these equations will be used later to show that the ideal norms of certain nonzero ideals of $\mathcal{O}_K$ coincide.

Equations, part C

$$
\begin{aligned}
d =& a_1^2 + c_1c_2^2c_3c_5c_6c_7c_{43}c_{53} = a_2^2 + c_3^2c_7c_9c_{16}c_{27}c_{46} = a_3^2 + c_3c_{11}c_{15}c_{39}c_{52}c_{53}\\
=& a_4^2 + c_2c_3c_7c_{12}c_{37}c_{40}c_{41} = a_5^2 + c_2^3c_3c_{12}c_{39}c_{48}c_{51} = a_6^2 + c_2c_3c_9c_{14}c_{23}c_{35}c_{47}\\
=& a_7^2 + c_1c_2c_3c_5c_{12}c_{31}c_{38}c_{46} = a_8^2 + c_1c_3^2c_4c_5c_{10}c_{52}c_{54} = a_9^2 + c_1c_2^2c_6c_{19}c_{39}c_{46}c_{48}\\
=& a_{10}^2 + c_2c_{14}c_{37}c_{46}c_{52}c_{55} = a_{11}^2 + c_1c_3c_6c_{25}c_{36}c_{37}c_{53} = a_{12}^2 + c_2c_3c_5c_{17}c_{28}c_{44}c_{48}\\
=& a_{13}^2 + c_1c_2c_3c_{10}c_{15}c_{21}c_{25}c_{44} = a_{14}^2 + c_2^2c_9c_{23}c_{30}c_{49}c_{53} = a_{15}^2 + c_1c_2c_3c_5c_8c_{34}c_{37}c_{55}\\
=& a_{16}^2 + c_1c_2^3c_5c_{18}c_{23}c_{33}c_{43} = a_{17}^2 + c_2c_3^2c_{30}c_{31}c_{34}c_{52} = a_{18}^2 + c_2^2c_{21}^2c_{33}c_{37}c_{45}\\
=& a_{19}^2 + c_2^2c_7c_8c_{13}c_{15}^2c_{37} = a_{20}^2 + c_1c_2^5c_3c_4c_{12}c_{26}c_{41} = a_{21}^2 + c_2^2c_{18}c_{19}c_{29}c_{38}c_{52}\\
=& a_{22}^2 + c_2c_4c_8c_{20}^2c_{28}c_{51} = a_{23}^2 + c_2c_5c_{13}c_{18}c_{22}^2c_{35} = a_{24}^2 + c_3^2c_4c_8c_{26}c_{33}c_{50}\\
=& a_{25}^2 + c_2c_5c_9c_{12}c_{18}c_{30}c_{54} = a_{26}^2 + c_2c_3c_{11}c_{12}c_{19}c_{36}c_{56} = a_{27}^2 + c_1c_2c_3^2c_{18}c_{29}c_{33}c_{49}\\
=& a_{28}^2 + c_1c_4c_{15}c_{22}c_{23}c_{25}c_{55} = a_{29}^2 + c_6c_7c_9c_{10}c_{13}c_{16}c_{19} = a_{30}^2 + c_1c_2c_4c_5c_{17}c_{24}c_{30}c_{42}\\
=& a_{31}^2 + c_1c_6c_{16}c_{17}c_{21}c_{36}^2 = a_{32}^2 + c_2c_3c_6c_{13}c_{26}c_{46}c_{52} = a_{33}^2 + c_2^2c_3c_6c_8c_{19}c_{34}c_{46}\\
=& a_{34}^2 + c_1c_2c_7c_{10}^2c_{15}c_{30}c_{36} = a_{35}^2 + c_5c_{11}c_{12}c_{39}c_{46}c_{55} = a_{36}^2 + c_1c_2c_3c_5c_{26}c_{28}c_{29}c_{32}\\
=& a_{37}^2 + c_1c_4^2c_5c_6c_{12}c_{23}c_{56} = a_{38}^2 + c_2c_6c_{10}c_{15}^2c_{39}c_{40} = a_{39}^2 + c_1c_7c_{14}c_{18}c_{20}c_{23}c_{54}\\
=& a_{40}^2 + c_1c_2c_3c_5c_9c_{25}c_{45}c_{49} = a_{41}^2 + c_1c_2^2c_6c_{21}c_{30}c_{49}c_{51} = a_{42}^2 + c_2c_3c_{11}c_{13}c_{23}c_{39}c_{46}\\
=& a_{43}^2 + c_2c_3c_7c_{15}c_{32}c_{38}c_{47} = a_{44}^2 + c_2c_4c_9c_{21}c_{23}c_{33}c_{34} = a_{45}^2 + c_2c_4c_{10}c_{18}c_{28}c_{31}c_{32}\\
=& a_{46}^2 + c_2^2c_4^2c_{15}c_{21}c_{27}c_{39} = a_{47}^2 + c_1c_2c_3c_6^2c_{30}c_{40}c_{56} = a_{48}^2 + c_4c_5c_{28}c_{39}c_{49}c_{52}\\
=& a_{49}^2 + c_1c_3c_6c_{27}c_{30}c_{46}c_{50} = a_{50}^2 + c_3c_7^2c_8c_9c_{24}c_{47} = a_{51}^2 + c_1c_3c_6c_{16}c_{44}c_{45}c_{53}\\
=& a_{52}^2 + c_2c_3c_5c_{22}c_{25}c_{42}c_{46} = a_{53}^2 + c_1c_2c_3c_4c_5c_6c_9c_{13}c_{39} = a_{54}^2 + c_1c_3c_4c_7c_{11}c_{12}c_{14}c_{47}\\
=& a_{55}^2 + c_2c_3c_9c_{17}c_{28}c_{31}c_{39} = a_{56}^2 + c_3^2c_4c_7c_{17}c_{47}^2 = a_{57}^2 + c_1c_2^2c_3c_{22}c_{40}c_{50}c_{53}.
\end{aligned}
$$

Observe that 2 and d are precisely the ramified primes of $\mathcal{O}_K$ (since $d \in \mathbb{P}$ and $4d$ is the discriminant of $\mathcal{O}_K$). Therefore, c_j splits in $\mathcal{O}_K$ for each $j \in [2, s]$. We infer that all positive powers of the maximal ideals P_1, P_j and $\overline{P}_j$ are pairwise comaximal for each $j \in [2, s]$. Next, we discuss an elementary fact that will subsequently be used. Let I and J be nonzero ideals of $\mathcal{O}_K$ such that $I \subseteq J$ and $\mathrm{N}(I) = \mathrm{N}(J)$. It follows by the theorem of Lagrange that $|\mathcal{O}_K/J| = \mathrm{N}(J) = \mathrm{N}(I) = |\mathcal{O}_K/I| = |\mathcal{O}_K/J||J/I|$, and thus $|J/I| = 1$. This implies that $I = J$.

By combining the equations in the parts A, B and C above, we obtain the next list of equations. Note that the equations in parts A and B together show that each left hand side is contained in each right hand side. The equations in part C prove that the ideal norms on each side coincide, and hence both

sides coincide. We will make these statements clearer, by giving a detailed proof for the first equality below. (The remaining equations can be proved in analogy.)

Equations, part D

$$\left(a_1+\sqrt{d}\right)\mathcal{O}_K = P_1\overline{P}_2^2\overline{P}_3P_5\overline{P}_6\overline{P}_7P_{43}\overline{P}_{53},\ \left(a_2+\sqrt{d}\right)\mathcal{O}_K = P_3^2\overline{P}_7\overline{P}_9\overline{P}_{16}\overline{P}_{27}\overline{P}_{46},$$
$$\left(a_3+\sqrt{d}\right)\mathcal{O}_K = P_3\overline{P}_{11}\overline{P}_{15}\overline{P}_{39}P_{52}P_{53},\ \left(a_4+\sqrt{d}\right)\mathcal{O}_K = P_2\overline{P}_3P_7\overline{P}_{12}P_{37}P_{40}\overline{P}_{41},$$
$$\left(a_5+\sqrt{d}\right)\mathcal{O}_K = P_2^3P_3\overline{P}_{12}P_{39}\overline{P}_{48}P_{51},\ \left(a_6+\sqrt{d}\right)\mathcal{O}_K = \overline{P}_2P_3P_9P_{14}\overline{P}_{23}P_{35}P_{47},$$
$$\left(a_7+\sqrt{d}\right)\mathcal{O}_K = P_1P_2\overline{P}_3\overline{P}_5\overline{P}_{12}P_{31}\overline{P}_{38}P_{46},\ \left(a_8+\sqrt{d}\right)\mathcal{O}_K = P_1P_3^2P_4P_5\overline{P}_{10}\overline{P}_{52}P_{54},$$
$$\left(a_9+\sqrt{d}\right)\mathcal{O}_K = P_1P_2^2P_6\overline{P}_{19}P_{39}\overline{P}_{46}P_{48},\ \left(a_{10}+\sqrt{d}\right)\mathcal{O}_K = P_2P_{14}P_{37}P_{46}\overline{P}_{52}\overline{P}_{55},$$
$$\left(a_{11}+\sqrt{d}\right)\mathcal{O}_K = P_1\overline{P}_3P_6\overline{P}_{25}\overline{P}_{36}\overline{P}_{37}P_{53},\ \left(a_{12}+\sqrt{d}\right)\mathcal{O}_K = \overline{P}_2P_3P_5P_{17}\overline{P}_{28}\overline{P}_{44}\overline{P}_{48},$$
$$\left(a_{13}+\sqrt{d}\right)\mathcal{O}_K = P_1\overline{P}_2\overline{P}_3\overline{P}_{10}P_{15}\overline{P}_{21}P_{25}P_{44},\ \left(a_{14}+\sqrt{d}\right)\mathcal{O}_K = P_2^2P_9P_{23}\overline{P}_{30}\overline{P}_{49}\overline{P}_{53},$$
$$\left(a_{15}+\sqrt{d}\right)\mathcal{O}_K = P_1\overline{P}_2P_3\overline{P}_5P_8\overline{P}_{34}P_{37}P_{55},\ \left(a_{16}+\sqrt{d}\right)\mathcal{O}_K = P_1P_2^3P_5P_{18}P_{23}\overline{P}_{33}\overline{P}_{43},$$
$$\left(a_{17}+\sqrt{d}\right)\mathcal{O}_K = \overline{P}_2\overline{P}_3^2\overline{P}_{30}P_{31}\overline{P}_{34}P_{52},\ \left(a_{18}+\sqrt{d}\right)\mathcal{O}_K = P_2^2\overline{P}_{21}^2P_{33}P_{37}\overline{P}_{45},$$
$$\left(a_{19}+\sqrt{d}\right)\mathcal{O}_K = \overline{P}_2^2\overline{P}_7P_8\overline{P}_{13}\overline{P}_{15}^2P_{37},\ \left(a_{20}+\sqrt{d}\right)\mathcal{O}_K = P_1P_2^5\overline{P}_3\overline{P}_4\overline{P}_{12}\overline{P}_{26}P_{41},$$
$$\left(a_{21}+\sqrt{d}\right)\mathcal{O}_K = P_2^2\overline{P}_{18}\overline{P}_{19}P_{29}\overline{P}_{38}P_{52},\ \left(a_{22}+\sqrt{d}\right)\mathcal{O}_K = \overline{P}_2P_4P_8\overline{P}_{20}^2P_{28}P_{51},$$
$$\left(a_{23}+\sqrt{d}\right)\mathcal{O}_K = P_2P_5P_{13}\overline{P}_{18}P_{22}^2\overline{P}_{35},\ \left(a_{24}+\sqrt{d}\right)\mathcal{O}_K = P_3^2\overline{P}_4\overline{P}_8\overline{P}_{26}\overline{P}_{33}\overline{P}_{50},$$
$$\left(a_{25}+\sqrt{d}\right)\mathcal{O}_K = \overline{P}_2P_5P_9\overline{P}_{12}P_{18}P_{30}\overline{P}_{54},\ \left(a_{26}+\sqrt{d}\right)\mathcal{O}_K = P_2P_3P_{11}\overline{P}_{12}P_{19}P_{36}\overline{P}_{56},$$
$$\left(a_{27}+\sqrt{d}\right)\mathcal{O}_K = P_1P_2\overline{P}_3^2P_{18}P_{29}P_{33}P_{49},\ \left(a_{28}+\sqrt{d}\right)\mathcal{O}_K = P_1P_4P_{15}\overline{P}_{22}\overline{P}_{23}P_{25}P_{55},$$
$$\left(a_{29}+\sqrt{d}\right)\mathcal{O}_K = P_6P_7P_9P_{10}P_{13}P_{16}P_{19},\ \left(a_{30}+\sqrt{d}\right)\mathcal{O}_K = P_1\overline{P}_2\overline{P}_4\overline{P}_5\overline{P}_{17}P_{24}P_{30}\overline{P}_{42},$$
$$\left(a_{31}+\sqrt{d}\right)\mathcal{O}_K = P_1\overline{P}_6\overline{P}_{16}\overline{P}_{17}P_{21}P_{36}^2,\ \left(a_{32}+\sqrt{d}\right)\mathcal{O}_K = \overline{P}_2\overline{P}_3P_6P_{13}P_{26}P_{46}P_{52},$$
$$\left(a_{33}+\sqrt{d}\right)\mathcal{O}_K = P_2^2P_3\overline{P}_6\overline{P}_8\overline{P}_{19}P_{34}P_{46},\ \left(a_{34}+\sqrt{d}\right)\mathcal{O}_K = P_1P_2\overline{P}_7\overline{P}_{10}^2P_{15}\overline{P}_{30}\overline{P}_{36},$$
$$\left(a_{35}+\sqrt{d}\right)\mathcal{O}_K = \overline{P}_5P_{11}P_{12}P_{39}P_{46}\overline{P}_{55},\ \left(a_{36}+\sqrt{d}\right)\mathcal{O}_K = P_1\overline{P}_2P_3\overline{P}_5P_{26}P_{28}\overline{P}_{29}\overline{P}_{32},$$
$$\left(a_{37}+\sqrt{d}\right)\mathcal{O}_K = P_1\overline{P}_4^2\overline{P}_5P_6P_{12}P_{23}P_{56},\ \left(a_{38}+\sqrt{d}\right)\mathcal{O}_K = P_2\overline{P}_6P_{10}\overline{P}_{15}^2\overline{P}_{39}\overline{P}_{40},$$
$$\left(a_{39}+\sqrt{d}\right)\mathcal{O}_K = P_1P_7P_{14}\overline{P}_{18}P_{20}\overline{P}_{23}P_{54},\ \left(a_{40}+\sqrt{d}\right)\mathcal{O}_K = P_1P_2P_3P_5\overline{P}_9P_{25}\overline{P}_{45}\overline{P}_{49},$$
$$\left(a_{41}+\sqrt{d}\right)\mathcal{O}_K = P_1\overline{P}_2^2\overline{P}_6\overline{P}_{21}\overline{P}_{30}P_{49}\overline{P}_{51},\ \left(a_{42}+\sqrt{d}\right)\mathcal{O}_K = P_2\overline{P}_3P_{11}P_{13}\overline{P}_{23}P_{39}\overline{P}_{46},$$
$$\left(a_{43}+\sqrt{d}\right)\mathcal{O}_K = P_2P_3\overline{P}_7P_{15}P_{32}P_{38}\overline{P}_{47},\ \left(a_{44}+\sqrt{d}\right)\mathcal{O}_K = \overline{P}_2P_4\overline{P}_9P_{21}P_{23}P_{33}P_{34},$$
$$\left(a_{45}+\sqrt{d}\right)\mathcal{O}_K = P_2\overline{P}_4P_{10}\overline{P}_{18}\overline{P}_{28}\overline{P}_{31}P_{32},\ \left(a_{46}+\sqrt{d}\right)\mathcal{O}_K = \overline{P}_2^2P_4^2P_{15}P_{21}P_{27}P_{39},$$
$$\left(a_{47}+\sqrt{d}\right)\mathcal{O}_K = P_1\overline{P}_2\overline{P}_3\overline{P}_6^2\overline{P}_{30}P_{40}\overline{P}_{56},\ \left(a_{48}+\sqrt{d}\right)\mathcal{O}_K = P_4P_5\overline{P}_{28}P_{39}\overline{P}_{49}P_{52},$$
$$\left(a_{49}+\sqrt{d}\right)\mathcal{O}_K = P_1\overline{P}_3\overline{P}_6\overline{P}_{27}P_{30}P_{46}\overline{P}_{50},\ \left(a_{50}+\sqrt{d}\right)\mathcal{O}_K = P_3P_7^2\overline{P}_8P_9\overline{P}_{24}P_{47},$$
$$\left(a_{51}+\sqrt{d}\right)\mathcal{O}_K = P_1\overline{P}_3\overline{P}_6\overline{P}_{16}P_{44}P_{45}\overline{P}_{53},\ \left(a_{52}+\sqrt{d}\right)\mathcal{O}_K = P_2\overline{P}_3\overline{P}_5\overline{P}_{22}\overline{P}_{25}P_{42}P_{46},$$
$$\left(a_{53}+\sqrt{d}\right)\mathcal{O}_K = P_1\overline{P}_2\overline{P}_3P_4\overline{P}_5\overline{P}_6\overline{P}_9\overline{P}_{13}\overline{P}_{39},\ \left(a_{54}+\sqrt{d}\right)\mathcal{O}_K = P_1P_3\overline{P}_4\overline{P}_7\overline{P}_{11}P_{12}\overline{P}_{14}P_{47},$$
$$\left(a_{55}+\sqrt{d}\right)\mathcal{O}_K = \overline{P}_2P_3\overline{P}_9P_{17}P_{28}\overline{P}_{31}P_{39},\ \left(a_{56}+\sqrt{d}\right)\mathcal{O}_K = \overline{P}_3^2\overline{P}_4P_7\overline{P}_{17}\overline{P}_{47}^2,$$
$$\left(a_{57}+\sqrt{d}\right)\mathcal{O}_K = P_1\overline{P}_2^2\overline{P}_3\overline{P}_{22}P_{40}P_{50}\overline{P}_{53}.$$

Now, we show that the first equation in part D holds. By the equations in part B, we have that

$$a_1+\sqrt{d}=-1516739c_1+e_1+\sqrt{d}\in c_1\mathcal{O}_K+\left(e_1+\sqrt{d}\right)\mathcal{O}_K=P_1 \text{ and}$$

$$a_1+\sqrt{d}=-131890c_3-\left(e_3-\sqrt{d}\right)\in c_3\mathcal{O}_K+\left(e_3-\sqrt{d}\right)\mathcal{O}_K=\overline{P}_3.$$

Furthermore, we have that $a_1+\sqrt{d}=-121339c_2^2-\left(2-\sqrt{d}\right)\in c_2^2\mathcal{O}_K+\left(2-\sqrt{d}\right)\mathcal{O}_K=\overline{P}_2^2$ by the equations in parts A and B. Along the same lines it can be proved that $a_1+\sqrt{d}\in P_5\cap\overline{P}_6\cap\overline{P}_7\cap P_{43}\cap\overline{P}_{53}$. This implies that $a_1+\sqrt{d}\in P_1\cap\overline{P}_2^2\cap\overline{P}_3\cap P_5\cap\overline{P}_6\cap\overline{P}_7\cap P_{43}\cap\overline{P}_{53}$. Therefore, $\left(a_1+\sqrt{d}\right)\mathcal{O}_K\subseteq P_1\cap\overline{P}_2^2\cap\overline{P}_3\cap P_5\cap\overline{P}_6\cap\overline{P}_7\cap P_{43}\cap\overline{P}_{53}=P_1\overline{P}_2^2\overline{P}_3P_5\overline{P}_6\overline{P}_7P_{43}\overline{P}_{53}$ (since the powers of the maximal ideals in the aforementioned intersection are pairwise comaximal).

We infer by the equations in part C that

$$\begin{aligned}\mathrm{N}\left(\left(a_1+\sqrt{d}\right)\mathcal{O}_K\right)=&|\mathrm{N}\left(a_1+\sqrt{d}\right)|=d-a_1^2=c_1c_2^2c_3c_5c_6c_7c_{43}c_{53}\\=&\mathrm{N}\left(P_1\right)\mathrm{N}\left(\overline{P}_2\right)^2\mathrm{N}\left(\overline{P}_3\right)\mathrm{N}\left(P_5\right)\mathrm{N}\left(\overline{P}_6\right)\mathrm{N}\left(\overline{P}_7\right)\mathrm{N}\left(P_{43}\right)\mathrm{N}\left(\overline{P}_{53}\right)\\=&\mathrm{N}\left(P_1\overline{P}_2^2\overline{P}_3P_5\overline{P}_6\overline{P}_7P_{43}\overline{P}_{53}\right).\end{aligned}$$

Therefore, $\left(a_1+\sqrt{d}\right)\mathcal{O}_K=P_1\overline{P}_2^2\overline{P}_3P_5\overline{P}_6\overline{P}_7P_{43}\overline{P}_{53}$.

We want to emphasize that the equations in part D show that the maximal ideals P_j and $\overline{P}_j$ for $j\in[1,s]$ are the only maximal ideals of $\mathcal{O}_K$ that can occur in the factorization of $\left(a_i+\sqrt{d}\right)\mathcal{O}_K$ into maximal ideals for each $i\in[1,r]$.

Next, we determine the factorization of $z\mathcal{O}_K$ into maximal ideals of $\mathcal{O}_K$. Let I be a nonzero ideal of $\mathcal{O}_K$ and let P be maximal ideal of $\mathcal{O}_K$. In what follows, let $\mathrm{v}_P(I)$ be the unique nonnegative integer k such that $I=P^kJ$ for some nonzero ideal J of $\mathcal{O}_K$ with $J\nsubseteq P$. Note that the equations in part E below consist of two equalities. Each of the first equalities is an immediate consequence of the equations in part D, while each of the second equalities is (a consequence of) elementary arithmetic.

Equations, part E

$\mathrm{v}_{P_1}(z\mathcal{O}_K)=b_1+b_7+b_8+b_9+b_{11}+b_{13}+b_{15}+b_{16}+b_{20}+b_{27}+b_{28}+b_{30}+b_{31}+b_{34}+b_{36}+b_{37}+b_{39}+b_{40}+b_{41}+b_{47}+b_{49}+b_{51}+b_{53}+b_{54}+b_{57}=2d_1,$
$\mathrm{v}_{P_2}(z\mathcal{O}_K)=b_4+3b_5+b_7+2b_9+b_{10}+2b_{14}+3b_{16}+2b_{18}+5b_{20}+2b_{21}+b_{23}+b_{26}+b_{27}+2b_{33}+b_{34}+b_{38}+b_{40}+b_{42}+b_{43}+b_{45}+b_{52}=d_2,$
$\mathrm{v}_{\overline{P}_2}(z\mathcal{O}_K)=2b_1+b_6+b_{12}+b_{13}+b_{15}+b_{17}+2b_{19}+b_{22}+b_{25}+b_{30}+b_{32}+b_{36}+2b_{41}+b_{44}+2b_{46}+b_{47}+b_{53}+b_{55}+2b_{57}=d_2,$
$\mathrm{v}_{P_3}(z\mathcal{O}_K)=2b_2+b_3+b_5+b_6+2b_8+b_{12}+b_{15}+2b_{24}+b_{26}+b_{33}+b_{36}+b_{40}+b_{43}+b_{50}+b_{54}+b_{55}=d_3,$
$\mathrm{v}_{\overline{P}_3}(z\mathcal{O}_K)=b_1+b_4+b_7+b_{11}+b_{13}+2b_{17}+b_{20}+2b_{27}+b_{32}+b_{42}+b_{47}+b_{49}+b_{51}+b_{52}+b_{53}+2b_{56}+b_{57}=d_3,$
$\mathrm{v}_{P_4}(z\mathcal{O}_K)=b_8+b_{22}+b_{28}+b_{44}+2b_{46}+b_{48}+b_{53}=d_4,$ $\mathrm{v}_{\overline{P}_4}(z\mathcal{O}_K)=b_{20}+b_{24}+b_{30}+2b_{37}+b_{45}+b_{54}+b_{56}=d_4,$
$\mathrm{v}_{P_5}(z\mathcal{O}_K)=b_1+b_8+b_{12}+b_{16}+b_{23}+b_{25}+b_{40}+b_{48}=d_5,$
$\mathrm{v}_{\overline{P}_5}(z\mathcal{O}_K)=b_7+b_{15}+b_{30}+b_{35}+b_{36}+b_{37}+b_{52}+b_{53}=d_5,$ $\mathrm{v}_{P_6}(z\mathcal{O}_K)=b_9+b_{11}+b_{29}+b_{32}+b_{37}=d_6,$
$\mathrm{v}_{\overline{P}_6}(z\mathcal{O}_K)=b_1+b_{31}+b_{33}+b_{38}+b_{41}+2b_{47}+b_{49}+b_{51}+b_{53}=d_6,$ $\mathrm{v}_{P_7}(z\mathcal{O}_K)=b_4+b_{29}+b_{39}+2b_{50}+b_{56}=d_7,$
$\mathrm{v}_{\overline{P}_7}(z\mathcal{O}_K)=b_1+b_2+b_{19}+b_{34}+b_{43}+b_{54}=d_7,$ $\mathrm{v}_{P_8}(z\mathcal{O}_K)=b_{15}+b_{19}+b_{22}=d_8,$
$\mathrm{v}_{\overline{P}_8}(z\mathcal{O}_K)=b_{24}+b_{33}+b_{50}=d_8,$ $\mathrm{v}_{P_9}(z\mathcal{O}_K)=b_6+b_{14}+b_{25}+b_{29}+b_{50}=d_9,$
$\mathrm{v}_{\overline{P}_9}(z\mathcal{O}_K)=b_2+b_{40}+b_{44}+b_{53}+b_{55}=d_9,$ $\mathrm{v}_{P_{10}}(z\mathcal{O}_K)=b_{29}+b_{38}+b_{45}=d_{10},$
$\mathrm{v}_{\overline{P}_{10}}(z\mathcal{O}_K)=b_8+b_{13}+2b_{34}=d_{10},$ $\mathrm{v}_{P_{11}}(z\mathcal{O}_K)=b_{26}+b_{35}+b_{42}=d_{11},$ $\mathrm{v}_{\overline{P}_{11}}(z\mathcal{O}_K)=b_3+b_{54}=d_{11},$
$\mathrm{v}_{P_{12}}(z\mathcal{O}_K)=b_{35}+b_{37}+b_{54}=d_{12},$ $\mathrm{v}_{\overline{P}_{12}}(z\mathcal{O}_K)=b_4+b_5+b_7+b_{20}+b_{25}+b_{26}=d_{12},$

$\mathrm{v}_{P_{13}}(z\mathcal{O}_K) = b_{23}+b_{29}+b_{32}+b_{42} = d_{13}$, $\mathrm{v}_{\overline{P}_{13}}(z\mathcal{O}_K) = b_{19}+b_{53} = d_{13}$, $\mathrm{v}_{P_{14}}(z\mathcal{O}_K) = b_6+b_{10}+b_{39} = d_{14}$,
$\mathrm{v}_{\overline{P}_{14}}(z\mathcal{O}_K) = b_{54} = d_{14}$, $\mathrm{v}_{P_{15}}(z\mathcal{O}_K) = b_{13}+b_{28}+b_{34}+b_{43}+b_{46} = d_{15}$, $\mathrm{v}_{\overline{P}_{15}}(z\mathcal{O}_K) = b_3+2b_{19}+2b_{38} = d_{15}$,
$\mathrm{v}_{P_{16}}(z\mathcal{O}_K) = b_{29} = d_{16}$, $\mathrm{v}_{\overline{P}_{16}}(z\mathcal{O}_K) = b_2+b_{31}+b_{51} = d_{16}$, $\mathrm{v}_{P_{17}}(z\mathcal{O}_K) = b_{12}+b_{55} = d_{17}$,
$\mathrm{v}_{\overline{P}_{17}}(z\mathcal{O}_K) = b_{30}+b_{31}+b_{56} = d_{17}$, $\mathrm{v}_{P_{18}}(z\mathcal{O}_K) = b_{16}+b_{25}+b_{27} = d_{18}$,
$\mathrm{v}_{\overline{P}_{18}}(z\mathcal{O}_K) = b_{21}+b_{23}+b_{39}+b_{45} = d_{18}$, $\mathrm{v}_{P_{19}}(z\mathcal{O}_K) = b_{26}+b_{29} = d_{19}$, $\mathrm{v}_{\overline{P}_{19}}(z\mathcal{O}_K) = b_9+b_{21}+b_{33} = d_{19}$,
$\mathrm{v}_{P_{20}}(z\mathcal{O}_K) = b_{39} = d_{20}$, $\mathrm{v}_{\overline{P}_{20}}(z\mathcal{O}_K) = 2b_{22} = d_{20}$, $\mathrm{v}_{P_{21}}(z\mathcal{O}_K) = b_{31}+b_{44}+b_{46} = d_{21}$,
$\mathrm{v}_{\overline{P}_{21}}(z\mathcal{O}_K) = b_{13}+2b_{18}+b_{41} = d_{21}$, $\mathrm{v}_{P_{22}}(z\mathcal{O}_K) = 2b_{23} = d_{22}$, $\mathrm{v}_{\overline{P}_{22}}(z\mathcal{O}_K) = b_{28}+b_{52}+b_{57} = d_{22}$,
$\mathrm{v}_{P_{23}}(z\mathcal{O}_K) = b_{14}+b_{16}+b_{37}+b_{44} = d_{23}$, $\mathrm{v}_{\overline{P}_{23}}(z\mathcal{O}_K) = b_6+b_{28}+b_{39}+b_{42} = d_{23}$, $\mathrm{v}_{P_{24}}(z\mathcal{O}_K) = b_{30} = d_{24}$,
$\mathrm{v}_{\overline{P}_{24}}(z\mathcal{O}_K) = b_{50} = d_{24}$, $\mathrm{v}_{P_{25}}(z\mathcal{O}_K) = b_{13}+b_{28}+b_{40} = d_{25}$, $\mathrm{v}_{\overline{P}_{25}}(z\mathcal{O}_K) = b_{11}+b_{52} = d_{25}$,
$\mathrm{v}_{P_{26}}(z\mathcal{O}_K) = b_{32}+b_{36} = d_{26}$, $\mathrm{v}_{\overline{P}_{26}}(z\mathcal{O}_K) = b_{20}+b_{24} = d_{26}$, $\mathrm{v}_{P_{27}}(z\mathcal{O}_K) = b_{46} = d_{27}$,
$\mathrm{v}_{\overline{P}_{27}}(z\mathcal{O}_K) = b_2+b_{49} = d_{27}$, $\mathrm{v}_{P_{28}}(z\mathcal{O}_K) = b_{22}+b_{36}+b_{55} = d_{28}$, $\mathrm{v}_{\overline{P}_{28}}(z\mathcal{O}_K) = b_{12}+b_{45}+b_{48} = d_{28}$,
$\mathrm{v}_{P_{29}}(z\mathcal{O}_K) = b_{21}+b_{27} = d_{29}$, $\mathrm{v}_{\overline{P}_{29}}(z\mathcal{O}_K) = b_{36} = d_{29}$, $\mathrm{v}_{P_{30}}(z\mathcal{O}_K) = b_{25}+b_{30}+b_{49} = d_{30}$,
$\mathrm{v}_{\overline{P}_{30}}(z\mathcal{O}_K) = b_{14}+b_{17}+b_{34}+b_{41}+b_{47} = d_{30}$, $\mathrm{v}_{P_{31}}(z\mathcal{O}_K) = b_7+b_{17} = d_{31}$, $\mathrm{v}_{\overline{P}_{31}}(z\mathcal{O}_K) = b_{45}+b_{55} = d_{31}$,
$\mathrm{v}_{P_{32}}(z\mathcal{O}_K) = b_{43}+b_{45} = d_{32}$, $\mathrm{v}_{\overline{P}_{32}}(z\mathcal{O}_K) = b_{36} = d_{32}$, $\mathrm{v}_{P_{33}}(z\mathcal{O}_K) = b_{18}+b_{27}+b_{44} = d_{33}$,
$\mathrm{v}_{\overline{P}_{33}}(z\mathcal{O}_K) = b_{16}+b_{24} = d_{33}$, $\mathrm{v}_{P_{34}}(z\mathcal{O}_K) = b_{33}+b_{44} = d_{34}$, $\mathrm{v}_{\overline{P}_{34}}(z\mathcal{O}_K) = b_{15}+b_{17} = d_{34}$,
$\mathrm{v}_{P_{35}}(z\mathcal{O}_K) = b_6 = d_{35}$, $\mathrm{v}_{\overline{P}_{35}}(z\mathcal{O}_K) = b_{23} = d_{35}$, $\mathrm{v}_{P_{36}}(z\mathcal{O}_K) = b_{26}+2b_{31} = d_{36}$,
$\mathrm{v}_{\overline{P}_{36}}(z\mathcal{O}_K) = b_{11}+b_{34} = d_{36}$, $\mathrm{v}_{P_{37}}(z\mathcal{O}_K) = b_4+b_{10}+b_{15}+b_{18}+b_{19} = d_{37}$, $\mathrm{v}_{\overline{P}_{37}}(z\mathcal{O}_K) = b_{11} = d_{37}$,
$\mathrm{v}_{P_{38}}(z\mathcal{O}_K) = b_{43} = d_{38}$, $\mathrm{v}_{\overline{P}_{38}}(z\mathcal{O}_K) = b_7+b_{21} = d_{38}$, $\mathrm{v}_{P_{39}}(z\mathcal{O}_K) = b_5+b_9+b_{35}+b_{42}+b_{46}+b_{48}+b_{55} = d_{39}$,
$\mathrm{v}_{\overline{P}_{39}}(z\mathcal{O}_K) = b_3+b_{38}+b_{53} = d_{39}$, $\mathrm{v}_{P_{40}}(z\mathcal{O}_K) = b_4+b_{47}+b_{57} = d_{40}$, $\mathrm{v}_{\overline{P}_{40}}(z\mathcal{O}_K) = b_{38} = d_{40}$,
$\mathrm{v}_{P_{41}}(z\mathcal{O}_K) = b_{20} = d_{41}$, $\mathrm{v}_{\overline{P}_{41}}(z\mathcal{O}_K) = b_4 = d_{41}$, $\mathrm{v}_{P_{42}}(z\mathcal{O}_K) = b_{52} = d_{42}$, $\mathrm{v}_{\overline{P}_{42}}(z\mathcal{O}_K) = b_{30} = d_{42}$,
$\mathrm{v}_{P_{43}}(z\mathcal{O}_K) = b_1 = d_{43}$, $\mathrm{v}_{\overline{P}_{43}}(z\mathcal{O}_K) = b_{16} = d_{43}$, $\mathrm{v}_{P_{44}}(z\mathcal{O}_K) = b_{13}+b_{51} = d_{44}$,
$\mathrm{v}_{\overline{P}_{44}}(z\mathcal{O}_K) = b_{12} = d_{44}$, $\mathrm{v}_{P_{45}}(z\mathcal{O}_K) = b_{51} = d_{45}$, $\mathrm{v}_{\overline{P}_{45}}(z\mathcal{O}_K) = b_{18}+b_{40} = d_{45}$,
$\mathrm{v}_{P_{46}}(z\mathcal{O}_K) = b_7+b_{10}+b_{32}+b_{33}+b_{35}+b_{49}+b_{52} = d_{46}$, $\mathrm{v}_{\overline{P}_{46}}(z\mathcal{O}_K) = b_2+b_9+b_{42} = d_{46}$,
$\mathrm{v}_{P_{47}}(z\mathcal{O}_K) = b_6+b_{50}+b_{54} = d_{47}$, $\mathrm{v}_{\overline{P}_{47}}(z\mathcal{O}_K) = b_{43}+2b_{56} = d_{47}$, $\mathrm{v}_{P_{48}}(z\mathcal{O}_K) = b_9 = d_{48}$,
$\mathrm{v}_{\overline{P}_{48}}(z\mathcal{O}_K) = b_5+b_{12} = d_{48}$, $\mathrm{v}_{P_{49}}(z\mathcal{O}_K) = b_{27}+b_{41} = d_{49}$, $\mathrm{v}_{\overline{P}_{49}}(z\mathcal{O}_K) = b_{14}+b_{40}+b_{48} = d_{49}$,
$\mathrm{v}_{P_{50}}(z\mathcal{O}_K) = b_{57} = d_{50}$, $\mathrm{v}_{\overline{P}_{50}}(z\mathcal{O}_K) = b_{24}+b_{49} = d_{50}$, $\mathrm{v}_{P_{51}}(z\mathcal{O}_K) = b_5+b_{22} = d_{51}$,
$\mathrm{v}_{\overline{P}_{51}}(z\mathcal{O}_K) = b_{41} = d_{51}$, $\mathrm{v}_{P_{52}}(z\mathcal{O}_K) = b_3+b_{17}+b_{21}+b_{32}+b_{48} = d_{52}$, $\mathrm{v}_{\overline{P}_{52}}(z\mathcal{O}_K) = b_8+b_{10} = d_{52}$,
$\mathrm{v}_{P_{53}}(z\mathcal{O}_K) = b_3+b_{11} = d_{53}$, $\mathrm{v}_{\overline{P}_{53}}(z\mathcal{O}_K) = b_1+b_{14}+b_{51}+b_{57} = d_{53}$, $\mathrm{v}_{P_{54}}(z\mathcal{O}_K) = b_8+b_{39} = d_{54}$,
$\mathrm{v}_{\overline{P}_{54}}(z\mathcal{O}_K) = b_{25} = d_{54}$, $\mathrm{v}_{P_{55}}(z\mathcal{O}_K) = b_{15}+b_{28} = d_{55}$, $\mathrm{v}_{\overline{P}_{55}}(z\mathcal{O}_K) = b_{10}+b_{35} = d_{55}$,
$\mathrm{v}_{P_{56}}(z\mathcal{O}_K) = b_{37} = d_{56}$, $\mathrm{v}_{\overline{P}_{56}}(z\mathcal{O}_K) = b_{26}+b_{47} = d_{56}$.

The equations in part E show that $\mathrm{v}_{P_1}(z\mathcal{O}_K) = 2d_1 = \mathrm{v}_{P_1}(n\mathcal{O}_K)$ and $\mathrm{v}_{P_j}(z\mathcal{O}_K) = \mathrm{v}_{\overline{P}_j}(z\mathcal{O}_K) = d_j = \mathrm{v}_{\overline{P}_j}(n\mathcal{O}_K) = \mathrm{v}_{P_j}(n\mathcal{O}_K)$ for each $j \in [2,s]$, and thus $\eta\mathcal{O}_K = \mathcal{O}_K$. We infer that $\eta \in \mathcal{O}_K^\times$. □(Claim 2)

Claim 3: $\eta \in \mathcal{O}_d^\times$.

Observe that $d > c_j$ and $c_j \in \mathbb{P}$ for each $j \in [1,s]$. It follows that $d \nmid n$ (since $d \in \mathbb{P}$ by Claim 1). Since $z \in \mathcal{O}_K$, there are some $g, h \in \mathbb{Z}$ such that $z = g + h\sqrt{d}$. Moreover, since $\eta \in \mathcal{O}_K$ (by Claim 2) and $\{1, \sqrt{d}\}$ is a $\mathbb{Z}$-basis of $\mathcal{O}_K$, we obtain that $n \mid g$, $n \mid h$ and $\eta = \frac{g}{n} + \frac{h}{n}\sqrt{d}$. Clearly, $\mathcal{O}_K^\times \cap \mathcal{O}_d = \mathcal{O}_d^\times$. Since $d \nmid n$, $d \in \mathbb{P}$ and $\eta \in \mathcal{O}_K^\times$ (by Claim 2), we infer that $\eta \in \mathcal{O}_d^\times$ if and only if $\eta \in \mathcal{O}_d$ if and only if $d \mid \frac{h}{n}$ if and only if $d \mid h$ if and only if $z \in \mathcal{O}_d$. Consequently, it remains to show that $z \in \mathcal{O}_d$.

Set $z' = \prod_{i=1}^{r}\left(a_i + b_i\sqrt{d}\right)$. Since $\left(u+\sqrt{d}\right)^v \equiv u^{v-1}\left(u+v\sqrt{d}\right) \mod d\mathcal{O}_K$ for all $u \in \mathbb{Z}$ and $v \in \mathbb{N}$, there is some $t \in \mathbb{Z}$ such that $z \equiv tz' \mod d\mathcal{O}_K$. For this reason, it is sufficient to show that $z' \in \mathcal{O}_d$. Next we determine $\prod_{i=1}^{k}\left(a_i + b_i\sqrt{d}\right)$ modulo $d\mathcal{O}_K$ step-by-step for each $k \in [2,r]$. Thereby, we use that $\left(v+w\sqrt{d}\right)\left(v'+w'\sqrt{d}\right) \equiv vv' + (wv'+vw')\sqrt{d} \mod d\mathcal{O}_K$ for each $v, v', w, w' \in \mathbb{Z}$. Note that the items below consist of two equations that give us the reductions of u_k and v_k modulo d (for each $k \in [2,r]$), where $u_k, v_k \in \mathbb{Z}$ are such that $\prod_{i=1}^{k}\left(a_i + b_i\sqrt{d}\right) = u_k + v_k\sqrt{d}$.

- $a_1a_2 = 6890530871592$,
  $b_1a_2 + a_1b_2 = -d + 21554509784529$.
- $6890530871592a_3 = -315587d + 28244816980645$,
  $21554509784529a_3 + 6890530871592b_3 = -972567d + 5178868509861$.
- $28244816980645a_4 = -1272650d + 15513835265240$,
  $5178868509861a_4 + 28244816980645b_4 = 2697384d + 4467486423321$.
- $15513835265240a_5 = -675500d + 35811027000980$,
  $4467486423321a_5 + 15513835265240b_5 = 652149d + 30537129644761$.
- $35811027000980a_6 = -1309781d + 20714594938939$,
  $30537129644761a_6 + 35811027000980b_6 = 6236863d + 16497568105021$.
- $20714594938939a_7 = -441941d + 21223637681572$,
  $16497568105021a_7 + 20714594938939b_7 = 28697d + 18837954382492$.
- $21223637681572a_8 = -407041d + 13338881989779$,
  $18837954382492a_8 + 21223637681572b_8 = 4721606d + 17458165870498$.
- $13338881989779a_9 = -230639d + 3661427934964$,
  $17458165870498a_9 + 13338881989779b_9 = 5070046d + 17002573835617$.
- $3661427934964a_{10} = -52048d + 15123266299360$,
  $17002573835617a_{10} + 3661427934964b_{10} = 130052d + 34143299535672$.
- $15123266299360a_{11} = -202084d + 22867483171996$,
  $34143299535672a_{11} + 15123266299360b_{11} = 6208881d + 10414810162433$.
- $22867483171996a_{12} = -282499d + 8083910734589$,
  $10414810162433a_{12} + 22867483171996b_{12} = 7832662d + 17492146626900$.
- $8083910734589a_{13} = -92186d + 37797607362521$,
  $17492146626900a_{13} + 8083910734589b_{13} = 1575020d + 23141664104777$.
- $37797607362521a_{14} = -411842d + 28944089196110$,
  $23141664104777a_{14} + 37797607362521b_{14} = 1939038d + 6609389335501$.
- $28944089196110a_{15} = -239446d + 16720583049264$,
  $6609389335501a_{15} + 28944089196110b_{15} = 2568152d + 34718516820815$.
- $16720583049264a_{16} = -120376d + 6912281518232$,
  $34718516820815a_{16} + 16720583049264b_{16} = 1211229d + 10227378309170$.
- $6912281518232a_{17} = -44790d + 5195414457466$,
  $10227378309170a_{17} + 6912281518232b_{17} = 741048d + 289652693936$.
- $5195414457466a_{18} = -30152d + 22465854532740$,
  $289652693936a_{18} + 5195414457466b_{18} = 265509d + 17460040771835$.
- $22465854532740a_{19} = -123878d + 18660863010082$,
  $17460040771835a_{19} + 22465854532740b_{19} = 2001631d + 19162392372661$.
- $18660863010082a_{20} = -99441d + 31621822715453$,
  $19162392372661a_{20} + 18660863010082b_{20} = 1834171d + 26774899887696$.
- $31621822715453a_{21} = -167069d + 18945559551357$,
  $26774899887696a_{21} + 31621822715453b_{21} = 4282685d + 14809237415832$.
- $18945559551357a_{22} = -93807d + 25490759313159$,
  $14809237415832a_{22} + 18945559551357b_{22} = 1035530d + 16642252647821$.
- $25490759313159a_{23} = -82380d + 31410566401668$,
  $16642252647821a_{23} + 25490759313159b_{23} = 5180714d + 7665156542464$.
- $31410566401668a_{24} = -97899d + 24750475711101$,
  $7665156542464a_{24} + 31410566401668b_{24} = 4386896d + 12345706811136$.

- $24750475711101a_{25} = -34513d + 4341133124105$,
  $12345706811136a_{25} + 24750475711101b_{25} = 8807546d + 33705248273090$.
- $4341133124105a_{26} = -5385d + 20420906899575$,
  $33705248273090a_{26} + 4341133124105b_{26} = 888730d + 19138455539665$.
- $20420906899575a_{27} = -21329d + 11628413963866$,
  $19138455539665a_{27} + 20420906899575b_{27} = 432256d + 14371072675706$.
- $11628413963866a_{28} = -10322d + 27536499676132$,
  $14371072675706a_{28} + 11628413963866b_{28} = 2230673d + 20483449818325$.
- $27536499676132a_{29} = -16214d + 11871313871146$,
  $20483449818325a_{29} + 27536499676132b_{29} = 10491877d + 23841147446157$.
- $11871313871146a_{30} = 1168d + 36708968638606$,
  $23841147446157a_{30} + 11871313871146b_{30} = 583202d + 29824187122383$.
- $36708968638606a_{31} = 27670d + 35292993605984$,
  $29824187122383a_{31} + 36708968638606b_{31} = 6287182d + 30828612734725$.
- $35292993605984a_{32} = 33791d + 32099367907223$,
  $30828612734725a_{32} + 35292993605984b_{32} = 2928174d + 15667131309718$.
- $32099367907223a_{33} = 31502d + 8686497691888$,
  $15667131309718a_{33} + 32099367907223b_{33} = 1722251d + 13069916858845$.
- $8686497691888a_{34} = 17842d + 32178145784978$,
  $13069916858845a_{34} + 8686497691888b_{34} = 1025368d + 38021671372563$.
- $32178145784978a_{35} = 68542d + 38082283375434$,
  $38021671372563a_{35} + 32178145784978b_{35} = 5937837d + 11732682959923$.
- $38082283375434a_{36} = 100145d + 35965182833267$,
  $11732682959923a_{36} + 38082283375434b_{36} = 6202247d + 37211608563518$.
- $35965182833267a_{37} = 106375d + 33169240097520$,
  $37211608563518a_{37} + 35965182833267b_{37} = 8936948d + 26020836879171$.
- $33169240097520a_{38} = 99259d + 17708055977779$,
  $26020836879171a_{38} + 33169240097520b_{38} = 10150666d + 32544641385778$.
- $17708055977779a_{39} = 70781d + 30770548150680$,
  $32544641385778a_{39} + 17708055977779b_{39} = 2202939d + 36915064492287$.
- $30770548150680a_{40} = 155955d + 11904176271315$,
  $36915064492287a_{40} + 30770548150680b_{40} = 2562988d + 165258901717$.
- $11904176271315a_{41} = 64624d + 25508831080099$,
  $165258901717a_{41} + 11904176271315b_{41} = 1347305d + 5907415462461$.
- $25508831080099a_{42} = 143454d + 872140076252$,
  $5907415462461a_{42} + 25508831080099b_{42} = 790654d + 34225185690977$.
- $872140076252a_{43} = 5137d + 10987048845161$,
  $34225185690977a_{43} + 872140076252b_{43} = 339647d + 26138731918415$.
- $10987048845161a_{44} = 74345d + 8152289751913$,
  $26138731918415a_{44} + 10987048845161b_{44} = 1871482d + 38377823723539$.
- $8152289751913a_{45} = 56296d + 17399000858672$,
  $38377823723539a_{45} + 8152289751913b_{45} = 295750d + 32817075202061$.
- $17399000858672a_{46} = 140874d + 13424492118010$,
  $32817075202061a_{46} + 17399000858672b_{46} = 2189576d + 33163475056782$.
- $13424492118010a_{47} = 116689d + 22068166729399$,
  $33163475056782a_{47} + 13424492118010b_{47} = 705436d + 24070914532262$.
- $22068166729399a_{48} = 266604d + 20874064459588$,
  $24070914532262a_{48} + 22068166729399b_{48} = 292240d + 15318085049644$.

- $20874064459588a_{49} = 267012d + 25167892634080$,
  $15318085049644a_{49} + 20874064459588b_{49} = 789145d + 30753673516637$.
- $25167892634080a_{50} = 384789d + 20538358542749$,
  $30753673516637a_{50} + 25167892634080b_{50} = 1701637d + 23065592907429$.
- $20538358542749a_{51} = 591526d + 21236595367747$,
  $23065592907429a_{51} + 20538358542749b_{51} = 3306393d + 31259701520158$.
- $21236595367747a_{52} = 628169d + 663951808753$,
  $31259701520158a_{52} + 21236595367747b_{52} = 1963739d + 37377022803680$.
- $663951808753a_{53} = 20667d + 11352564579816$,
  $37377022803680a_{53} + 663951808753b_{53} = 1565314d + 35652369557844$.
- $11352564579816a_{54} = 408337d + 12677902910401$,
  $35652369557844a_{54} + 11352564579816b_{54} = 6095135d + 24252482581547$.
- $12677902910401a_{55} = 632652d + 37493335194470$,
  $24252482581547a_{55} + 12677902910401b_{55} = 2876153d + 28715685383254$.
- $37493335194470a_{56} = 1912828d + 27624385992608$,
  $28715685383254a_{56} + 37493335194470b_{56} = 11211960d + 33470792628624$.
- $27624385992608a_{57} = 1505206d + 34038147456710$,
  $33470792628624a_{57} + 27624385992608b_{57} = 6487920d$.

Consequently, $z' \equiv 34038147456710 \mod d\mathcal{O}_K$, and thus $z' \in \mathcal{O}_d$. □(Claim 3)

Claim 4: $\eta \neq 1$.

First, we show that $z \equiv \sqrt{d} \mod 3\mathcal{O}_K$. We have that $d \equiv 2 \mod 3\mathcal{O}_K$ and $\left(0+\sqrt{d}\right)^2 \equiv \left(1+\sqrt{d}\right)^4 \equiv \left(2+\sqrt{d}\right)^4 \equiv 2 \mod 3\mathcal{O}_K$. Moreover, $\left(0+\sqrt{d}\right)^4 \equiv \left(1+\sqrt{d}\right)^8 \equiv \left(2+\sqrt{d}\right)^8 \equiv 1 \mod 3\mathcal{O}_K$. Using this, we infer that

$$\begin{aligned}
z \equiv{}& \left(0+\sqrt{d}\right)^3 \left(2+\sqrt{d}\right)^6 \left(0+\sqrt{d}\right)^2 \left(1+\sqrt{d}\right)^3 \left(2+\sqrt{d}\right)^6 \left(0+\sqrt{d}\right)^2 \left(0+\sqrt{d}\right)^0 \left(1+\sqrt{d}\right)^2 \left(0+\sqrt{d}\right)^3 \\
&\left(2+\sqrt{d}\right)^6 \left(2+\sqrt{d}\right)^2 \left(0+\sqrt{d}\right)^1 \left(2+\sqrt{d}\right)^5 \left(2+\sqrt{d}\right)^7 \left(2+\sqrt{d}\right)^1 \left(1+\sqrt{d}\right)^3 \left(2+\sqrt{d}\right)^2 \left(2+\sqrt{d}\right)^3 \\
&\left(0+\sqrt{d}\right)^1 \left(2+\sqrt{d}\right)^3 \left(1+\sqrt{d}\right)^7 \left(0+\sqrt{d}\right)^3 \left(1+\sqrt{d}\right)^2 \left(1+\sqrt{d}\right)^4 \left(1+\sqrt{d}\right)^0 \left(0+\sqrt{d}\right)^3 \left(1+\sqrt{d}\right)^3 \\
&\left(0+\sqrt{d}\right)^3 \left(0+\sqrt{d}\right)^1 \left(0+\sqrt{d}\right)^3 \left(1+\sqrt{d}\right)^7 \left(0+\sqrt{d}\right)^1 \left(1+\sqrt{d}\right)^2 \left(1+\sqrt{d}\right)^3 \left(1+\sqrt{d}\right)^2 \left(0+\sqrt{d}\right)^2 \\
&\left(1+\sqrt{d}\right)^7 \left(2+\sqrt{d}\right)^1 \left(1+\sqrt{d}\right)^6 \left(2+\sqrt{d}\right)^5 \left(1+\sqrt{d}\right)^5 \left(2+\sqrt{d}\right)^3 \left(2+\sqrt{d}\right)^3 \left(1+\sqrt{d}\right)^1 \left(1+\sqrt{d}\right)^7 \\
&\left(2+\sqrt{d}\right)^0 \left(0+\sqrt{d}\right)^0 \left(1+\sqrt{d}\right)^4 \left(1+\sqrt{d}\right)^2 \left(2+\sqrt{d}\right)^7 \left(0+\sqrt{d}\right)^0 \left(2+\sqrt{d}\right)^7 \left(0+\sqrt{d}\right)^2 \left(2+\sqrt{d}\right)^6 \\
&\left(2+\sqrt{d}\right)^7 \left(2+\sqrt{d}\right)^6 \left(2+\sqrt{d}\right)^6 \\
\equiv{}& \left(0+\sqrt{d}\right)^2 \left(1+\sqrt{d}\right)^6 \left(2+\sqrt{d}\right)^4 \equiv 2^3 \left(1+\sqrt{d}\right)^2 \equiv \sqrt{d} \mod 3\mathcal{O}_K.
\end{aligned}$$

Since $z \equiv \sqrt{d} \mod 3\mathcal{O}_K$, it follows that $z \notin \mathbb{Z}$, and hence $\eta \neq 1$. □(Claim 4)

It remains to show that $d \mid y$ (where y was defined at the beginning of Section 3). Observe that for each $j \in [1, r]$, $1 \leq -3033477 + \sqrt{d} \leq a_j + \sqrt{d} \leq 2126573 + \sqrt{d} \leq 2^{100}$. Moreover, $\max\{b_i \mid i \in [1, r]\} = 23621570 \leq 10^8$ and $r \leq 100$, and thus

$$\eta \le \prod_{i=1}^{r} \left(a_i + \sqrt{d}\right)^{b_i} \le \prod_{i=1}^{r} \left(2^{100}\right)^{10^8} = 2^{10^{10} r} \le 2^{10^{12}} < 2^d \le \left(1 + \sqrt{d}\right)^d \le \varepsilon^d.$$

Also note that for each $j \in [1, s]$, $c_j \le 2^{10}$. Furthermore, $\max\{d_i \mid i \in [1, s]\} = 146634276 \le 10^9$ and $s \le 100$, and hence

$$\eta^{-1} \le \prod_{j=1}^{s} c_j^{d_j} \le \prod_{j=1}^{s} \left(2^{10}\right)^{10^9} = 2^{10^{10} s} \le 2^{10^{12}} < 2^d \le \left(1 + \sqrt{d}\right)^d \le \varepsilon^d.$$

It follows from Claims 2 and 4 that $\eta \in \mathcal{O}_K^\times \setminus \{1\}$. Since $\varepsilon^{-d} < \eta < \varepsilon^d$, there is some $k \in \mathbb{Z} \setminus \{0\}$ such that $-d < k < d$ and $\eta = \varepsilon^k$. Set $\operatorname{ord}\left(\varepsilon \mathcal{O}_d^\times\right) = \min\{k \in \mathbb{N} \mid \left(\varepsilon \mathcal{O}_d^\times\right)^k = \mathcal{O}_d^\times\}$. Observe that $\operatorname{ord}\left(\varepsilon \mathcal{O}_d^\times\right) = \left(\mathcal{O}_K^\times : \mathcal{O}_d^\times\right) \mid d$ (because $\mathcal{O}_K^\times / \mathcal{O}_d^\times$ is generated by $\varepsilon \mathcal{O}_d^\times$ and $d \in \mathbb{P}$ is ramified in $\mathcal{O}_K$). Since $\left(\varepsilon \mathcal{O}_d^\times\right)^k = \eta \mathcal{O}_d^\times = \mathcal{O}_d^\times$ by Claim 3, we have that $\operatorname{ord}\left(\varepsilon \mathcal{O}_d^\times\right) \mid k$, and thus $\left(\mathcal{O}_K^\times : \mathcal{O}_d^\times\right) = 1$ (since $d \in \mathbb{P}$). We infer that $\varepsilon \in \mathcal{O}_d$, and hence $d \mid y$. □(Theorem 1.2)

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