A COUNTEREXAMPLE TO THE PELLIAN EQUATION CONJECTURE OF MORDELL

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Abstract. Let $d \geq 2$ be a squarefree integer, let $\omega \in \{\sqrt{d}, \frac{1+\sqrt{d}}{2}\}$ be such that $\mathbb{Z}[\omega]$ is the ring of algebraic integers of the real quadratic number field $\mathbb{Q}(\sqrt{d})$, let $\varepsilon > 1$ be the fundamental unit of $\mathbb{Z}[\omega]$ and let x and y be the unique nonnegative integers with $\varepsilon = x + y\omega$. In this note, we extend and study the list of known squarefree integers $d \geq 2$, for which y is divisible by d (cf. OEIS A135735). As a byproduct, we present a counterexample to a conjecture of L. J. Mordell.

1. Introduction, conjectures and terminology

Let $\mathbb{P}$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$ denote the sets of prime numbers, positive integers, nonnegative integers, integers and rational numbers, respectively. Let $f \in \mathbb{N}$. We say that f is *squarefree* if $p^2 \nmid f$ for each $p \in \mathbb{P}$. Moreover, f is called *powerful* (also called *squareful*) if for each $p \in \mathbb{P}$ with $p \mid f$, we have that $p^2 \mid f$. Observe that f is powerful if and only if $f = a^2b^3$ for some $a, b \in \mathbb{N}$.

Let $d \in \mathbb{N}_{\geq 2}$ be squarefree, let $K = \mathbb{Q}(\sqrt{d})$ and let $\mathcal{O}_K$ be the ring of algebraic integers of K. We set

$$\omega = \begin{cases} \sqrt{d} & \text{if } d \equiv 2,3 \mod 4, \\ \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \mod 4, \end{cases} \quad \text{and} \quad \mathsf{d}_K = \begin{cases} 4d & \text{if } d \equiv 2,3 \mod 4, \\ d & \text{if } d \equiv 1 \mod 4. \end{cases}$$

It is well known that $\mathcal{O}_K = \mathbb{Z}[\omega] = \mathbb{Z} \oplus \omega\mathbb{Z}$. Let $\varepsilon \in \mathcal{O}_K$ be the (unique) fundamental unit with $\varepsilon > 1$ (i.e., $\{\pm\varepsilon^k \mid k \in \mathbb{Z}\}$ is the unit group of $\mathcal{O}_K$). Observe that there always exist unique $x, y \in \mathbb{N}_0$ such that $\varepsilon = x + y\omega$, and if $d \neq 5$, then $x, y \in \mathbb{N}$.

So far, there are 17 known squarefree integers $d \in \mathbb{N}_{\geq 2}$ with $d \mid y$ (see [18] or OEIS A135735). In this note, we extend the list of known squarefree integers $d \in \mathbb{N}_{\geq 2}$ with $d \mid y$ to 21 members in total and one of the newly found numbers happens to be a counterexample to the Pellian equation conjecture of Mordell. For the readers' convenience, we include the complete list here.

$$46, 430, 1817, 58254, 209991, 1752299, 3124318, 4099215, 5374184665, 6459560882, 16466394154,$$
$$20565608894, 25666082990, 117477414815, 125854178626, 1004569189366, 1188580642033,$$
$$15826129757609, 18803675974841, 20256129307923, 39028039587479$$

For more details, we refer to the last part of this note (which contains several tables that summarize the properties of these numbers). Next, we want to discuss the importance of the squarefree integers $d \in \mathbb{N}_{\geq 2}$ for which $d \mid y$. Indeed, there are several conjectures and results that are tied to these numbers. In what follows, we present these conjectures and results and restate them by using the aforementioned terminology.

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The conjecture of Ankeny, Artin and Chowla (or **(AAC)-conjecture** for short):
This conjecture states that if $d \in \mathbb{P}$ and $d \equiv 1 \mod 4$, then $d \nmid y$. The (AAC)-conjecture was first mentioned by N. C. Ankeny, E. Artin and S. Chowla in 1952 ([1, page 480]) and has subsequently been studied by various authors. For instance, L. J. Mordell provided a characterization of the (AAC)-conjecture (if $d \equiv 5 \mod 8$) that involves Bernoulli numbers ([14, 15]). Also note that the (AAC)-conjecture plays some role in the study of direct-sum cancellation for modules over orders in real quadratic number fields ([10]). For more recent work involving the (AAC)-conjecture, we refer to [2, 19]. The (AAC)-conjecture has been verified (for all primes $d \equiv 1 \mod 4$) up to $2 \cdot 10^{11}$ ([16, 17]).

The Pellian equation conjecture of Mordell:
The Pellian equation conjecture states that if $d \in \mathbb{P}$ and $d \equiv 3 \mod 4$, then $d \nmid y$. It was first formulated by L. J. Mordell in 1961 ([15, page 283]) who also established a connection of this conjecture to Euler numbers. The conjecture of Mordell has recently been studied in a series of papers ([2, 5, 19]) and has been verified (for all primes $d \equiv 3 \mod 4$) up to $1.6 \cdot 10^9$ ([2]). (Although, we want to mention that both the (AAC)-conjecture and the Mordell conjecture (technically) have been verified up to $1.5 \cdot 10^{12}$ in [18].) Later we provide a counterexample to the Mordell conjecture (Example 2.6).

The conjecture of Erdös, Mollin and Walsh (or **(EMW)-conjecture** for short):
It states that for each $a \in \mathbb{N}$, there is some $b \in \{a, a+1, a+2\}$ such that b is not powerful (i.e., there are no three consecutive powerful numbers). The (EMW)-conjecture was first mentioned in a paper of P. Erdös ([6]) and has subsequently been rediscovered by R. A. Mollin and P. G. Walsh ([13]) who also provided a characterization of the conjecture in terms of fundamental units ([11, 13]). This conjecture has wide implications (if it is true), like the existence of infinitely many primes that are not Wieferich primes ([8]).

Now we want to discuss various results that involve the squarefree integers $d \in \mathbb{N}_{\geq 2}$ with $d \mid y$. To do so, we need some more terminology. For $s, r, t \in \mathbb{N}_0$, let $[r, s] = \{z \in \mathbb{N}_0 \mid r \leq z \leq s\}$ and $\mathbb{N}_{\geq t} = \{z \in \mathbb{N}_0 \mid z \geq t\}$. Let $\mathrm{N} : K \to \mathbb{Q}$ defined by $\mathrm{N}(a + b\sqrt{d}) = a^2 - db^2$ for each $a, b \in \mathbb{Q}$ be the norm map on K. A subring $\mathcal{O}$ of K with quotient field K is called an *order* in K if it is a finitely generated $\mathbb{Z}$-module. For each $f \in \mathbb{N}$, let $\mathcal{O}_f = \mathbb{Z} + f\mathcal{O}_K$ and note that $\mathcal{O}_f$ is the unique order in K with conductor f (i.e., $\{z \in \mathcal{O}_f \mid x\mathcal{O}_K \subseteq \mathcal{O}_f\} = f\mathcal{O}_K$). Let $\mathrm{Pic}(\mathcal{O})$ be the Picard group of $\mathcal{O}$ for each order $\mathcal{O}$ in K. We let $\mathrm{h}(d) = |\mathrm{Pic}(\mathcal{O}_K)|$ denote the class number of K. For each $a, b \in \mathbb{Z}$, let $\left(\frac{a}{b}\right) \in \{-1, 0, 1\}$ denote the Kronecker symbol of a modulo b. If $p \in \mathbb{P}$, then p is called *inert*, *ramified*, *split* (in $\mathcal{O}_K$) if $\left(\frac{\mathsf{d}_K}{p}\right) = -1$, $\left(\frac{\mathsf{d}_K}{p}\right) = 0$, $\left(\frac{\mathsf{d}_K}{p}\right) = 1$, respectively. We will use well known properties of the Kronecker symbol (like the quadratic reciprocity law) throughout this note without further mention.

Relationships between the (EMW)-conjecture and fundamental units:
We say that d *induces a counterexample to the (EMW)-conjecture* (or d *satisfies (C)* for short) if $d \equiv 7 \mod 8$ and there are some $k, u, v \in \mathbb{N}_0$ such that u is powerful, k and v are odd, $\varepsilon^k = u + v\sqrt{d}$ and $d \mid v$. Furthermore, we say that d *induces a strong counterexample to the (EMW)-conjecture* (or d *satisfies (SC)* for short) if $d \equiv 7 \mod 8$, x is powerful, y is odd and $d \mid y$. Clearly, if d satisfies (SC), then d satisfies (C). It is shown in [13] that the (EMW)-conjecture holds if and only if there is no squarefree $d \in \mathbb{N}_{\geq 2}$ that satisfies (C).

Connections to conductors of relative class number one:
The integer d is said to *have no nontrivial conductors of relative class number one* (or *to satisfy (RC)* for short) if $\{f \in \mathbb{N} \mid \mathrm{h}(d) = |\mathrm{Pic}(\mathcal{O}_f)|\} = \{1\}$. The first systematic study (of which we are aware) of this condition was done in [7]. Following this, the problem of describing (RC) gained more traction ([12]) and

was finally solved in [4]. We present the connection of (RC) to squarefree integers $d \in \mathbb{N}_{\geq 2}$ with $d \mid y$ in Proposition 2.2.

Unusual orders in real quadratic number fields and the condition $d \mid y$:
Let $f \in \mathbb{N}$. We say that f is an *unusual conductor of* d if f is squarefree, f is divisible by a ramified prime, f is not divisible by a split prime, $\mathrm{h}(d) = |\mathrm{Pic}(\mathcal{O}_f)| = 2$ and for each ramified $p \in \mathbb{P}$ with $p \mid f$ and all $a, b \in \mathbb{Z}$, $|pa^2 - \frac{\mathsf{d}_K}{p} b^2| \neq 4$. Let D_d be the set of unusual conductors of d. The definition of an unusual conductor seems artificial, but becomes clear in view of the results of [3, 18] (since these results provide a link to an important property in factorization theory). We discuss the relationships of unusual orders to squarefree integers $d \in \mathbb{N}_{\geq 2}$ with $d \mid y$ in Proposition 2.4 and Theorem 2.5 below.

Throughout this note, let $d \in \mathbb{N}_{\geq 2}$ be squarefree and let K, $\mathcal{O}_K$, ω, d_K, ε and $\mathcal{O}_f$ for each $f \in \mathbb{N}$ be defined (as above) with respect to d. Furthermore, let $x, y \in \mathbb{N}_0$ be such that $\varepsilon = x + y\omega$.

2. Results and examples

We start with a lemma that will be useful in the subsequent discussion of the conditions (C) and (SC) (concerning the 21 members of the list).

Lemma 2.1. *Let d satisfy (C). Then y is odd.*

Proof. There are $k, u, v \in \mathbb{N}_0$ such that k and v are odd and $\varepsilon^k = u + v\sqrt{d}$. Since $d \equiv 7 \mod 8$, we have that $x^2 - dy^2 = \mathrm{N}(\varepsilon) = 1$, and hence xy is even. Therefore, $v = \sum_{i=0, i \equiv 1 \mod 2}^{k} \binom{k}{i} x^{k-i} y^i d^{\frac{i-1}{2}} \equiv y^k d^{\frac{k-1}{2}} \equiv y \mod 2$, and thus y is odd. □

Our next result is a variant of the main theorem of [4]. It establishes a connection between the condition (RC) and the divisibility of y by d.

Proposition 2.2. *d satisfies* (RC) *if and only if* $\mathrm{N}(\varepsilon) = 1$, *$d \not\equiv 1 \mod 8$, y is even and $d \mid y$.*

Proof. First we recall some definitions of [4]. Clearly, there exist unique $\alpha_0, \beta_0 \in \mathbb{Q}$ such that $\varepsilon = \alpha_0 + \beta_0\sqrt{d}$. Note that $2\alpha_0, 2\beta_0 \in \mathbb{N}_0$. We set $\tilde{\beta}_0 = \begin{cases} \beta_0 & \text{if } \varepsilon \in \mathbb{Z}[\sqrt{d}], \\ 2\beta_0 & \text{if } \varepsilon \notin \mathbb{Z}[\sqrt{d}]. \end{cases}$ Observe that $\tilde{\beta}_0 \in \mathbb{N}_0$ and $y = \begin{cases} \tilde{\beta}_0 & \text{if } d \not\equiv 1 \mod 4 \text{ or } \varepsilon \notin \mathbb{Z}[\sqrt{d}], \\ 2\tilde{\beta}_0 & \text{if } d \equiv 1 \mod 4 \text{ and } \varepsilon \in \mathbb{Z}[\sqrt{d}]. \end{cases}$ In particular, $\tilde{\beta}_0 \mid y$ and if $y \neq \tilde{\beta}_0$, then $y = 2\tilde{\beta}_0$ and d is odd.

Now let d satisfy (RC). We infer by [4, Proposition 3.4] that $\mathrm{N}(\varepsilon) = 1$. Moreover, it follows from [4, Theorem 4.1] that $d \mid \tilde{\beta}_0$, and hence $d \mid y$. If d is even, then clearly $d \not\equiv 1 \mod 8$ and y is even (since $d \mid y$). Now let d be odd. Then [4, Theorem 4.1] implies that $d \not\equiv 1 \mod 8$ and $\tilde{\beta}_0$ is even. Therefore, y is even.

Conversely, let $\mathrm{N}(\varepsilon) = 1$, let $d \not\equiv 1 \mod 8$, let y be even and let $d \mid y$. We obtain that $d \mid \tilde{\beta}_0$. Next we show that $\tilde{\beta}_0$ is even. Without restriction, we can assume that $d \equiv 1 \mod 4$ and $\varepsilon \in \mathbb{Z}[\sqrt{d}]$. Note that $\alpha_0, \beta_0 \in \mathbb{N}_0$ and $\beta_0 = \tilde{\beta}_0$. Consequently, $1 = \mathrm{N}(\varepsilon) = \alpha_0^2 - d\tilde{\beta}_0^2$, and thus $\alpha_0^2 \equiv 1 + \tilde{\beta}_0^2 \mod 4$. If $\tilde{\beta}_0$ is odd, then $\alpha_0^2 \equiv 2 \mod 4$, a contradiction. This implies that $\tilde{\beta}_0$ is even. It is now an immediate consequence of [4, Theorem 4.1] that d satisfies (RC). □

Lemma 2.3. *Let $p, q \in \mathbb{P}$ be such that $p \equiv 1 \mod 4$, $q \equiv 3 \mod 4$ and $d = pq$. If y is even, there are some $a, b \in \mathbb{Z}$ such that $|pa^2 - qb^2| = 1$. If y is odd, then there are some $a, b \in \mathbb{Z}$ such that $|a^2 - db^2| = 2$ or there are some $a, b \in \mathbb{Z}$ such that $|pa^2 - qb^2| = 2$.*

Proof. This is well known and can be shown by investigating the norm of the fundamental unit. A detailed proof can be found in case 3 of the proof of [18, Theorem 4.4]. □

In [18, Theorem 5.4] it was shown that the set of real quadratic number fields that posses an order with an unusual conductor can (naturally) be divided into 41 disjoint subsets. It was also proved in [18] that all (but possibly one of) these subsets are nonempty. The squarefree integers that define the real quadratic number fields in the aforementioned exceptional subset are called the squarefree integers of type 4/form 1. Note that the squarefree integers d that satisfy the conditions in Proposition 2.4 are precisely the squarefree integers d of type 4/form 1. The hitherto open problem of their existence was the driving factor for the search conducted in [18]. Recall that $\mathrm{h}(d)$ denotes the class number of K.

Proposition 2.4. *Let $p, q \in \mathbb{P}$ be such that $p \equiv 1 \mod 4$, $q \equiv 3 \mod 4$, $d = pq$ and $\mathrm{h}(d) = 2$. The following conditions are equivalent.*

(1) $D_d = \{2\}$.
(2) $p \equiv 5 \mod 8$, *y is odd and* $d \mid y$.
(3) $p \equiv 5 \mod 8$, $\left(\frac{p}{q}\right) = -1$ *and* $d \mid y$.

Proof. First, we show that if $p \equiv 5 \mod 8$, then y is odd if and only if $\left(\frac{p}{q}\right) = -1$. Let $p \equiv 5 \mod 8$.

Let y be odd. If there are some $a, b \in \mathbb{Z}$ such that $|a^2 - db^2| = 2$, then $\left(\frac{2}{p}\right) = 1$, which contradicts the fact that $p \equiv 5 \mod 8$. We infer by Lemma 2.3 that there are some $a, b \in \mathbb{Z}$ such that $|pa^2 - qb^2| = 2$. Consequently, $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) = \left(\frac{2}{p}\right) = -1$.

Now let y be even. By Lemma 2.3, there are some $a, b \in \mathbb{Z}$ such that $|pa^2 - qb^2| = 1$. This implies that $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) = 1$.

(1) $\Rightarrow$ (2) Since $2 \in D_d$, it follows from [18, Theorem 4.4] that $\mathrm{h}(d) = |\mathrm{Pic}(\mathcal{O}_2)|$ and $\left(\frac{2}{p}\right) = -1$. Therefore, $p \equiv 5 \mod 8$ and y is odd by [9, Theorem 5.9.7.4]. Since $\left(\frac{p}{q}\right) = -1$, it follows that $\left(\frac{\alpha d/p}{p}\right) = \left(\frac{-\alpha q}{p}\right) = -1$ for each $\alpha \in \{-1, 1\}$, and since $p, q \notin D_d$, we infer by [18, Theorem 4.4] that $\mathrm{h}(d) \neq |\mathrm{Pic}(\mathcal{O}_r)|$ for each $r \in \{p, q\}$. Therefore, $r \mid y$ for each $r \in \{p, q\}$ by [9, Theorem 5.9.7.4], and thus $d \mid y$.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) Since y is odd and $d \mid y$, we infer by [9, Theorem 5.9.7.4] that $\mathrm{h}(d) = |\mathrm{Pic}(\mathcal{O}_2)|$ and $|\mathrm{Pic}(\mathcal{O}_p)| \neq \mathrm{h}(d) \neq |\mathrm{Pic}(\mathcal{O}_q)|$. Since $p \equiv 5 \mod 8$, it follows from [18, Theorem 4.4] that $2 \in D_d$ and $p, q \notin D_d$. Therefore, $D_d = \{2\}$ by [18, Theorem 5.4]. $\square$

Finally, we present the main result of this note. It was the main motivation (besides Proposition 2.4) for the computer search discussed below.

Theorem 2.5. *Let $\mathrm{h}(d) = 2$ and let one of the following conditions be satisfied.*

(a) *There are some distinct $p, q \in \mathbb{P}$ such that $p \equiv q \equiv 1 \mod 4$, $d = pq$ and $\mathrm{N}(\varepsilon) = -1$.*
(b) *There are some $p, q \in \mathbb{P}$ such that $p \equiv 1 \mod 8$, $q \equiv 3 \mod 4$, $d = pq$ and y is odd.*
(c) *There are some distinct $p, q \in \mathbb{P}$ such that $p \equiv q \equiv 3 \mod 8$ and $d = 2pq$.*
(d) *There are some $p, q \in \mathbb{P}$ such that $p \equiv 1 \mod 8$, $q \equiv 3 \mod 4$, $\left(\frac{p}{q}\right) = -1$ and $d = 2pq$.*

Then $D_d = \emptyset$ if and only if $d \mid y$.

Proof. It is a simple consequence of [9, Theorem 5.9.7.4] that for each ramified $r \in \mathbb{P}$, $\mathrm{h}(d) \neq |\mathrm{Pic}(\mathcal{O}_r)|$ if and only if $r \mid y$. In what follows, we use this fact without further mention.

CASE 1: Condition (a) is satisfied. Obviously, $\{p, q\}$ is the set of ramified primes. It follows immediately from [18, Corollary 3.10(2)] that $D_d = \emptyset$ if and only if $\mathrm{h}(d) \neq |\mathrm{Pic}(\mathcal{O}_r)|$ for each $r \in \{p, q\}$ if and only if $r \mid y$ for each $r \in \{p, q\}$ if and only if $d \mid y$.

CASE 2: Condition (b) holds. Clearly, $\{2,p,q\}$ is the set of ramified primes. Since $p \equiv 1 \mod 8$, we have that $\left(\frac{2}{p}\right) = 1$, and hence $2 \notin D_d$ by [18, Theorem 4.4]. Moreover, $\left(\frac{p}{q}\right) = -1$ by [18, Lemma 4.3], and thus for each $a,b \in \mathbb{Z}$, $|pa^2 - qb^2| \neq 1$. This implies that for each $r \in \{p,q\}$ and all $a,b \in \mathbb{Z}$, $|ra^2 - \frac{\mathsf{d}_K}{r}b^2| \neq 4$. We infer by [18, Corollary 3.10(1)] that $D_d = \emptyset$ if and only if $\mathrm{h}(d) \neq |\mathrm{Pic}(\mathcal{O}_r)|$ for each $r \in \{p,q\}$ if and only if $r \mid y$ for each $r \in \{p,q\}$ if and only if $d \mid y$.

CASE 3: Condition (c) is satisfied. Observe that $\{2,p,q\}$ is the set of ramified primes and $x^2 - dy^2 = 1$. Therefore, y is even, and hence $2 \notin D_d$ by [18, Theorem 4.4]. Let $\alpha \in \{-1,1\}$. Then $\left(\frac{\alpha d/p}{p}\right) = \left(\frac{2\alpha}{p}\right)\left(\frac{q}{p}\right) = -\alpha\left(\frac{q}{p}\right) \neq -\alpha\left(\frac{p}{q}\right) = \left(\frac{-\alpha p}{q}\right)$, and hence $\left(\frac{\alpha d/p}{p}\right) = -1$ or $\left(\frac{-\alpha p}{q}\right) = -1$. It follows by analogy that $\left(\frac{\alpha d/q}{q}\right) = -1$ or $\left(\frac{-\alpha q}{p}\right) = -1$. We infer by [18, Theorem 4.4] that for each $r \in \{p,q\}$, $r \notin D_d$ if and only if $r \mid y$. Since y is even and $2 \notin D_d$, we obtain by [18, Theorem 2.6(3)] that $D_d = \emptyset$ if and only if $r \notin D_d$, for each $r \in \{p,q\}$ if and only if $r \mid y$ for each $r \in \{p,q\}$ if and only if $d \mid y$.

CASE 4: Condition (d) holds. Note that $\{2,p,q\}$ is the set of ramified primes and $x^2 - dy^2 = 1$. We infer that y is even, and thus $2 \notin D_d$ by [18, Theorem 4.4]. Let $\alpha \in \{-1,1\}$. Observe that $\left(\frac{\alpha d/p}{p}\right) = \left(\frac{2q}{p}\right) = \left(\frac{p}{q}\right) = -1$ and $\left(\frac{-\alpha q}{p}\right) = \left(\frac{q}{p}\right) = -1$. Then [18, Theorem 4.4] implies that for each $r \in \{p,q\}$, $r \notin D_d$ if and only if $r \mid y$. Since y is even and $2 \notin D_d$, it follows from [18, Theorem 2.6(3)] that $D_d = \emptyset$ if and only if $r \notin D_d$, for each $r \in \{p,q\}$ if and only if $r \mid y$ for each $r \in \{p,q\}$ if and only if $d \mid y$. $\square$

In what follows, let $X,Y \in \mathbb{N}_0$ be such that $X + Y\sqrt{d}$ is the fundamental unit of $\mathbb{Z}[\sqrt{d}]$ (i.e., $X + Y\sqrt{d}$ is the unique unit η of $\mathbb{Z}[\sqrt{d}]$ such that $\eta > 1$ and $\{\pm\eta^k \mid k \in \mathbb{Z}\}$ is the unit group of $\mathbb{Z}[\sqrt{d}]$). Observe that $X + Y\sqrt{d} \in \{\varepsilon, \varepsilon^3\}$. Let $\alpha \in \{0,1\}$ be such that $\alpha \equiv y \mod 2$ and let $\beta \in [0,7]$ be such that $\beta \equiv d \mod 8$. Moreover, let $s = |\{p \in \mathbb{P} \mid d \equiv 0 \mod p\}|$ (i.e., s is the number of distinct prime divisors of d). It follows from Proposition 2.2 that d satisfies (RC) if and only if $d \mid y$, $\alpha \neq 1 \neq \beta$ and $\mathrm{N}(\varepsilon) = 1$. Obviously, if d satisfies (C), then $\alpha = 1$ (by Lemma 2.1).

Next we want to briefly discuss two algorithms to find squarefree $d \in \mathbb{N}_{\geq 2}$ with $d \mid y$. The first algorithm is called the small step algorithm. We use it to determine whether a squarefree integer $d \in \mathbb{N}_{\geq 2}$ satisfies $d \mid y$. The second algorithm is the large step algorithm. It is utilized to identify the squarefree integers $d \in \mathbb{N}_{\geq 1000000}$ with $d \mid Y$. It is well known that if $d \mid y$, then $d \mid Y$. Moreover, if $d \mid Y$, then $d \mid 3y$. Also note that if $d \mid Y$ and $d \nmid y$, then $d \equiv 5 \mod 8$, $3 \mid d$ and $\varepsilon \notin \mathbb{Z}[\sqrt{d}]$. It is mentioned in [18] (and can also be derived from the tables below) that if $d = 17451248829$, then d is squarefree, $d \mid Y$ and $d \nmid y$. The large step algorithm is mainly used for search purposes (due to its better time complexity), while the small step algorithm is used for independent verification (and to handle the corner case with $d \mid Y$ and $d \nmid y$ that was mentioned before). For more details on the prior remarks and the used algorithms, we refer to [20]. Since the interval $[2, 1.5 \cdot 10^{12}]$ has already been searched ([18]), we now focus solely on the squarefree integers $d \geq 1.5 \cdot 10^{12}$.

The main purpose of the following part is to present the results of our recent computer search. For this search, we used two implementations of the large step algorithm, a scalar implementation and a (partially) vectorized implementation with AVX-512. The vectorized version (with AVX-512) provides about 40% more throughput than the scalar version on Zen 4 based CPUs. The programs were written in C and compiled with GCC-12.3.0 (with the compiler flag -O3). As a side note, we only used privately owned hardware for this computer search. We used 162 CPU cores (with hyperthreading and a clock rate around 4.1 GHz on average). Among these CPU cores are 74 cores with AVX-512 support (while the remaining 88 cores support AVX2). We did an exhaustive search on the squarefree integers $d \in [1.5 \cdot 10^{12}, 4.15 \cdot 10^{13}]$ (to find those that satisfy $d \mid Y$) and we spent approximately 2500 hours for this search in total.

Despite the fact, that we performed an exhaustive search, we do not claim that the newly found numbers (four numbers in total) are all the squarefree integers d with $d \mid Y$ in the search interval. The reason is twofold. On the one hand, we used an aggressive setting (-O3) to compile our programs. On the other hand, we have currently not enough available computational resources for an independent double check (of all squarefree integers in the search interval).

Nevertheless, we tested each of the squarefree integers (in the tables) below with our (old and new) implementations of the small step algorithm and the large step algorithm. Furthermore, we used both Mathematica 12.0.0 and Pari/GP 2.15.2 to compute $\alpha, \beta, s, \mathrm{N}(\varepsilon)$ and $\mathrm{h}(d)$ in the tables below and to provide independent checks of the squarefree integers involved. Also note that our verifications with Mathematica and Pari/GP did not use the small step algorithm or the large step algorithm. These verifications were done by computing the fundamental unit of $\mathcal{O}_K$ (respectively $\mathbb{Z}[\sqrt{d}]$) in full, by extracting the component y (respectively Y) and by using the "mod operation" to check whether $d \mid y$ (respectively $d \mid Y$).

d	46	430	1817	58254	209991	1752299	3124318	4099215	5374184665	6459560882	16466394154
$d \mid Y$	true	true	true	true	true	true	true	true	true	true	true
$d \mid y$	true	true	true	true	true	true	true	true	true	true	true
(RC)	true	true	false	true	true	true	true	false	false	true	true
α	0	0	0	0	0	0	0	1	0	0	0
β	6	6	1	6	7	3	6	7	1	2	2
s	2	3	2	5	2	3	2	3	2	4	4
$\mathrm{N}(\varepsilon)$	1	1	1	1	1	1	1	1	-1	1	1
$\mathrm{h}(d)$	1	2	1	8	2	4	1	4	2	4	32

d	17451248829	20565608894	25666082990	117477414815	125854178626	1004569189366
$d \mid Y$	true	true	true	true	true	true
$d \mid y$	false	true	true	true	true	true
(RC)	false	true	true	true	true	true
α	1	0	0	0	0	0
β	5	6	6	7	2	6
s	4	3	4	4	4	2
$\mathrm{N}(\varepsilon)$	1	1	1	1	1	1
$\mathrm{h}(d)$	4	2	8	8	8	1

d	11885806420033	15826129757609	18803675974841	20256129307923	39028039587479
$d \mid Y$	true	true	true	true	true
$d \mid y$	true	true	true	true	true
(RC)	false	false	false	false	false
α	0	0	0	1	1
β	1	1	1	3	7
s	3	2	3	4	1
$\mathrm{N}(\varepsilon)$	1	1	1	1	1
$\mathrm{h}(d)$	2	1	2	16	1

It follows from Lemma 2.1 that if d is a squarefree integer of the tables above that satisfies (C), then $d \in \{4099215, 39028039587479\}$. If $d = 4099215$, then d does not satisfy (SC), since $701 \in \mathbb{P}$, $701 \mid x$ and $701^2 \nmid x$. Moreover, if $d = 39028039587479$, then d does not satisfy (SC), since $5 \in \mathbb{P}$, $5 \mid x$ and $5^2 \nmid x$. In particular, none of the squarefree integers d in the tables above satisfies (SC). We do not know if any $d \in \{4099215, 39028039587479\}$ satisfies (C). Next we want to present the aforementioned counterexample (which can easily be derived from the tables above). We state it explicitly for the readers' convenience.

Example 2.6. (The counterexample to Mordell's Pellian equation conjecture)
Let $d = 39028039587479$. Then $d \in \mathbb{P}$, $d \equiv 3 \mod 4$ and $d \mid y$.

We do not know (with reasonable certainty) whether the example above is the smallest counterexample to Mordell's Pellian equation conjecture. Furthermore, we want to emphasize that (to the best of our knowledge) the (AAC)-conjecture and the (EMW)-conjecture are still open. Besides that, it is also unknown (now as before) whether squarefree integers of type 4/form 1 exist.

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