COMMUTATIVE RINGS WITH ONE-ABSORBING FACTORIZATION

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Abstract. Let R be a commutative ring with nonzero identity. A. Yassine et al. defined in the paper (Yassine, Nikmehr and Nikandish, 2020), the concept of 1-absorbing prime ideals as follows: a proper ideal I of R is said to be a 1-absorbing prime ideal if whenever $xyz \in I$ for some nonunit elements $x, y, z \in R$, then either $xy \in I$ or $z \in I$. We use the concept of 1-absorbing prime ideals to study those commutative rings in which every proper ideal is a product of 1-absorbing prime ideals (we call them OAF-rings). Any OAF-ring has dimension at most one and local OAF-domains (D, M) are atomic such that M^2 is universal.

1. Introduction

Throughout this paper, all rings are commutative with nonzero identity and all modules are unital. Let $\mathbb{N}$ denote the set of positive integers. For $m \in \mathbb{N}$, let $[1, m] = \{n \in \mathbb{N} \mid 1 \leq n \leq m\}$. Let R be a ring. An ideal I of R is said to be *proper* if $I \neq R$. The *radical* of I is denoted by $\sqrt{I} = \{x \in R \mid x^n \in I$ for some $n \in \mathbb{N}\}$. We denote by $\text{Min}(I)$ the *set of minimal prime ideals* over the ideal I. The concept of prime ideals plays an important role in ideal theory and there are many ways to generalize it.

In [9] Badawi introduced and studied the concept of 2-absorbing ideals which is a generalization of prime ideals. An ideal I of R is a 2-*absorbing ideal* if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In this case $\sqrt{I} = P$ is a prime ideal with $P^2 \subseteq I$ or $\sqrt{I} = P_1 \cap P_2$ where P_1, P_2 are incomparable prime ideals with $P_1P_2 \subseteq I$, cf. [9, Theorem 2.4]. In [8] Anderson and Badawi introduced the concept of n-absorbing ideals as a generalization of prime ideals where n is a positive integer. An ideal I of R is called an n-*absorbing ideal* of R, if whenever $a_1, a_2, \ldots, a_{n+1} \in R$ and $\prod_{i=1}^{n+1} a_i \in I$, then there are n of the a_i's whose product is in I. In this case, due to Choi and Walker [13, Theorem 1], $(\sqrt{I})^n \subseteq I$.

In [23] M. Mukhtar et al. studied the commutative rings whose ideals have a TA-factorization. A proper ideal is called a TA-*ideal* if it is a 2-absorbing ideal. By a TA-*factorization* of a proper ideal I we mean an expression of I as a product $\prod_{i=1}^{r} J_i$ of TA-ideals. M. Mukhtar et al. prove that any TAF-ring has dimension at most one and the local TAF-domains are atomic pseudo-valuations domains. Recently in [1], M. T. Ahmed et al. studied commutative rings whose proper ideals have an n-absorbing factorization.

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Let I be a proper ideal of R. By an n-*absorbing factorization* of I we mean an expression of I as a product $\prod_{i=1}^{r} I_i$ of proper n-absorbing ideals of R. M. T. Ahmed et al. called AF-dim(R) (*absorbing factorization dimension*) the minimum positive integer n such that every ideal of R has an n-absorbing factorization. If no such n exists, set $AF\text{-dim}(R) = \infty$. An FAF-*ring* (*finite absorbing factorization ring*) is a ring such that $AF\text{-dim}(R) < \infty$. Recall that a *general* ZPI-*ring* is a ring whose proper ideals can be written as a product of prime ideals. Therefore, AF-dim(R) measures, in some sense, how far R is from being a general ZPI-ring, cf. [1, Proposition 3]. By dim(R) we denote the *Krull dimension* of R.

In [25], A. Yassine et al. introduced the concept of a 1-absorbing prime ideal which is a generalization of a prime ideal. A proper ideal I of R is a 1-*absorbing prime ideal* (our abbreviation OA-*ideal*) if whenever we take nonunit elements $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $c \in I$. In this case $\sqrt{I} = P$ is a prime ideal, cf. [25, Theorem 2.3]. And if R is a ring in which exists an OA-ideal that is not prime, then R is a local ring, that is a ring with one maximal ideal.

Let I be a proper ideal of R. By an OA-*factorization* of I we mean an expression of I as a product $\prod_{i=1}^{n} J_i$ of OA-ideals. The aim of this note is to study the commutative rings whose proper ideals (resp., proper principal ideals, resp., proper 2-generated ideals) have an OA-factorization.

We call R a 1-*absorbing prime factorization ring* (OAF-*ring*) if every proper ideal has an OA-factorization. An OAF-*domain* is a domain which is an OAF-ring. Our paper consists of five sections (including the introduction).

In the next section, we characterize OA-ideals (Lemma 2.1) and we prove that if I is an OA-ideal, then I is a primary ideal. We also show that the OAF-ring property is stable under factor ring (resp., fraction ring) formation (Propositions 2.2 and 2.3). Furthermore, we investigate OAF-rings with respect to direct products (Corollary 2.5) and polynomial ring extensions (Corollary 2.6). We prove that the general ZPI-rings are exactly the arithmetical OAF-rings (Theorem 2.8).

The third section consists of a collection of preparational results which will be of major importance in the fourth section. For instance, we show that the Krull dimension of an OAF-ring is at most one (Theorem 3.5).

The fourth section contains the main results of our paper. Among other results, we provide characterizations of OAF-rings (Theorem 4.2), rings whose proper principal ideals have an OA-factorization (Corollary 4.3) and rings whose proper (principal) ideals are OA-ideals (Proposition 4.5).

In the last section, we study the transfer of the various OA-factorization properties to the trivial ring extension.

2. Characterization of OA-ideals and simple facts

We start with a characterization of OA-ideals. Recall that a ring R is a Q-*ring* (cf. [3]) if every proper ideal of R is a product of primary ideals.

Lemma 2.1. *Let R be a ring with Jacobson radical M and I be an ideal of R.*

(1) *If R is not local, then I is an OA-ideal if and only if I is a prime ideal.*
(2) *If R is local, then I is an OA-ideal if and only if I is a prime ideal or $M^2 \subseteq I \subseteq M$.*
(3) *Every OA-ideal is a primary TA-ideal. In particular, every OAF-ring is both a Q-ring and a TAF-ring.*

Proof. (1) This follows from [25, Theorem 2.4].

(2) Let R be local. Then M is the maximal ideal of R.

($\Rightarrow$) Let I be an OA-ideal such that I is not a prime ideal. Since I is proper, we infer that $I \subseteq M$. Since I is not prime, there are $a, b \in M \setminus I$ such that $ab \in I$. To prove that $M^2 \subseteq I$, it suffices to show that $xy \in I$ for all $x, y \in M$. Let $x, y \in M$. Then $xyab \in I$. Since $xy, a, b \in M$, $b \notin I$ and I is an OA-ideal, it follows that $xya \in I$. Again, since $x, y, a \in M$, $a \notin I$ and I is an OA-ideal, we have that $xy \in I$.

($\Leftarrow$) Clearly, if I is a prime ideal, then I is an OA-ideal. Now let $M^2 \subseteq I \subseteq M$. Then I is proper. Let $a, b, c \in M$ be such that $abc \in I$. Then $ab \in M^2 \subseteq I$. Therefore, I is an OA-ideal.

(3) Let I be an OA-ideal. It is an immediate consequence of (1) and (2) that I is a primary ideal. Now let $a, b, c \in R$ be such that $abc \in I$. We have to show that $ab \in I$ or $ac \in I$ or $bc \in I$.

First let a or b or c be a unit of R. Without restriction let a be unit of R. Since $abc \in I$, we infer that $bc \in I$.

Now let a, b and c be nonunits. Then $ab \in I$ or $c \in I$. If $c \in I$, then $ac \in I$. The in particular statement is clear. □

Proposition 2.2. *Let R be an OAF-ring and I be a proper ideal of R. Then R/I is an OAF-ring.*

Proof. Let J be a proper ideal of R which contains I. Let $J = \prod_{i=1}^{m} J_i$ be an OA-factorization. Then $J/I = \prod_{i=1}^{m}(J_i/I)$. It suffices to show that J_i/I is an OA-ideal for each $i \in [1, m]$. Let $i \in [1, m]$ and let $a, b, c \in R$ be such that $\bar{a}, \bar{b}, \bar{c}$ are three nonunit elements of R/I and $\bar{a}\bar{b}\bar{c} \in J_i/I$. Clearly, a, b, c are nonunit elements of R and $abc \in J_i$. Since J_i is an OA-ideal of R, we get that $ab \in J_i$ or $c \in J_i$ which implies that $\bar{a}\bar{b} \in J_i/I$ or $\bar{c} \in J_i/I$. Therefore, R/I is an OAF-ring. □

Proposition 2.3. *Let S be a multiplicatively closed subset of $R \setminus \mathbf{0}$. If R is an OAF-ring, then $S^{-1}R$ is an OAF-ring. In particular, R_M is an OAF-ring for every maximal ideal M of R.*

Proof. Let J be a proper ideal of $S^{-1}R$. Then $J = S^{-1}I$ for some proper ideal I of R with $I \cap S = \varnothing$. Let $I = \prod_{i=1}^{m} I_i$ be an OA-factorization. Then $J = \prod_{i=1}^{m}(S^{-1}I_i)$ where each $S^{-1}I_i$ which is proper is an OA-ideal by [25, Theorem 2.18]. Thus $S^{-1}R$ is an OAF-ring. The in particular statement is clear. □

Let R be a ring. Then R is said to be a *π-ring* if every proper principal ideal of R is a product of prime ideals. We say that R is a *unique factorization ring* (in the sense of Fletcher, cf. [4]) if every proper principal ideal of R is a product of principal prime ideals. A *unique factorization domain* is an integral domain which is a unique factorization ring.

Remark 2.4. Let R be a non local ring.

(1) R is a general ZPI-ring if and only if R is an OAF-ring.
(2) R is a π-ring if and only if each proper principal ideal of R has an OA-factorization.
(3) R is a unique factorization ring if and only if each proper principal ideal of R is a product of principal OA-ideals.

Proof. This is an immediate consequence of Lemma 2.1(1). □

In the light of the above remark we give the next result.

Corollary 2.5. *Let R_1 and R_2 be two rings and $R = R_1 \times R_2$ be their direct product. The following statements are equivalent.*

(1) *R is an OAF-ring.*
(2) *R is a general ZPI-ring.*
(3) *R_1 and R_2 are general ZPI-rings.*

Proof. This follows from Remark 2.4(1) and [21, Exercise 6(g), page 223]. □

Let R be a ring. Then R is called a *von Neumann regular ring* if for each $x \in R$ there is some $y \in R$ with $x = x^2y$. The ring R is von Neumann regular if and only if R is a zero-dimensional reduced ring (see [19, Theorem 3.1, page 10]).

Corollary 2.6. *Let R be a ring. The following statements are equivalent.*

(1) *$R[X]$ is an OAF-ring.*
(2) *R is a Noetherian von Neumann regular ring.*
(3) *R is a finite direct product of fields.*

Proof. Observe that the polynomial ring $R[X]$ is never local, since X and $1-X$ are nonunit elements of $R[X]$, but their sum is a unit. Consequently, $R[X]$ is an OAF-ring if and only if $R[X]$ is a general ZPI-ring by Remark 2.4(1). The rest is now an easy consequence of [2, Theorem 6 and Corollary 6.1], [21, Exercise 10, page 225] and Hilbert's basis theorem. □

Let R be a ring and I be an ideal of R. Then I is called *divided* if I is comparable to every ideal of R (or equivalently, I is comparable to every principal ideal of R).

Lemma 2.7. *Let R be a local ring with maximal ideal M such that M^2 is divided. The following statements are equivalent.*

(1) *Each two principal OA-ideals which contain M^2 are comparable.*
(2) *For each OA-ideal I of R, we have that I is a prime ideal or $I = M^2$.*

Proof. (1) $\Rightarrow$ (2): Let I be an OA-ideal of R such that I is not a prime ideal of R. Then $M^2 \subseteq I \subset M$ by Lemma 2.1(2). Assume that $M^2 \subset I$. Let $x \in I \setminus M^2$ and let $y \in M \setminus I$. Then $x, y \notin M^2$, and thus $M^2 \subseteq xR, yR$ (since M^2 is divided). It follows that xR and yR are (principal) OA-ideals of R by Lemma 2.1(2). Since $y \notin xR$ and xR and yR are comparable, we infer that $xR \subset yR$. Consequently, there is some $z \in M$ such that $x = yz$, and hence $x \in M^2$, a contradiction. Therefore, $I = M^2$.

(2) $\Rightarrow$ (1): This is obvious. □

Let R be a ring. An ideal I of R is called 2-*generated* if $I = xR + yR$ for some (not necessarily distinct) $x, y \in R$. Note that every principal ideal of R is 2-generated. We say that R is a *chained ring* if each two ideals of R are comparable under inclusion. Moreover, R is said to be an *arithmetical ring* if R_M is a chained ring for each maximal ideal M of R.

Theorem 2.8. *Let R be a ring. The following statements are equivalent.*

(1) *R is a general ZPI-ring*
(2) *R is an arithmetical OAF-ring.*
(3) *R is an arithmetical ring and each proper principal ideal of R has an OA-factorization.*

Proof. First we show that if R is an arithmetical π-ring, then R is a general ZPI-ring. Let R be an arithmetical π-ring and let M be a maximal ideal of R. It is straightforward to show that R_M is a π-ring. Moreover, R_M is a chained ring, and hence every 2-generated ideal of R_M is principal. Therefore, every proper 2-generated ideal of R_M is a product of prime ideals of R_M. Consequently, R_M is a general ZPI-ring by [22, Theorem 3.2]. This implies that $\dim(R_M) \leq 1$ by [21, page 205]. We infer that $\dim(R) \leq 1$, and thus R is a general ZPI-ring by [16, Theorems 39.2, 46.7, and 46.11].

(1) $\Rightarrow$ (2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (1): It is sufficient to show that R is a π-ring. If R is not local, then R is a π-ring by Remark 2.4(2). Therefore, we can assume that R is local with maximal ideal M. Since R is local, we have that R is a chained ring. Therefore, M^2 is divided and each two OA-ideals of R are comparable. We infer by Lemma 2.7 that each OA-ideal of R is a product of prime ideals. Now it clearly follows that R is a π-ring. $\square$

3. Preparational results

From Lemma 2.1(3), we have that $|\text{Min}(I)| = 1$ for every OA-ideal I of R. In view of this remark, we obtain the following result.

Proposition 3.1. *Let R be a ring and I be a proper ideal of R. If I has an OA-factorization, then* $\text{Min}(I)$ *is finite.*

Proof. Let $I = \prod_{i=1}^{n} I_i$ be an OA-factorization. It follows that $\text{Min}(I) \subseteq \bigcup_{i=1}^{n} \text{Min}(I_i)$, and thus $|\text{Min}(I)| \leq n$. $\square$

Let R be a ring and I be an ideal of R. Then I is called a *multiplication ideal* of R if for each ideal J of R with $J \subseteq I$, there is some ideal L of R such that $J = IL$.

Lemma 3.2. *Let R be a local ring such that each proper principal ideal of R has an OA-factorization. Then each nonmaximal minimal prime ideal of R is principal.*

Proof. Let P be a nonmaximal minimal prime ideal of R. By [2, Theorem 1] it is sufficient to show that P is a multiplication ideal.

Let $x \in P$ and let $xR = \prod_{i=1}^{n} I_i$ be an OA-factorization. There is some $j \in [1, n]$ such that $I_j \subseteq P$. By Lemma 2.1(2) we have that $P = I_j$, and hence $xR = PJ$ for some ideal J of R. We infer that $xR = P(xR : P)$.

Now let I be an ideal of R such that $I \subseteq P$. Then $I = \sum_{y\in I} yR = \sum_{y\in I} P(yR : P) = P\sum_{y\in I}(yR : P)$, and thus P is a multiplication ideal. □

The next result is a generalization of [16, Theorem 46.8] and its proof is based on the proof of the same result.

Proposition 3.3. *Let R be a local ring with maximal ideal M such that $\dim(R) \geq 1$ and every proper principal ideal of R has an OA-factorization. Then R is an integral domain and if $\dim(R) \geq 2$, then R is a unique factorization domain.*

Proof. Let N be the nilradical of R. It follows from Proposition 3.1 and Lemma 3.2 that $\mathrm{Min}(\mathbf{0})$ is finite and each $P \in \mathrm{Min}(\mathbf{0})$ is principal.

Claim: Every proper principal ideal of R/N has an OA-factorization. Let I be a proper principal ideal of R/N. Then $I = (xR + N)/N$ for some $x \in M$. Let $xR = \prod_{i=1}^{n} I_i$ be an OA-factorization. We infer that $I = (xR)/N = (\prod_{i=1}^{n} I_i)/N = \prod_{i=1}^{n}(I_i/N)$. It suffices to show that I_i/N is an OA-ideal of R/N for each $i \in [1, n]$. Let $i \in [1, n]$. If I_i is a prime ideal of R, then $N \subseteq I_i$, and hence I_i/N is a prime ideal of R/N. Now let I_i be not a prime ideal of R. By Lemma 2.1(2), we have that $M^2 \subseteq I_i \subseteq M$. Note that R/N is local with maximal ideal M/N. Since $(M/N)^2 = M^2/N \subseteq I_i/N \subseteq M/N$, it follows by Lemma 2.1(2) that I_i/N is an OA-ideal of R/N. This proves the claim.

Case 1: R is one-dimensional. We prove that R is an integral domain. If every OA-ideal of R is a prime ideal, then R is π-ring, and hence R is an integral domain by [16, Theorem 46.8]. Now let not every OA-ideal of R be a prime ideal. It follows from Lemma 2.1(2) that M is not idempotent. Set $L = M^2 \cup \bigcup_{Q\in \mathrm{Min}(\mathbf{0})} Q$. Next we prove that $M^2 \subseteq xR$ for each $x \in R \setminus L$. Let $x \in R \setminus L$. Without restriction let x be a nonunit. Note that xR cannot be a product of more than one OA-ideal, and hence xR is an OA-ideal. By Lemma 2.1(2) we have that $M^2 \subseteq xR$.

Now we show that $P \subseteq M^2$ for each $P \in \mathrm{Min}(\mathbf{0})$. Let $P \in \mathrm{Min}(\mathbf{0})$. Assume that $P \nsubseteq M^2$. Let $w \in R \setminus P$. Then $P + wR \nsubseteq L$ by the prime avoidance lemma, and thus there is some $v \in (P + wR) \setminus L$. It follows that $M^2 \subseteq vR \subseteq P + wR$. Since P is a nonmaximal prime ideal, we have that R/P has no simple R/P-submodules, and hence $\bigcap_{y\in R\setminus P}(P + yR) = P$. (Note that if $\bigcap_{y\in R\setminus P}(P + yR) \neq P$, then $\bigcap_{y\in R\setminus P}(P + yR)/P$ is a simple R/P-submodule of R/P.) This implies that $M^2 \subseteq \bigcap_{y\in R\setminus P}(P + yR) = P$, and thus $P = M$, a contradiction.

Let $Q \in \mathrm{Min}(\mathbf{0})$. By the prime avoidance lemma, there is some $z \in M \setminus L$. We infer that $Q \subset M^2 \subset zR$. Consequently, $Q = zQ$. Since Q is principal, it follows that $Q = \mathbf{0}$ (e.g. by Nakayama's lemma), and hence R is an integral domain.

Case 2: $\dim(R) \geq 2$ and R is reduced. We show that R is a unique factorization domain. There is some nonmaximal nonminimal prime ideal Q of R. By the prime avoidance lemma, there is some $x \in Q \setminus \bigcup_{P\in \mathrm{Min}(\mathbf{0})} P$. Since R is reduced, we have that $\bigcap_{L\in \mathrm{Min}(\mathbf{0})} L = \mathbf{0}$. If $y \in R$ is nonzero with $xy = 0$, then $y \notin L$ and $xy \in L$ for some $L \in \mathrm{Min}(\mathbf{0})$, and hence $x \in L$, a contradiction. We infer that x is a regular element of R. Let $xR = \prod_{i=1}^{n} I_i$ be an OA-factorization. Then $I_j \subseteq Q$ for some $j \in [1, n]$. Since x is regular, I_j is invertible, and hence I_j is a regular principal ideal (because invertible ideals of a local ring are regular principal ideals). Since $I_j \subseteq Q$ and $Q \neq M$, we have that I_j is a prime ideal by Lemma 2.1(2). Consequently, $P \subseteq I_j$ for some $P \in \mathrm{Min}(\mathbf{0})$. Since I_j is regular, we infer that $P \subset I_j$, and hence $P = PI_j$ (since I_j is principal). It follows (e.g. from Nakayama's lemma) that $P = \mathbf{0}$ (since P is principal). We obtain that R is an integral domain.

To show that R is a unique factorization domain, it suffices to show by [4, Theorem 2.6] that every nonzero prime ideal of R contains a nonzero principal prime ideal. Since $\dim(R) \geq 2$ and R is local, we only need to show that every nonzero nonmaximal prime ideal of R contains a nonzero principal prime ideal. Let L be a nonzero nonmaximal prime ideal of R and let $z \in L$ be nonzero. Let $zR = \prod_{k=1}^{m} J_k$ be an OA-factorization. Then $J_\ell \subseteq L$ for some $\ell \in [1, m]$. Since R is an integral domain, zR is invertible, and hence J_ℓ is invertible. Therefore, J_ℓ is nonzero and principal (since R is local). Since $L \neq M$, it follows from Lemma 2.1(2) that J_ℓ is a prime ideal.

Case 3: $\dim(R) \geq 2$. We have to show that R is a unique factorization domain. Note that R/N is a reduced local ring with maximal ideal M/N and $\dim(R/N) \geq 2$. Moreover, each proper principal ideal of R/N has an OA-factorization by the claim. It follows by Case 2 that R/N is a unique factorization domain, and thus N is the unique minimal prime ideal of R. Since R/N is a unique factorization domain and $\dim(R/N) \geq 2$, R/N possesses a nonzero nonmaximal principal prime ideal. We infer that there is some nonminimal nonmaximal prime ideal Q of R such that Q/N is a principal ideal of R/N. Consequently, there is some $q \in Q$ such that $Q = qR + N$. Let $qR = \prod_{i=1}^{n} I_i$ be an OA-factorization. Then $I_j \subseteq Q$ for some $j \in [1, n]$. Since $Q \neq M$, we infer by Lemma 2.1(2) that I_j is a prime ideal of R. Therefore, $Q = qR + N \subseteq I_j \subseteq Q$, and hence $I_j = Q$.

Assume that $Q \neq qR$. Then $qR = QJ$ for some proper ideal J of R. It follows that $q \in qR = (qR + N)J \subseteq qJ + N$, and thus $q(1 - a) \in N$ for some $a \in J$. Since a is a nonunit of R, we obtain that $q \in N$. This implies that $Q = qR + N = N$, a contradiction. We infer that $Q = qR$. Since $N \subset Q$ and N is a prime ideal of R, we have that $N = NQ$. Consequently, $N = \mathbf{0}$ (e.g. by Nakayama's lemma, since N is principal), and thus $R \cong R/N$ is a unique factorization domain. □

Proposition 3.4. *Let R be a local ring with maximal ideal M such that each proper 2-generated ideal of R has an OA-factorization. Then* $\dim(R) \leq 2$ *and each nonmaximal prime ideal of R is principal.*

Proof. First we show that $\dim(R_P) \leq 1$ for each nonmaximal prime ideal P of R. Let P be a nonmaximal prime ideal and let I be a proper 2-generated ideal of R_P. Observe that $I = J_P$ for some 2-generated ideal J of R with $J \subseteq P$. Let $J = \prod_{i=1}^{n} J_i$ be an OA-factorization. Then $I = J_P = \prod_{i=1}^{n} (J_i)_P = \prod_{i=1, J_i \subseteq P}^{n} (J_i)_P$. If $i \in [1, n]$ is such that $J_i \subseteq P$, then J_i is a prime ideal of R by Lemma 2.1(2), and thus $(J_i)_P$ is a prime ideal of R_P. We infer that I is a product of prime ideals of R_P. It follows from [22, Theorem 3.2], that R_P is a general ZPI-ring. It is an easy consequence of [21, page 205] that $\dim(R_P) \leq 1$.

This implies that $\dim(R) \leq 2$. It remains to show that every nonmaximal prime ideal of R is principal. Without restriction let $\dim(R) \geq 1$. It follows from Proposition 3.3 that R is either a one-dimensional domain or a two-dimensional unique factorization domain. In any case we have that each nonmaximal prime ideal of R is principal. □

In the next result we will prove a generalization of the fact that every OAF-ring has Krull dimension at most one.

Theorem 3.5. *Let R be a ring such that every proper 2-generated ideal of R has an OA-factorization. Then $\dim(R) \leq 1$.*

Proof. If every OA-ideal of R is a prime ideal, then R is a general ZPI-ring by [22, Theorem 3.2], and hence $\dim(R) \leq 1$ by [21, page 205]. Now let not every OA-ideal of R be a prime ideal. We infer by Lemma 2.1 that R is local and the maximal ideal of R is not idempotent. Let M be the maximal ideal of R. It suffices to show that if Q is a nonmaximal prime ideal of R, then $Q = \mathbf{0}$. Let Q be a nonmaximal prime ideal of R.

Assume that $Q \not\subseteq M^2$. Since $\dim(R) \leq 2$ by Proposition 3.4, there is some prime ideal P of R such that $Q \subseteq P$ and $\dim(R/P) = 1$. Next we show that $M^2 \subseteq P + yR$ for each $y \in R \setminus P$. Let $y \in R \setminus P$ and set $J = P + yR$. Without restriction let $J \subset M$. Note that J is 2-generated by Proposition 3.4. Since $J \not\subseteq M^2$, J cannot be a product of more than one OA-ideal, and thus J is an OA-ideal of R. Since $P \subset J \subset M$, we have that J is not a prime ideal of R, and thus $M^2 \subseteq J$ by Lemma 2.1(2). Moreover, R/P is an integral domain that is not a field. Consequently, R/P does not have any simple R/P-submodules, which implies that $P = \bigcap_{x \in R \setminus P}(P + xR)$. (Observe that if $\bigcap_{x \in R \setminus P}(P + xR) \neq P$, then $\bigcap_{x \in R \setminus P}(P + xR)/P$ is a simple R/P-submodule of R/P.) Therefore, $M^2 \subseteq \bigcap_{x \in R \setminus P}(P + xR) = P$, and hence $P = M$, a contradiction. We infer that $Q \subseteq M^2$.

There is some $z \in M \setminus M^2$ (since M is not idempotent). Since zR is a product of OA-ideals, we have that zR is an OA-ideal of R. As shown before, $L \subseteq M^2$ for each nonmaximal prime ideal L of R, and thus zR is not a nonmaximal prime ideal. Consequently, $Q \subset M^2 \subset zR$ by Lemma 2.1(2), and hence $Q = zQ$. Since Q is principal by Proposition 3.4, it follows (e.g. by Nakayama's lemma) that $Q = \mathbf{0}$. □

Lemma 3.6. *Let D be a local domain with maximal ideal M. Then each proper principal ideal of D has an OA-factorization if and only if D is atomic and each irreducible element generates an OA-ideal. If these equivalent conditions are satisfied, then $\bigcap_{n \in \mathbb{N}} P^n = \mathbf{0}$ for each height-one prime ideal P of D.*

Proof. ($\Rightarrow$) Let each proper principal ideal of D have an OA-factorization. If D is a unique factorization domain, then D is atomic and each irreducible element generates a prime ideal. Now let D be not a unique factorization domain. Then $\dim(D) = 1$ by Proposition 3.3.

Assume that M^2 is principal. Then M is invertible, and hence M is principal (since D is local). Note that D is a DVR (since $\dim(D) = 1$), and hence D is a unique factorization domain, a contradiction.

We infer that M^2 is not principal. We show that D is atomic. Let $y \in D$ be a nonzero nonunit. Then $yD = \prod_{i=1}^{n} I_i$ for some principal OA-ideals I_i. There are nonzero nonunits $x_i \in D$ such that $y = \prod_{i=1}^{n} x_i$ and $I_j = x_j D$ for each $j \in [1, n]$. Let $i \in [1, n]$. If I_i is a prime ideal, then x_i is a prime element, and thus x_i is irreducible. Now let I_i not be a prime ideal. It follows from Lemma 2.1(2) that $M^2 \subseteq I_i$. Since M^2 is not principal, we have that $x_i \notin M^2$. Therefore, x_i is irreducible.

Finally, let $z \in D$ be irreducible. Then $zD = \prod_{j=1}^{m} J_j$ for some principal OA-ideals J_j. Since zD is maximal among the proper principal ideals of D, we obtain that $zD = J_j$ for some $j \in [1, n]$.

($\Leftarrow$) Let D be atomic such that each irreducible element generates an OA-ideal. Let I be a proper principal ideal of D. Without restriction let I be nonzero. Then $I = xD$ for some nonzero nonunit $x \in D$. Observe that $x = \prod_{i=1}^{n} x_i$ for some irreducible elements $x_i \in D$. It follows that $\prod_{i=1}^{n} x_i D$ is an OA-factorization of I.

Now let the equivalent conditions be satisfied and let P be a height-one prime ideal of D. First let $P \neq M$. Then D is a unique factorization domain by Proposition 3.3, and hence P is principal. Therefore, $\bigcap_{n\in\mathbb{N}} P^n$ is a prime ideal of D by [5, Theorem 2.2(1)]. Since $\bigcap_{n\in\mathbb{N}} P^n \subset P$, we infer that $\bigcap_{n\in\mathbb{N}} P^n = \mathbf{0}$.

Now let $P = M$. Assume that $\bigcap_{n\in\mathbb{N}} M^n \neq \mathbf{0}$ and let $x \in \bigcap_{n\in\mathbb{N}} M^n$ be nonzero. Then xD is a product of m OA-ideals of D for some positive integer m. We infer by Lemma 2.1(2) that $M^{2m} \subseteq xD$, and hence $M^{2m} \subseteq xD \subseteq M^{4m} \subseteq M^{2m}$. This implies that $xD = M^{2m} = M^{4m} = x^2 D$, and thus x is a unit of D, a contradiction. Therefore, $\bigcap_{n\in\mathbb{N}} M^n = \mathbf{0}$. $\square$

Lemma 3.7. *Let R be a local ring with maximal ideal M such that M^2 is divided and such that either M is nilpotent or R is an integral domain with $\bigcap_{n\in\mathbb{N}} M^n = \mathbf{0}$. Then R is an OAF-ring and every proper principal ideal of R is a product of principal OA-ideals.*

Proof. If M is idempotent, then $M = \mathbf{0}$, and hence R is a field and both statements are clearly satisfied. Now let M be not idempotent. There is some $x \in M \setminus M^2$. In what follows, we freely use the fact that if N is an ideal of R and $z \in R$ such that $N \subseteq zR$, then $N = z(N : zR)$, and hence $N = zJ$ for some ideal J of R.

Next we prove that $M^2 = xM$ and xR is an OA-ideal of R. Since $x \notin M^2$ and M^2 is divided, we have that $M^2 \subseteq xR \subseteq M$. Therefore, xR is an OA-ideal by Lemma 2.1(2). Since $M^2 \subset xR$, there is some proper ideal J of R with $M^2 = xJ$, and thus $M^2 \subseteq xM$. Obviously, $xM \subseteq M^2$, and hence $M^2 = xM$.

Now we show that R is an OAF-ring. Let I be a proper ideal of R. First let $I = \mathbf{0}$. If M is nilpotent, then I is obviously a product of OA-ideals. If R is an integral domain, then I is an OA-ideal. Now let I be nonzero. In any case there is a largest positive integer n such that $I \subseteq M^n$. Observe that $I \subseteq M^n = x^{n-1}M \subseteq x^{n-1}R$. Consequently, $I = x^{n-1}L = (xR)^{n-1}L$ for some proper ideal L of R. Assume that $L \subseteq M^2$. Note that $L \subseteq M^2 = xM \subseteq xR$. This implies that $L = xA$ for some proper ideal A of R, and hence $I = x^n A \subseteq x^n M = M^{n+1}$, a contradiction. We infer that $M^2 \subseteq L$

(since M^2 is divided). It follows from Lemma 2.1(2) that L is an OA-ideal. In any case, I is a product of OA-ideals.

Finally, we prove that every proper principal ideal of R is a product of principal OA-ideals. Let $y \in M$. First let $y = 0$. If M is nilpotent, then $x^k = 0$ for some $k \in \mathbb{N}$, and thus $yR = (xR)^k$ is a product of principal OA-ideals. If R is an integral domain, then yR is a principal OA-ideal. Now let y be nonzero. There is some greatest $\ell \in \mathbb{N}$ such that $y \in M^\ell$. Therefore, $y = x^{\ell-1}z$ for some $z \in M$. If $z \in M^2$, then $z = xv$ for some $v \in M$, and hence $y = x^\ell v \in M^{\ell+1}$, a contradiction. We infer that $z \notin M^2$, and thus $M^2 \subseteq zR \subseteq M$. It follows from Lemma 2.1(2) that zR is an OA-ideal of R. Consequently, $yR = (xR)^{\ell-1}(zR)$ is a product of principal OA-ideals. □

4. Characterization of OAF-rings and related concepts

First we recall several definitions and discuss the factorization theoretical properties of local one-dimensional OAF-domains. Let D be an integral domain with quotient field K. Then $\widehat{D} = \{x \in K \mid$ there is some nonzero $c \in D$ such that $cx^n \in D$ for all $n \in \mathbb{N}\}$ is called the *complete integral closure* of D. Let $(D : \widehat{D}) = \{x \in D \mid x\widehat{D} \subseteq D\}$ be the *conductor* of D in $\widehat{D}$. The domain D is called *completely integrally closed* if $D = \widehat{D}$ and D is said to be *seminormal* if for all $x \in K$ such that $x^2, x^3 \in D$, it follows that $x \in D$. Note that every completely integrally closed domain is seminormal. We say that D is a *finitely primary domain of rank one* if D is a local one-dimensional domain such that $\widehat{D}$ is a DVR and $(D : \widehat{D}) \neq \mathbf{0}$. For each subset $X \subseteq K$ let $X^{-1} = \{x \in K \mid xX \subseteq D\}$ and $X_v = (X^{-1})^{-1}$. An ideal I of D is called *divisorial* if $I_v = I$. Moreover, D is called a *Mori domain* if D satisfies the ascending chain condition on divisorial ideals. It is well known that every unique factorization domain and every Noetherian domain is a Mori domain (see [14, Corollary 2.3.13] and [11, page 57]). We say that D is *half-factorial* if D is atomic and each two factorizations of each nonzero element of D into irreducible elements are of the same length. Finally, D is called a *C-domain* if the monoid of nonzero elements of D (i.e., $D \setminus \mathbf{0}$) is a C-monoid. For the precise definition of C-monoids we refer to [14, Definition 2.9.5].

Let D be a local domain with quotient field K and maximal ideal M. Set $(M : M) = \{x \in K \mid xM \subseteq M\}$. Then $(M : M)$ is called the *ring of multipliers* of M. Moreover, M^2 is said to be *universal* if $M^2 \subseteq uD$ for each irreducible element $u \in D$.

Theorem 4.1. *Let D be a local domain with maximal ideal M such that D is not a field. The following statements are equivalent.*

1. *D is an OAF-domain.*
2. *D is a TAF-domain.*
3. *D is one-dimensional and every proper principal ideal has an OA-factorization.*
4. *D is one-dimensional and atomic and every irreducible element generates an OA-ideal.*
5. *D is atomic such that M^2 is universal.*
6. *$(M : M)$ is a DVR with maximal ideal M.*
7. *D is a seminormal finitely primary domain of rank one.*

If these equivalent conditions are satisfied, then D is a half-factorial C-domain and a Mori domain.

Proof. (1) ⇒ (2): This follows from Lemma 2.1(3).

(1) ⇒ (3): By Theorem 3.5, D is one-dimensional. The rest of assertion (3) is clear.

(2) ⇔ (5) ⇔ (6): This follows from [23, Theorem 4.3].

(3) ⇔ (4): This is an immediate consequence of Lemma 3.6.

(4) ⇒ (5): Let $y \in D$ be an irreducible element. Since yD is an OA-ideal and $\sqrt{yD} = M$, we deduce from Lemma 2.1(2) that $M^2 \subseteq yD$. Hence M^2 is universal.

(5)+(6) ⇒ (1): It follows from [6, Theorem 5.1] that M^2 is comparable to every principal ideal of D, and thus M^2 is divided. Since $(M : M)$ is a DVR with maximal ideal M, we have that $\bigcap_{n\in\mathbb{N}} M^n = \mathbf{0}$. Consequently, D is an OAF-domain by Lemma 3.7.

(5)+(6) ⇒ (7): First we show that D is finitely primary of rank one. Let P be a nonzero prime ideal of D. Then P contains an irreducible element $y \in D$, and hence $M^2 \subseteq yD \subseteq P$. Therefore, $P = M$, and thus D is one-dimensional. It remains to show that $\widehat{D}$ is a DVR and $(D : \widehat{D}) \neq \mathbf{0}$. Since $(M : M)$ is a DVR, we have that $(M : M)$ is completely integrally closed. Observe that $D \subseteq (M : M) \subseteq \widehat{D}$, and hence $\widehat{D} \subseteq \widehat{(M : M)} = (M : M)$. Therefore, $\widehat{D} = (M : M)$ is a DVR. Since $M\widehat{D} = M(M : M) \subseteq M \subseteq D$ and $M \neq \mathbf{0}$, we infer that $(D : \widehat{D}) \neq \mathbf{0}$.

Next we show that D is seminormal. Let V be the group of units of $\widehat{D}$. Let K be the field of quotients of D and let $x \in K$ be such that $x^2, x^3 \in D$. Then $x^2, x^3 \in \widehat{D}$. Since $\widehat{D}$ is a DVR, $\widehat{D}$ is seminormal, and thus $x \in \widehat{D}$. In particular, $x \in M$ or $x \in V$. If $x \in M$, then $x \in D$. Now let $x \in V$. Note that $V \cap D$ is the group of units of D (by [24, Corollary 1.4] and [12, Proposition 2.1]), and thus x^2 and x^3 are units of D. Therefore, $x = x^{-2}x^3$ is a unit of D, and hence $x \in D$.

(7) ⇒ (6): By [15, Lemma 3.3.3], we have that M is the maximal ideal of $\widehat{D}$. If $x \in \widehat{D}$, then $xM \subseteq M$ (since M is an ideal of $\widehat{D}$). It is straightforward to show that $(M : M) \subseteq \widehat{D}$. We infer that $(M : M) = \widehat{D}$ is a DVR.

Now let the equivalent statements of Theorem 4.1 be satisfied. It remains to show that D is a half-factorial C-domain and a Mori domain. It follows from [6, Theorem 6.2] that D is a half-factorial domain. Obviously, V is a subgroup of finite index of V and $VM \subseteq \widehat{D}M = (M : M)M \subseteq M$. It follows from [18, Corollary 2.8] and [14, Corollary 2.9.8] that D is a C-domain. Moreover, D is a Mori domain by [18, Proposition 2.5.1]. □

We want to point out that a local one-dimensional OAF-domain need not be Noetherian. Let $K \subseteq L$ be a field extension such that $[L : K] = \infty$ and let $D = K + XL[\![X]\!]$. Then D is a local one-dimensional domain with maximal ideal $M = XL[\![X]\!]$ and $(M : M) = L[\![X]\!]$ is a DVR with maximal ideal M. Consequently, D is an OAF-domain by Theorem 4.1. Since $[L : K] = \infty$, it follows that D is not Noetherian.

An integral domain D is called a *Cohen-Kaplansky domain* if D is atomic and D has only finitely many irreducible elements up to associates. It follows from [6, Example 6.7] that there exists a local half-factorial Cohen-Kaplansky domain with maximal ideal M for which M^2 is not universal. We infer by Theorem 4.1 that the aforementioned domain is not an OAF-domain.

Theorem 4.2. *Let R be a ring with Jacobson radical M. The following statements are equivalent.*

(1) *R is an OAF-ring.*
(2) *Each proper 2-generated ideal of R has an OA-factorization.*
(3) $\dim(R) \leq 1$ *and each proper principal ideal has an OA-factorization.*
(4) *R satisfies one of the following conditions.*
 (A) *R is a general ZPI-ring.*
 (B) *R is a local domain, M^2 is divided and $\bigcap_{n\in\mathbb{N}} M^n = \mathbf{0}$.*
 (C) *R is local, M^2 is divided and M is nilpotent.*

Proof. (1) $\Rightarrow$ (2): This is obvious.

(2) $\Rightarrow$ (3): This is an immediate consequence of Theorem 3.5.

(3) $\Rightarrow$ (4): First let each OA-ideal of R be a prime ideal. Then R is a π-ring. By [16, Theorems 39.2, 46.7, and 46.11], R is a general ZPI-ring. Now let there be an OA-ideal of R which is not a prime ideal. It follows from Lemma 2.1 that R is local with maximal ideal M and M is not idempotent. Note that if $x \in M \setminus M^2$, then xR cannot be a product of more than one OA-ideal, and hence xR is an OA-ideal.

Case 1: R is zero-dimensional. Let $x \in M \setminus M^2$. Then xR is an OA-ideal. We infer by Lemma 2.1(2) that $M^2 \subseteq xR$. Consequently, M^2 is divided. It follows from Lemma 2.1 that $M^2 \subseteq I$ for each OA-ideal I of R. Since $\mathbf{0}$ is a product of OA-ideals, we have that $\mathbf{0}$ contains a power of M. This implies that M is nilpotent.

Case 2: R is one-dimensional. It follows from Proposition 3.3 that R is an integral domain, and hence $\bigcap_{n\in\mathbb{N}} M^n = \mathbf{0}$ by Lemma 3.6. It remains to show that M^2 is divided. Let $x \in R \setminus M^2$. Without restriction let x be a nonunit. Then xR is an OA-ideal. By Lemma 2.1(2) we have that $M^2 \subseteq xR$.

(4) $\Rightarrow$ (1): Clearly, every general ZPI-ring is an OAF-ring. The rest follows from Lemma 3.7. □

Corollary 4.3. *Let R be a ring with Jacobson radical M. The following statements are equivalent.*

(1) *Each proper principal ideal of R has an OA-factorization.*
(2) *R is a π-ring or an OAF-ring.*
(3) *R satisfies one of the following conditions.*
 (A) *R is a π-ring.*
 (B) *R is a local domain, M^2 is divided and $\bigcap_{n\in\mathbb{N}} M^n = \mathbf{0}$.*
 (C) *R is local, M^2 is divided and M is nilpotent.*

Proof. (1) $\Rightarrow$ (2): If R is not local, then R is a π-ring by Remark 2.4(2). Now let R be local. If $\dim(R) \geq 2$, then R is a unique factorization domain by Proposition 3.3, and hence R is a π-ring. If $\dim(R) \leq 1$, then R is an OAF-ring by Theorem 4.2.

(2) ⇒ (1): This is obvious.

(2) ⇔ (3): This is an immediate consequence of Theorem 4.2 and the fact that every general ZPI-ring is a π-ring. □

Corollary 4.4. *Let R be a ring with Jacobson radical M. The following statements are equivalent.*

(1) *Each proper principal ideal of R is a product of principal OA-ideals.*
(2) *R satisfies one of the following conditions.*
  (A) *R is a unique factorization ring.*
  (B) *R is a local domain, M^2 is divided and $\bigcap_{n\in\mathbb{N}} M^n = \mathbf{0}$.*
  (C) *R is local, M^2 is divided and M is nilpotent.*

Proof. (1) ⇒ (2): If R is not local, then R is a unique factorization ring by Remark 2.4(3). If R is local, then the statement follows from Corollary 4.3 and the fact that every local π-ring is a unique factorization ring ([4, Corollary 2.2]).

(2) ⇒ (1): Obviously, if R is a unique factorization ring, then each proper principal ideal of R is a product of principal OA-ideals. The rest is an immediate consequence of Lemma 3.7. □

In Lemma 2.1, we saw that if R is a local ring with maximal ideal M and I is an ideal of R such that $M^2 \subseteq I$, then I is an OA-ideal of R. Now we will give a characterization of the rings for which every proper (principal) ideal is an OA-ideal.

Proposition 4.5. *Let R be a ring with Jacobson radical M. The following statements are equivalent.*

(1) *Every proper ideal of R is an OA-ideal.*
(2) *Every proper principal ideal of R is an OA-ideal.*
(3) *R is local and $M^2 = \mathbf{0}$.*

Proof. (1) ⇒ (2): This is obvious.

(2) ⇒ (3): Assume that R is not local. Then every proper principal ideal of R is a prime ideal by Lemma 2.1(1). Consequently, R is an integral domain. If $x \in R$ is a nonunit, then x^2R is a prime ideal, and hence $x^2R = xR$ and $x = 0$. Therefore, R is a field, a contradiction. This implies that R is local with maximal ideal M. We infer by Lemma 2.1(2) that $\mathbf{0}$ is a prime ideal or $M^2 = \mathbf{0}$.

Assume that $M^2 \neq \mathbf{0}$. Then R is an integral domain and there is some nonzero $x \in M^2$. It follow from Lemma 2.1(2) that x^2R is a prime ideal or $M^2 \subseteq x^2R$. If x^2R is a prime ideal, then $x^2R = xR$. If $M^2 \subseteq x^2R$, then $M^2 \subseteq x^2R \subseteq xR \subseteq M^2$, and thus $x^2R = xR$. In any case we have that $x^2R = xR$, and hence x is a unit (since x is regular), a contradiction.

(3) ⇒ (1): This is an immediate consequence of Lemma 2.1(2). □

5. OA-factorization properties and trivial ring extensions

Let A be a ring and E be an A-module. Then $A \propto E$, the *trivial* (*ring*) *extension of A by E*, is the ring whose additive structure is that of the external direct sum $A \oplus E$ and whose multiplication is defined by $(a, e)(b, f) = (ab, af + be)$ for all $a, b \in A$ and all $e, f \in E$. (This construction is also known by other terminology and other notation, such as the *idealization* $A(+)E$.) The basic properties of trivial ring extensions are summarized in the textbooks [17, 19]. Trivial ring extensions have been studied or generalized extensively, often because of their usefulness in constructing new classes of examples of rings satisfying various properties (cf. [7, 10, 20]). We say that E is *divisible* if $E = aE$ for each regular element $a \in A$.

We start with the following lemma.

Lemma 5.1. *Let A be a ring, I be an ideal of A and E be an A-module. Let $R = A \propto E$ be the trivial ring extension of A by E.*

1. *$I \propto E$ is an OA-ideal of R if and only if I is an OA-ideal of A.*
2. *Assume that A contains a nonunit regular element and E is a divisible A-module. Then the OA-ideals of R have the form $L \propto E$ where L is an OA-ideal of A.*

Proof. (1) This follows immediately from [25, Theorem 2.20].

(2) Let J be an OA-ideal of R. Our aim is to show that $\mathbf{0} \propto E \subseteq J$. Let $e \in E$ and let $a \in A$ be a nonunit regular element. Then $e = af$ for some $f \in E$ and thus $(a, 0)(0, f)(0, e) = (0, 0) \in J$. Since J is an OA-ideal, we conclude that $(a, 0)(0, f) = (0, e) \in J$ or $(0, e) \in J$ which implies that $\mathbf{0} \propto E \subseteq J$. Therefore, $J = L \propto E$ with $L = \{b \in A \mid (b, g) \in J$ for some $g \in E\}$ and L is an ideal of A by [7, Theorems 3.1 and 3.3(1)]. Now the result follows from (1). □

Corollary 5.2. *Let A be an integral domain that is not a field, E be a divisible A-module and $R = A \propto E$. Then the OA-ideals of R have the form $I \propto E$ where I is an OA-ideal of A.*

Next, we study the transfer of the OAF-ring property to the trivial ring extension.

Theorem 5.3. *Let A be a ring with Jacobson radical M, E be an A-module and $R = A \propto E$.*

1. *R is an OAF-ring if and only if one of the following conditions is satisfied.*
   - (A) *A is a general ZPI-ring, E is cyclic and the annihilator of E is a (possibly empty) product of idempotent maximal ideals of A.*
   - (B) *A is local, M^2 is divided, $E = \mathbf{0}$ and either M is nilpotent or A is a domain with $\bigcap_{n \in \mathbb{N}} M^n = \mathbf{0}$.*
   - (C) *A is local, $M^2 = \mathbf{0}$, $ME = aE$ for each nonzero $a \in M$ and $ME = Mx$ for each $x \in E \setminus ME$.*

   In particular, if R is an OAF-ring, then A is an OAF-ring.
2. *Every proper ideal of R is an OA-ideal if and only if A is local, $M^2 = \mathbf{0}$ and $ME = \mathbf{0}$.*

Proof. (1) ($\Rightarrow$) First let R be an OAF-ring. By Theorem 4.2, it follows that (a) R is a general ZPI-ring or (b) R is local with maximal ideal N, N^2 is divided and (N is nilpotent or R is a domain such that $\bigcap_{n \in \mathbb{N}} N^n = \mathbf{0}$). If R is a general ZPI-ring, then condition (A) is satisfied by [7, Theorem 4.10].

From now on let R be local with maximal ideal N such that N^2 is divided. Observe that A is local with maximal ideal M and $N = M \propto E$ by [7, Theorem 3.2(1)]. If R is a domain such that $\bigcap_{n\in\mathbb{N}} N^n = \mathbf{0}$, then $E = \mathbf{0}$ (for if $z \in E$ is nonzero, then $(0, z)$ is a nonzero zero-divisor of R), and hence $A \cong R$ is a domain, M^2 is divided and $\bigcap_{n\in\mathbb{N}} M^n = \mathbf{0}$.

Now let N be nilpotent. If $E = \mathbf{0}$, then $A \cong R$, and thus M^2 is divided and M is nilpotent. From now on let E be nonzero. There is some $k \in \mathbb{N}$ such that $N^k = \mathbf{0}$. Note that $N^2 = M^2 \propto ME$ and $N^k = M^k \propto M^{k-1}E$, and thus $M^k = \mathbf{0}$. Since N^2 is divided, we have that $\mathbf{0} \propto E \subseteq N^2$ or $N^2 \subseteq \mathbf{0} \propto E$. If $\mathbf{0} \propto E \subseteq N^2$, then $E = ME$, and hence $E = M^kE = \mathbf{0}$, a contradiction. Therefore, $N^2 \subseteq \mathbf{0} \propto E$, which implies that $M^2 = \mathbf{0}$.

Let $a \in M$ be nonzero. Then $(a, 0) \notin N^2$, and hence $N^2 \subseteq (a, 0)R = aA \propto aE$. Consequently, $ME \subseteq aE$, and thus $ME = aE$. Finally, let $x \in E \setminus ME$. Then $(0, x) \notin N^2$. We infer that $N^2 \subseteq (0, x)R = \mathbf{0} \propto Ax$. This implies that $ME \subseteq Ax$. If $ME \nsubseteq Mx$, then $bx \in ME$ for some unit $b \in A$, and hence $x \in ME$, a contradiction. It follows that $ME \subseteq Mx$, which clearly implies that $ME = Mx$.

($\Leftarrow$) Next we prove the converse. If condition (A) is satisfied, then R is a general ZPI-ring by [7, Theorem 4.10], and thus R is an OAF-ring. If condition (B) is satisfied, then A is an OAF-ring by Theorem 4.2, and hence $R \cong A$ is an OAF-ring. Now let condition (C) be satisfied. Set $N = M \propto E$. Then R is local with maximal ideal N by [7, Theorem 3.2(1)]. By Theorem 4.2, it suffices to show that N is nilpotent and N^2 is divided. Since $M^2 = \mathbf{0}$, we obtain that $N^3 = M^3 \propto M^2E = \mathbf{0}$, and thus N is nilpotent. It remains to show that $N^2 \subseteq (a, x)R$ for each $(a, x) \in R \setminus N^2$. Let $a \in A$ and $x \in E$ be such that $(a, x) \notin N^2$. Since $N^2 = \mathbf{0} \propto ME$, we have to show that $\mathbf{0} \propto ME \subseteq (a, x)R$. If a is a unit of A, then (a, x) is a unit of R by [7, Theorem 3.7] and the statement is clearly true. Let $z \in \mathbf{0} \propto ME$. Then $z = (0, y)$ for some $y \in ME$.

Case 1: a is a nonzero nonunit. Since $ME = aE$, there is some $v \in E$ such that $y = av$. Observe that $z = (0, av) = (a, x)(0, v) \in (a, x)R$.

Case 2: $a = 0$. Then $x \in E \setminus ME$ (since $(a, x) \notin N^2$). Since $ME = Mx$, there is some $b \in M$ such that $y = bx$. It follows that $z = (0, bx) = (a, x)(b, 0) \in (a, x)R$.

The in particular statement now follows from Theorem 4.2.

(2) First let every proper ideal of R be an OA-ideal. By Proposition 4.5, we have that R is local with maximal ideal N and $N^2 = \mathbf{0}$. It follows that A is local with maximal ideal M and $N = M \propto E$ by [7, Theorem 3.2(1)]. Moreover, $\mathbf{0} = N^2 = M^2 \propto ME$, and hence $M^2 = \mathbf{0}$ and $ME = \mathbf{0}$.

Conversely, let A be local, $M^2 = \mathbf{0}$ and $ME = \mathbf{0}$. Set $N = M \propto E$. Then R is local with maximal ideal N by [7, Theorem 3.2(1)] and $N^2 = M^2 \propto ME = \mathbf{0}$. We infer by Proposition 4.5 that each proper ideal of R is an OA-ideal. □

Corollary 5.4. *Let A be an integral domain, E be a nonzero A-module and $R = A \propto E$. The following statements are equivalent.*

(1) *R is an OAF-ring.*
(2) *A is a field.*
(3) *Every proper ideal of R is an OA-ideal.*

Proof. (1) ⇒ (2): It follows from Theorem 5.3(1) that A is a general ZPI-ring and the annihilator of E is a product of idempotent maximal ideals of A or that A is local with maximal ideal M such that $M^2 = \mathbf{0}$.

First let A be a general ZPI-ring such that the annihilator of E is a product of idempotent maximal ideals of A. Note that A is a Dedekind domain, and thus the only proper idempotent ideal of A is the zero ideal. Since E is nonzero, the annihilator of E is a proper ideal of A, and hence A must possess an idempotent maximal ideal. We infer that the zero ideal is a maximal ideal of A, and thus A is a field.

Now let A be local with maximal ideal M such that $M^2 = \mathbf{0}$. Since A is an integral domain, it follows that $M = \mathbf{0}$, and hence A is a field.

(2) ⇒ (3): Set $M = \mathbf{0}$. Then A is local with maximal ideal M, $M^2 = \mathbf{0}$ and $ME = \mathbf{0}$. Now the statement follows from Theorem 5.3(2).

(3) ⇒ (1): This is obvious. □

Remark 5.5. In general, if A is an OAF-ring and E is an A-module, then $A \propto E$ need not be an OAF-ring. Indeed, let A be an OAF-domain that is not a field and let E be a nonzero A-module. By Corollary 5.4, $A \propto E$ is not an OAF-ring.

Corollary 5.6. *Let A be a local ring with maximal ideal M and E be a nonzero A-module such that $ME = \mathbf{0}$. Set $R = A \propto E$. The following statements are equivalent.*

1. *R is an OAF-ring.*
2. *$M^2 = \mathbf{0}$.*
3. *Every proper ideal of R is an OA-ideal.*

Proof. (1) ⇒ (2): Assume that $M^2 \neq \mathbf{0}$. By Theorem 5.3(1), A is a local general ZPI-ring and M is idempotent (since the annihilator of E is a nonempty product of idempotent maximal ideals of A and M is the only maximal ideal of A). We infer by [21, Corollary 9.11] that A is a Dedekind domain or each proper ideal of A is a power of M (because local rings are indecomposable). If A is a Dedekind domain, then clearly $M^2 = M = \mathbf{0}$ (since M is idempotent and a Dedekind domain has no nonzero proper idempotent ideals). Moreover, if every proper ideal of A is a power of M, then again $M^2 = M = \mathbf{0}$ (since M is idempotent). In any case, we obtain that $M^2 = \mathbf{0}$, a contradiction.

(1) ⇐ (2) ⇔ (3): This follows from Theorem 5.3. □

Example 5.7. Let A be a local principal ideal ring with maximal ideal M such that A is not a field and $M^2 = \mathbf{0}$ (e.g. $A = \mathbb{Z}/4\mathbb{Z}$). Set $R = A \propto A$. Then R is an OAF-ring, and yet not every proper ideal of R is an OA-ideal.

Proof. Since $M \neq \mathbf{0}$, it follows from Theorem 5.3(2) that not every proper ideal of R is an OA-ideal. By Theorem 5.3(1) it remains to show that $M = aA$ for each nonzero $a \in M$ and $M = Mx$ for each $x \in A \setminus M$. Note that $M = zA$ for some $z \in M$. If $a \in M$ is nonzero, then $a = zb$ for some $b \in A$. Clearly, $b \notin M$, and thus b is a unit of A, which clearly implies that $M = zA = aA$. Finally, if $x \in A \setminus M$, then x is a unit of A, and thus $M = Mx$. □

Remark 5.8. Let A be a ring with Jacobson radical M, E be an A-module and $R = A \propto E$. Then each proper principal ideal of R has an OA-factorization if and only if one of the following conditions is satisfied.

(1) A is a π-ring, E is cyclic and the annihilator of E is a (possibly empty) product of idempotent maximal ideals of A.
(2) A is local, M^2 is divided, $E = \mathbf{0}$ and either M is nilpotent or A is a domain with $\bigcap_{n\in\mathbb{N}} M^n = \mathbf{0}$.
(3) A is local, $M^2 = \mathbf{0}$, $ME = aE$ for each nonzero $a \in M$ and $ME = Mx$ for each $x \in E \setminus ME$.

Proof. This can be proved along similar lines as Theorem 5.3(1) by using Corollary 4.3 and [7, Theorems 3.2(1) and 4.10]. □

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