A COUNTEREXAMPLE TO THE CONJECTURE OF ANKENY, ARTIN AND CHOWLA

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ABSTRACT. Let p be a prime number with $p \equiv 1 \mod 4$, let $\varepsilon > 1$ be the fundamental unit of $\mathbb{Z}[\frac{1+\sqrt{p}}{2}]$ and let x and y be the unique nonnegative integers with $\varepsilon = x + y\frac{1+\sqrt{p}}{2}$. The Ankeny-Artin-Chowla-Conjecture states that p is not a divisor of y. In this note, we provide and discuss a counterexample to this conjecture.

1. Introduction

Let $\mathbb{P}$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$ denote the sets of prime numbers, positive integers, nonnegative integers, integers and rational numbers, respectively. Let $f \in \mathbb{N}$. Then f is called *squarefree* if $p^2 \nmid f$ for each $p \in \mathbb{P}$.

For each $r, s, t \in \mathbb{N}_0$, let $[r, s] = \{z \in \mathbb{N}_0 : r \le z \le s\}$ and $\mathbb{N}_{\ge t} = \{z \in \mathbb{N}_0 : z \ge t\}$. Let $d \in \mathbb{N}_{\ge 2}$ be squarefree, let $K = \mathbb{Q}(\sqrt{d})$ and let $\mathcal{O}_K$ be the ring of algebraic integers of K. We set

$$\omega = \begin{cases} \sqrt{d} & \text{if } d \equiv 2,3 \mod 4, \\ \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \mod 4, \end{cases} \quad \textit{and} \quad \mathsf{d}_K = \begin{cases} 4d & \text{if } d \equiv 2,3 \mod 4, \\ d & \text{if } d \equiv 1 \mod 4. \end{cases}$$

It is well-known that $\mathcal{O}_K = \mathbb{Z}[\omega] = \mathbb{Z} \oplus \omega\mathbb{Z}$. Let $\varepsilon \in \mathcal{O}_K$ be the fundamental unit with $\varepsilon > 1$ (i.e., $\varepsilon \in \mathcal{O}_K$ is such that $\varepsilon > 1$ and $\{\pm\varepsilon^k : k \in \mathbb{Z}\}$ is the unit group of $\mathcal{O}_K$). Note that there are unique $x, y \in \mathbb{N}_0$ such that $\varepsilon = x + y\omega$, and if $d \neq 5$, then $x, y \in \mathbb{N}$. If not stated otherwise, then from now on let ε be the fundamental unit of $\mathcal{O}_K$ and let $x, y \in \mathbb{N}_0$ be such that $\varepsilon = x + y\omega$.

Let $\mathrm{N} : K \to \mathbb{Q}$ defined by $\mathrm{N}(a + b\sqrt{d}) = a^2 - db^2$ for all $a, b \in \mathbb{Q}$ be the norm map on K. We say that a subring $\mathcal{O}$ of K with quotient field K is an *order* in K if it is a finitely generated abelian group with respect to addition. For each $f \in \mathbb{N}$, let $\mathcal{O}_f = \mathbb{Z} + f\mathcal{O}_K$ and observe that $\mathcal{O}_f$ is the unique order in K with conductor f (i.e., $\{z \in \mathcal{O}_f : z\mathcal{O}_K \subseteq \mathcal{O}_f\} = f\mathcal{O}_K$). Let $\mathrm{Pic}(\mathcal{O})$ denote the Picard group of $\mathcal{O}$ and let $\mathcal{O}^\times$ denote the unit group of $\mathcal{O}$ for each order $\mathcal{O}$ in K. Let $\mathrm{h}(d) = |\mathrm{Pic}(\mathcal{O}_K)|$ be the class number of K. For all $a, b \in \mathbb{Z}$, let $\left(\frac{a}{b}\right) \in \{-1, 0, 1\}$ be the Kronecker symbol of a modulo b. If $p \in \mathbb{P}$, then p is called *inert*, *ramified*, *split* (in $\mathcal{O}_K$) if $\left(\frac{\mathsf{d}_K}{p}\right) = -1$, $\left(\frac{\mathsf{d}_K}{p}\right) = 0$, $\left(\frac{\mathsf{d}_K}{p}\right) = 1$, respectively. This note revolves around the following conjecture.

Conjecture 1.1 (The Conjecture of Ankeny, Artin and Chowla, or AAC-Conjecture)**.** If $d \in \mathbb{P}$ and $d \equiv 1 \mod 4$, then $d \nmid y$.

The AAC-Conjecture goes back to a paper of N. C. Ankeny, E. Artin and S. Chowla in 1952 (see [2, page 480]) where the authors posed Conjecture 1.1 as a question. This question was later rephrased as a conjecture by A. A. Kiselev and I. Sh. Slavutskiĭ in 1959 [16] and independently by L. J. Mordell in 1960 [19]. The main problem that motivated this conjecture is the computation of the class number $\mathrm{h}(d)$

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in terms of rational numbers. For instance, it is known [29] that if $d \in \mathbb{P}$ with $d \equiv 1 \mod 4$ and $u, v \in \mathbb{N}$ are such that $\varepsilon = \frac{u+v\sqrt{d}}{2}$, then $\mathrm{h}(d)v \equiv uB_{(d-1)/2} \mod d$, where B_n is the nth Bernoulli number for each $n \in \mathbb{N}_0$. Also note that $v = y$, $\mathrm{h}(d) < d$ [29] and $B_n \in \mathbb{Q}$ for each $n \in \mathbb{N}_0$. If $d \nmid y$, then we can determine $\mathrm{h}(d)$ from the aforementioned congruence.

The Conjecture of Ankeny, Artin and Chowla has a long history and was investigated in a large number of research papers. For instance, see [1, 3, 4, 8, 9, 11, 12, 13, 16, 18, 19, 22, 23, 28, 30] (to mention a few) and the survey article by I. Sh. Slavutskiĭ [29]. The AAC-Conjecture was known to be true up to $2 \cdot 10^{11}$ (see the tables below) and for a long time it was believed to hold in general. Despite substantial progress towards a proof, there were also good reasons to believe that it is false. First doubts came up by a heuristic argument ([33, page 82]) which indicates that infinitely many counterexamples may exist. Moreover, the analogue of the AAC-Conjecture fails for fake real quadratic orders [32]. Our own doubts grew bigger after having obtained a counterexample to a related conjecture in [26, Example 3.1] and [27, Theorem 2.2] (see the discussion in Section 3). Thus, we started with an "AAC-Conjecture specific" computer search which ended up with Theorem 2.3. We end this section with a brief historic overview on the verification of the AAC-Conjecture.

Verification history for the AAC-Conjecture

Upper bound for d	Investigator(s)	Date
2000	Ankeny, Artin, Chowla [2]	1952
100000	Goldberg [19, 20]	1954
6270714	Beach, Williams, Zarnke [3]	1971
10^9	Stephens, Williams [30]	1988
10^{11}	van der Poorten, te Riele, Williams [22]	2001
$2 \cdot 10^{11}$	van der Poorten, te Riele, Williams [23]	2003
$1.5 \cdot 10^{12}$	Reinhart* [25]	2023
$5.325 \cdot 10^{13}$	Reinhart* [26]	2024

(*) Note that no independent double check of the search interval was done here.

2. Main theorem

Lemma 2.1. *Let $a, c \in \mathbb{N}$ be such that $a^2 > c > 1$, $a \mid c-1$, $2^{c-1} \equiv 1 \mod c$ and for each $p \in \mathbb{P}$ with $p \mid a$, $\gcd\left(2^{\frac{c-1}{p}} - 1, c\right) = 1$. Then $c \in \mathbb{P}$.*

Proof. This follows from [27, Proposition 2.1] (and is based on the more general results of [5] and [21]). □

Lemma 2.2. *Let $d \in \mathbb{P}$ and $\eta \in \mathcal{O}_d^\times$ be such that $1 < \eta < \varepsilon^d$. Then $d \mid y$.*

Proof. Obviously, $\eta \in \mathcal{O}_K^\times$. Since $1 < \eta < \varepsilon^d$, there is some $k \in \mathbb{N}$ such that $k < d$ and $\eta = \varepsilon^k$. Since $d \mid \mathsf{d}_K$, we have that d is ramified. It is a simple consequence of [10, Theorem 5.9.7.4] that $(\mathcal{O}_K^\times : \mathcal{O}_d^\times) \mid d$. Note that $\mathcal{O}_K^\times/\mathcal{O}_d^\times$ is a cyclic group generated by $\varepsilon\mathcal{O}_d^\times$, and thus $(\mathcal{O}_K^\times : \mathcal{O}_d^\times) = \min\{r \in \mathbb{N} : \varepsilon^r \in \mathcal{O}_d^\times\}$. Since $\varepsilon^k \in \mathcal{O}_d^\times$, we infer that $(\mathcal{O}_K^\times : \mathcal{O}_d^\times) \mid \gcd(k, d) = 1$, and hence $\varepsilon \in \mathcal{O}_K^\times = \mathcal{O}_d^\times$. This clearly implies that $d \mid y$. □

Next we present the main theorem of this note. We prove it with computer assistance (in several ways) by using Mathematica 13.2.0 and Pari/GP 2.15.2. Readers who are interested in a proof that may be checked without computer assistance, can study the proof in [27]. (A proof for Theorem 2.3 below can be obtained along similar lines.)

Theorem 2.3 (The counterexample to the Ankeny-Artin-Chowla-Conjecture)**.** *Let* $d = 331914313984493$*. Then* $d \equiv 1 \mod 4$*,* $d \in \mathbb{P}$ *and* $d \mid y$*.*

Proof. Clearly, since $93 = 4 \cdot 23 + 1$, we have that $d \equiv 93 \equiv 1 \mod 4$. To show that $d \in \mathbb{P}$ we can use PrimeQ[d] in Mathematica or isprime(d) in Pari/GP (since $d \leq 2^{64}$). A classical alternative is to test whether any $p \in \mathbb{P}$ with $p^2 \leq d$ (or even whether any $p \in \mathbb{N}$ with $1 < p^2 \leq d$) divides d. For instance, use

```
d=331914313984493;
forprime(p=2,floor(sqrt(d)),if(d%p==0,printp(p)))
```

to show that $d \in \mathbb{P}$ with Pari/GP.
Another way to prove that $d \in \mathbb{P}$ is based on Lemma 2.1. (For any $b, e \in \mathbb{N}$ we compute 2^b modulo e and $\gcd(b, e)$ with Mathematica or Pari/GP.) Observe that $\{5, 17, 31, 37\} \subseteq \mathbb{P}$ by the sieve of Eratosthenes.
Set $a_1 = 5 \cdot 17$ and $c_1 = 4591$. Then $a_1^2 > c_1 > 1$, $a_1 \mid c_1 - 1$, $2^{c_1-1} \equiv 1 \mod c_1$ and $\gcd(2^{(c_1-1)/5} - 1, c_1) = \gcd(2^{(c_1-1)/17} - 1, c_1) = 1$. Therefore, $4591 \in \mathbb{P}$ by Lemma 2.1.
Set $a_2 = 31 \cdot 37$ and $c_2 = 68821$. Then $a_2^2 > c_2 > 1$, $a_2 \mid c_2 - 1$ and $2^{c_2-1} \equiv 1 \mod c_2$. Furthermore, $\gcd(2^{(c_2-1)/31} - 1, c_2) = \gcd(2^{(c_2-1)/37} - 1, c_2) = 1$. Consequently, $68821 \in \mathbb{P}$ by Lemma 2.1.
Set $a_3 = 4591 \cdot 68821$ and $c_3 = 2242664283679$. Then $a_3^2 > c_3 > 1$, $a_3 \mid c_3 - 1$, $2^{c_3-1} \equiv 1 \mod c_3$ and $\gcd(2^{(c_3-1)/4591} - 1, c_3) = \gcd(2^{(c_3-1)/68821} - 1, c_3) = 1$. We infer by Lemma 2.1 that $2242664283679 \in \mathbb{P}$.
Finally, note that $c_3^2 > d > 1$, $c_3 \mid d - 1$, $2^{d-1} \equiv 1 \mod d$ and $\gcd(2^{(d-1)/c_3} - 1, d) = 1$. It follows from Lemma 2.1 that $d \in \mathbb{P}$.

It remains to show that $d \mid y$. This can be done in various ways. One way is to use "standard functions" in Mathematica or Pari/GP. In Mathematica, we can use

```
d=331914313984493;
Mod[2*NumberFieldFundamentalUnits[Sqrt[d]][[1]][[2]][[2]],d]
```

to check that the remainder of the division of y by d is 0. In Pari/GP we can write the lines

```
default(parisize,10^8);
d=331914313984493;
imag(quadunit(d))%d
```

to perform the same task. Another approach (to prove $d \mid y$) is to implement the continued fraction algorithm in Mathematica or Pari/GP. Thereby, we determine the continued fraction expansion of ω and compute the denominators of the convergents of ω modulo d. This is done until we reach the last entry of the period of the continued fraction expansion of ω. In Mathematica, we can write the lines

```
d=331914313984493;f[a_,b_,c_,h_]:={k=Floor[(h+b)/a];l=k*a-b;m=(c-l^2)/a;{m,l,k,c,h}}[[1]];
g[u_]:={x=1;y=0;v=Floor[Sqrt[u]];z=f[2,1,u,v];e=2*Floor[(1+v)/2]-1;While[z[[3]]!=e,{n=y,
      y=Mod[y*z[[3]]+x,u],x=n,z=f[z[[1]],z[[2]],z[[4]],z[[5]]]}];y}[[1]];
g[d]
```

and then check that the last value is 0. To do the same with Pari/GP, we can apply the commands

```
d=331914313984493;
f(a,b,c,h)=[k=floor((h+b)/a),l=k*a-b,m=(c-l^2)/a,[m,l,k,c,h]][4];
g(u)=[x=1,y=0,v=floor(sqrt(u)),z=f(2,1,u,v),e=2*floor((1+v)/2)-1,while(z[3]!=e,[n=y,
      y=(y*z[3]+x)%u,x=n,z=f(z[1],z[2],z[4],z[5])]),y][7];
g(d)
```

and then verify that the last value is 0.
Finally, we present a method to prove $d \mid y$ that does not rely on continued fraction expansions. The first step is to find $u, v \in \mathbb{N}$ such that $u < 10^{d-1}$, $v < 10^{d-16}$ and $u^2 - d^3v^2 = -4$. With Pari/GP this can be done, for instance, with the commands

```
default(parisize,10^8);
d=331914313984493;
u=2*real(quadunit(d))+imag(quadunit(d));v=imag(quadunit(d))/d;
```

Next we store the values u and v in a save file (or we print the values u and v on paper). Now we show that $d \mid y$ (without continued fraction expansions). (Note that it does not matter how we obtained u and v.) Let $u, v \in \mathbb{N}$ be from the save file/printout before. Check that $u < 10^{d-1}$, $v < 10^{d-16}$ and $u^2 - d^3v^2 = -4$. (We only require a computer to check the last equality.) Clearly, $u \equiv v \mod 2$ and $vd < 10^{d-1}$. Set $\eta = \frac{u+vd\sqrt{d}}{2}$. Since $u + vd$ and $u - vd$ are even, we have that $\eta = \frac{u-vd}{2} + vd\omega \in \mathcal{O}_d$ and $\frac{u-vd\sqrt{d}}{2} = \frac{u+vd}{2} - vd\omega \in \mathcal{O}_d$. Moreover, $\eta\frac{u-vd\sqrt{d}}{2} = -1$, and thus $\eta \in \mathcal{O}_d^\times$. Observe that $1 < \eta < 10^{d-1}(\frac{1+\sqrt{d}}{2}) \leq (\frac{1+\sqrt{d}}{2})^d \leq \varepsilon^d$. Therefore, $d \mid y$ by Lemma 2.2. □

For the computer search that lead to the discovery of $d = 331914313984493$, we implemented two algorithms that are discussed in [30]. They are called the *small step algorithm* and the *large step algorithm*. For more information on these algorithms, we refer to [30]. On the one hand, we used plain versions (i.e., versions without optimizations) of both the small step algorithm and the large step algorithm. On the other hand, we used more advanced versions of the large step algorithm (with many optimizations) including a vectorized version (that involves AVX-512 instructions). All programs for the search were written in C and they were compiled with either GCC-12.2.0 or GCC-12.3.0 (with the compiler flag -O3). We only used privately owned hardware to perform the computations below. Next we summarize all types of computations that we performed in the search for squarefree $d \in \mathbb{N}_{\geq 2}$ with $d \mid y$ so far.

- Search for all squarefree $d \in [2, 10^{10}]$ with the plain version of the small step algorithm in 50 hours with 16 cores.
- Search for all squarefree $d \in [2, 1.5 \cdot 10^{12}]$ with the plain version of the large step algorithm in 650 hours with 140 cores.
- Search for all squarefree $d \in [1.5 \cdot 10^{12}, 10^{14}]$ with the advanced version/AVX-512 version of the large step algorithm in 7500 hours with 162 cores.
- Search for all prime $d \in [10^{14}, 3.4 \cdot 10^{14}]$ with $d \equiv 1 \mod 4$ with the advanced version/AVX-512 version of the large step algorithm adapted for primes in 1200 hours with 162 cores.

3. Related problems and conjectures

Next we discuss the counterexample in Theorem 2.3. First we provide an analysis along the lines of [26, page 93].
Let $d \in \mathbb{N}_{\geq 2}$ be squarefree. The integer d is said *to satisfy* (RC) if $\{f \in \mathbb{N} : \mathrm{h}(d) = |\mathrm{Pic}(\mathcal{O}_f)|\} = \{1\}$ (i.e., $\mathcal{O}_K$ is the only order in K with relative class number 1). It was proved in [26, Proposition 2.2] (based on results of [6]) that d satisfies (RC) if and only if $\mathrm{N}(\varepsilon) = 1$, $d \not\equiv 1 \mod 8$, y is even and $d \mid y$. Let ε' be the smallest positive power of ε such that $\varepsilon' \in \mathbb{Z}[\sqrt{d}]$ and let $X, Y \in \mathbb{N}$ be such that $\varepsilon' = X + Y\sqrt{d}$. It is well-known [30] that $\varepsilon' \in \{\varepsilon, \varepsilon^3\}$. If $d \mid y$, then $d \mid Y$. If $d \mid Y$, then $d \mid 3y$. Let $\alpha \in \{0, 1\}$ be such that $y \equiv \alpha \mod 2$ and let $\beta \in [0, 7]$ be such that $d \equiv \beta \mod 8$. Set $s = |\{p \in \mathbb{P} : p \mid d\}|$. In what follows, we use the tables of [26] without further mention. The entries in the table below were computed with Mathematica and Pari/GP. To compute the class number $\mathrm{h}(d)$ (for the specific d below), we used the commands qfbclassno(d) and quadclassunit(d)[1] in Pari/GP and we used the command NumberFieldClassNumber[Sqrt[d]] in Mathematica. All three results matched. Recall that $\mathrm{N}(\varepsilon)$ denotes the norm of the fundamental unit of $\mathcal{O}_K$.

d	331914313984493
$d \mid Y$	true
$d \mid y$	true
(RC)	false
α	1
β	5
s	1
$\mathrm{N}(\varepsilon)$	-1
$\mathrm{h}(d)$	3

Observe that $\beta \in \{1, 2, 3, 5, 6, 7\}$. Let Ω be the set consisting of the 22 known values of squarefree $d \in \mathbb{N}_{\geq 2}$ for which $d \mid y$. Set $\Omega_\beta = \{x \in \Omega : x \equiv \beta \mod 8\}$. For each $\ell \in \mathbb{N}$, let $\mathrm{H}_\ell = \{x \in \Omega : \mathrm{h}(x) = \ell\}$.

β	1	2	3	5	6	7
$\lvert\Omega_\beta\rvert$	5	3	2	1	7	4

ℓ	1	2	3	4	8	16	32
$\lvert\mathrm{H}_\ell\rvert$	6	6	1	3	4	1	1

Note that the counterexample to the AAC-Conjecture above is remarkable in several aspects. It is the first known example of a squarefree $d \in \mathbb{N}_{\geq 2}$ such that $d \mid y$ and $d \equiv 5 \mod 8$. (That said, there is the known "near example" $d = 17451248829$ for which $d \equiv 5 \mod 8$, $d \mid 3y$ and $d \nmid y$.) Moreover, it is one of two known examples of a squarefree $d \in \mathbb{N}_{\geq 2}$ such that $d \mid y$ and $\mathrm{N}(\varepsilon) = -1$ (where $d = 5374184665$ is the other known example). Besides that, it is the only known example of a squarefree $d \in \mathbb{N}_{\geq 2}$ such that $d \mid y$ and such that the odd part of the class number $\mathrm{h}(d)$ is larger than 1. Last but not least, it is probably the smallest counterexample to the AAC-Conjecture. We want to emphasize, however, that the search interval $[10^{10}, 3.4 \cdot 10^{14}]$ has not been (independently) double checked. (For this reason, we do not want to claim that it is the smallest counterexample.) We end this note with a few remarks, questions and open problems. Next, we recall a closely related conjecture that was recently settled. For other variants and analogues of the AAC-Conjecture we refer to [14, 31, 34].

Mordell's Pellian Equation Conjecture. If $d \in \mathbb{P}$ and $d \equiv 3 \mod 4$, then $d \nmid y$.
This conjecture was first formulated (together with the AAC-Conjecture as the extended AAC-Conjecture) by A. A. Kiselev and I. Sh. Slavutskiĭ in 1959 (see [16]). It was later stated independently by L. J. Mordell in 1961 (see [20, page 283]). Mordell's Pellian Equation Conjecture was disproved in [26, 27].

The Ankeny-Artin-Chowla-Conjecture for pure cubic number fields. Let $p \in \mathbb{P}$ be such that $p \geq 5$, let $d \in \{p, 2p\}$, let $L = \mathbb{Q}(\sqrt[3]{d})$, let $\eta > 1$ be the fundamental unit of $\mathcal{O}_L$ (i.e., the ring of algebraic integers of L) and let $a, b, c \in \mathbb{N}$ be the unique positive integers such that $\eta = \frac{1}{3}(a + b\sqrt[3]{d} + c\sqrt[3]{d^2})$. Then $p \nmid b$.
This conjecture was introduced in [14] and is still open. In analogy to the results of [30], we want to provide the list of all squarefree $d \in [2, 10^7]$ with $d \mid b$ (where b is defined as above). Note that the first 8 values below are well-known (see Western Number Theory Problems, 17 to 19 Dec 2019, 019:03).

Remark 3.1. Let $d \in \mathbb{N}_{\geq 2}$ be squarefree, let $L = \mathbb{Q}(\sqrt[3]{d})$, let $\mathcal{O}_L$ be the ring of algebraic integers of L, let $\eta > 1$ be the fundamental unit of $\mathcal{O}_L$ and let $a, b, c \in \mathbb{N}$ be the unique positive integers with $\eta = \frac{1}{3}(a + b\sqrt[3]{d} + c\sqrt[3]{d^2})$. We have $\{d \in [2, 10^7] : d \text{ is squarefree and } d \mid b\} = \{d \in [2, 10^7] : d \text{ is squarefree and } 3d \mid b\} = \{3, 6, 15, 39, 42, 57, 330, 1185, 28131, 47019, 89411, 125265, 144147, 435498, 1688610, 4580214, 5123415\}$.
Note that 89411 is the only number in the set before that is not divisible by 3.

The values in Remark 3.1 were determined with Pari/GP by using the following commands (where the loop below was split among 162 cores/instances of Pari/GP).

```
default(parisize,4*10^8);
f(d)=(3*polcoeff(bnfinit(X^3-d,1).fu[1].pol,1))%d;
forsquarefree(x=2,10^7,if(f(x[1])==0,printp(x[1])))
```

Next we want to provide an independent check that $3d \mid b$ holds for each of the 17 numbers in Remark 3.1 with Pari/GP. In general, the function "g(d)" below determines the unique $a', b', c', a'', b'', c'' \in \mathbb{Q}$ such that $\eta = a' + b'\sqrt[3]{d} + c'\sqrt[3]{d^2}$ and $\eta^{-1} = a'' + b''\sqrt[3]{d} + c''\sqrt[3]{d^2}$. For an independent check, we put this function to the test and show by other means that $\eta = a' + b'\sqrt[3]{d} + c'\sqrt[3]{d^2}$. Let d be any of the 17 numbers and compute g(d). It is easy to verify that $a', b', c', a'', b'', c'' \in \mathbb{Z}$, and thus $a' + b'\sqrt[3]{d} + c'\sqrt[3]{d^2} \in \mathcal{O}_L$ and $a'' + b''\sqrt[3]{d} + c''\sqrt[3]{d^2} \in \mathcal{O}_L$. The function "w(d)" checks whether $(a' + b'\sqrt[3]{d} + c'\sqrt[3]{d^2})(a'' + b''\sqrt[3]{d} + c''\sqrt[3]{d^2}) = 1$. We verify this equality by computing w(d) and we infer that $a' + b'\sqrt[3]{d} + c'\sqrt[3]{d^2} \in \mathcal{O}_L^\times$. The function "fg(d)" now determines whether $a' + b'\sqrt[3]{d} + c'\sqrt[3]{d^2}$ is the fundamental unit > 1 of $\mathcal{O}_L$ by using an algorithm from [15]. After checking fg(d) (such that "parisize" and "realprecision" are high enough), we now get $\eta = a' + b'\sqrt[3]{d} + c'\sqrt[3]{d^2}$. Observe that $b = 3b'$. It is now straightforward to check that $d \mid b'$, and hence $3d \mid b$.

```
g(d)=[k=bnfinit(X^3-d,1).fu[1],l=k.pol,if(polcoeff(l,2)<0,k=-k),l=k.pol,
    if(polcoeff(l,0)<0||polcoeff(l,1)<0,k=k^-1),l=k.pol,if(polcoeff(l,2)<0,k=-k),
    l=k.pol,[polcoeff(l,0),polcoeff(l,1),polcoeff(l,2),polcoeff((k^-1).pol,0),
    polcoeff((k^-1).pol,1),polcoeff((k^-1).pol,2)]][9];
w(d)=[c=g(d),c[1]*c[4]+c[2]*c[6]*d+c[3]*c[5]*d==1&&c[1]*c[5]+c[2]*c[4]+c[3]*c[6]*d==0&&
    c[1]*c[6]+c[2]*c[5]+c[3]*c[4]==0][2];
fg(d)=[p=g(d),m=p[1]+p[2]*d^(1/3)+p[3]*d^(2/3),if(d%9==1||d%9==8,
    L=(1+d^(1/3)+d^(2/3))/3,L=1+d^(1/3)+d^(2/3)),q=floor(log(m)/log(L)),
    s=true,forprime(z=2,q,[n=m^(1/z),u=ceil(n-11/4*n^(-1/2)),
    v=floor(n+11/4*n^(-1/2)),for(z=u,v,[r=z*n-n^2+n^-1,
    s=s&&!(abs(z-n)<11/4*n^(-1/2)&&floor(r)==r)])]),s&&m>1][7];
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Despite the fact that the AAC-Conjecture and Mordell's Pellian Equation Conjecture have been refuted, there are still interesting open problems involving squarefree $d \in \mathbb{N}_{\geq 2}$ with $d \mid y$. In what follows, we want to mention a few of them. Let $f \in \mathbb{N}$. Then f is said to be *powerful* if for each $p \in \mathbb{P}$ with $p \mid f$, we have that $p^2 \mid f$. Clearly, f is powerful if and only if $f = a^2b^3$ for some $a, b \in \mathbb{N}$.

The Conjecture of Erdös, Mollin and Walsh or EMW-Conjecture. For each $a \in \mathbb{N}$, there is some $b \in \{a, a+1, a+2\}$ such that b is not powerful.

The EMW-Conjecture was first mentioned in [7] and has subsequently been rediscovered in [17]. It was shown in [17] that there is some $a \in \mathbb{N}$ such that a, $a+1$ and $a+2$ are powerful if and only if there are some $d, k, u, v \in \mathbb{N}$ such that d is squarefree, $d \equiv 7 \mod 8$, k and v are odd, u is powerful, $d \mid v$ and $\varepsilon^k = u + v\sqrt{d}$. Observe that the counterexample in Theorem 2.3 cannot be used to construct a counterexample to the EMW-Conjecture, since it is congruent 5 modulo 8. The EMW-Conjecture is still open (to the best of our knowledge). We end this paper with a brief discussion on the problem that motivated us (originally) to search for squarefree $d \in \mathbb{N}_{\geq 2}$ with $d \mid y$.

Sets of distances. Let L be an algebraic number field, let $\mathcal{O}$ be an order in L and let $\mathcal{O}^\times$ be the unit group of $\mathcal{O}$. A nonzero nonunit $u \in \mathcal{O}$ is called an *atom* of $\mathcal{O}$ if u is not a product of two nonunits of $\mathcal{O}$. Since $\mathcal{O}$ is Noetherian, we have that $\mathcal{O}$ is atomic (i.e., every nonzero nonunit of $\mathcal{O}$ is a finite product of atoms of $\mathcal{O}$). For each nonzero nonunit $a \in \mathcal{O}$, let $\mathsf{L}(a) = \{k \in \mathbb{N} : a \text{ is a product of } k \text{ atoms of } \mathcal{O}\}$, called the *set of lengths of* a and let $\Delta(a) = \{s - r : r, s \in \mathsf{L}(a), r < s, \mathsf{L}(a) \cap [r, s] = \{r, s\}$, called the *set of distances of* a. We set $\Delta(\mathcal{O}) = \bigcup_{b \in \mathcal{O} \setminus (\{0\} \cup \mathcal{O}^\times)} \Delta(b)$, called the *set of distances of* $\mathcal{O}$. We say that

$\mathcal{O}$ *half-factorial* if $\Delta(\mathcal{O}) = \emptyset$ and in this case we set $\min \Delta(\mathcal{O}) = 0$. The half-factorial orders have been characterized recently in [24]. It is natural to ask how large $\min \Delta(\mathcal{O})$ can get. If $\mathcal{O}$ is seminormal (i.e., for all $x \in L$ with $x^2, x^3 \in \mathcal{O}$, it follows that $x \in \mathcal{O}$), then $\min \Delta(\mathcal{O}) \leq 1$ (see [25]). Also if L is an imaginary quadratic number field, then $\min \Delta(\mathcal{O}) \leq 1$ (see [25]).
Now let $L = K$ be a real quadratic number field. Then $\min \Delta(\mathcal{O}) \leq 2$ (see [25]). We say that $\mathcal{O}$ is *unusual* if $\min \Delta(\mathcal{O}) = 2$. Unusual orders have been completely characterized in [25, Theorems 2.9 and 4.4] and it was shown in [25, Example 3.2] that unusual orders do exist. In some situations (see [26, Proposition 2.4 and Theorem 2.5]), it is possible to connect unusual orders to the condition $d \mid y$. But apart from $d = 5374184665$, we do not know if any of these situations can actually occur. This leads us to the following open problem.

Open problem. Find squarefree $d \in \mathbb{N}_{\geq 2}$ that satisfy one of the following conditions.

(a) There are $p, q \in \mathbb{P}$ such that $p \equiv 5 \mod 8$, $q \equiv 3 \mod 4$, $d = pq$, $\mathrm{h}(d) = 2$, y is odd and $d \mid y$.
(b) There are distinct $p, q \in \mathbb{P}$ such that $p \equiv q \equiv 1 \mod 4$, $d = pq \equiv 5 \mod 8$, $\mathrm{h}(d) = 2$, $\mathrm{N}(\varepsilon) = -1$ and $d \mid y$.
(c) There are $p, q \in \mathbb{P}$ such that $p \equiv 1 \mod 8$, $q \equiv 3 \mod 4$, $d = pq$, $\mathrm{h}(d) = 2$, y is odd and $d \mid y$.
(d) There are distinct $p, q \in \mathbb{P}$ such that $p \equiv q \equiv 3 \mod 8$, $d = 2pq$, $\mathrm{h}(d) = 2$ and $d \mid y$.
(e) There are $p, q \in \mathbb{P}$ such that $p \equiv 1 \mod 8$, $q \equiv 3 \mod 4$, $\left(\frac{p}{q}\right) = -1$, $d = 2pq$, $\mathrm{h}(d) = 2$ and $d \mid y$.

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