Available online at www.sciencedirect.com
SCIENCE DIRECT®
Physica D 217 (2006) 193–201
PHYSICA D
www.elsevier.com/locate/physd

Pumping effects in models of periodically forced flow configurations

Georg Propst*

Institut für Mathematik und Wissenschaftliches Rechnen, Karl-Franzens-Universität Graz, Heinrichstraße 36, A-8010 Graz, Austria

Received 18 October 2004; received in revised form 5 April 2006; accepted 6 April 2006
Available online 8 May 2006
Communicated by Y. Kuramoto

Abstract

A periodically forced system of differential equations is defined to be a pump, if there exists an asymptotically periodic solution with non-equilibrium mean. It is proved that such systems exist. The definition is based on physical and numerical observations of pumping in (models of) asymmetric flow configurations. For models with rigid pipes and tanks, physical explanations for the pumping effects are derived. One of the pumps is an internally forced linear system. For externally forced nonlinear rigid pipe models, necessary and sufficient conditions for pumping are given. It is then demonstrated in a general setting that no externally forced linear pump exists.

Keywords: Valveless pumping; Periodic forcing; Asymmetric pressure losses

1. Introduction

The term "valveless pumping" is used for the conveyance of liquid fluids in mechanical systems that have no valves to ensure the preferential direction of flow. We mention four examples. Blood circulation in the cardiovascular system is maintained to some extent even when the heart's valves fail [1]. Valveless circulation can be found in early stages of human embryonic life as well as in invertebrates [2]. Valveless pumping is of interest in applications of nanotechnology [3]. At large, the earth's oceans are a valveless system of moving fluid. In all examples the pumping is caused by periodic excitation.

In view of the physiological examples, mathematical models have been developed with 2-dimensional annular geometry of an elastic tube that contains the fluid [4,5]. This leads to systems of partial differential equations. In other distributed parameter models and simulations, two tanks are connected by an elastic tube [6–8]. Models with ordinary differential equations for rigid pipe–tank systems are considered in [8–10]. In all models the periodic excitation is applied at a location that is not symmetric with respect to the system's geometry or its material properties. In the case of symmetry there is no pumping. In the quoted modelling and simulation papers no physical explanation or analytical analysis of the pumping phenomena is given.

In the present article we consider configurations with tanks that are connected by rigid pipes. The models are systems of ordinary differential equations with quadratic terms for the pipe–tank junctions. We give necessary and sufficient conditions for pumping effects. Whether or not these conditions are met in a particular physical configuration depends on the specific laboratory setup. Our results contribute to the understanding of the above-mentioned phenomena of valveless pumping.

The aim is to give an explanation of the pumping phenomenon for systems with three tanks. We proceed gradually, starting with one tank and one pipe; the basic mechanism for pumping then obviously is the pressure loss due to acceleration at the pipe–tank junction. For a configuration with two tanks, the necessity of unequal pressure losses at the junctions becomes apparent. Finally the 3-tank model is interpreted as a combination of two 2-tank models (Section 2).

Physical explanations aside, another objective of the paper is a mathematical one. A major motivation to study these models is the interest in nonlinear dynamics. Apparently, pumping in nonlinear systems so far has not been investigated in terms of theoretical analysis. We state definitions of periodically forced pumps and, as a special case, of externally forced periodic pumps. Using the perturbation theory of noncritical systems

* Tel.: +43 316 380 5168; fax: +43 316 380 9815.
E-mail address: georg.propst@uni-graz.at.

0167-2789/$ - see front matter
doi:10.1016/j.physd.2006.04.007

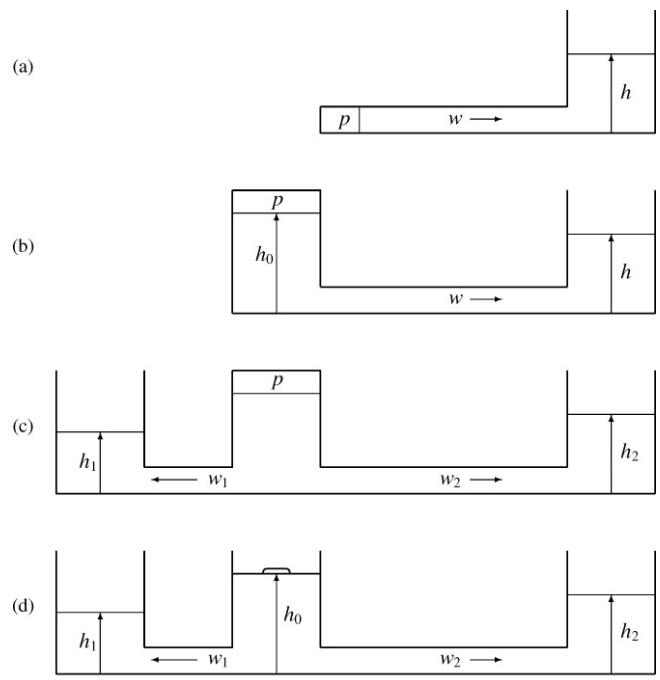

Fig. 1. Four different flow configurations with rigid pipes and tanks: the cross sections of the horizontal pipes are small in comparison to the cross sections of the vertical tanks. w, w_1, w_2 are flow velocities in the pipes, h, h_0, h_1, h_2 are level heights in the tanks. In (a), (b), (c) the pressure p outside a massless and frictionless piston is forced; in (d) the level height h_0 in the middle tank is forced.

in [11], we show that periodic solutions exist (Section 3). The rigid pipe models are externally forced periodic pumps, as long as physically reasonable relations among the model parameters are satisfied (Section 4). These relations exclude geometric or material symmetries. One of the models is transformed into a linear one with time dependent coefficient matrix. This is a contribution to the discussion whether or not there exist linear pumps, [6–8,12,13]. Furthermore, it is proven that there is no externally forced linear pump, not even with infinite-dimensional models.

2. The models

Our objective is to explain pumping effects in models with three tanks. We start with a 1-tank configuration and look at various versions of two tanks connected by one pipe. The 3-tank configuration will then be seen as a combination of two 2-tank configurations.

In the configuration of Fig. 1(a), a pipe of cross section A_p is closed by a frictionlessly moveable massless piston at distance ℓ from a tank of cross section A_t that is filled with fluid of density ρ up to the level height h. Let p be the pressure in the pipe outside of the piston; the environmental pressure above the fluid in the open tank is set to zero. The system is in equilibrium if $p \equiv \rho g h$, g being the gravitational acceleration.

Suppose that $A_t \gg A_p$ and that the forcing function p varies in time t. Neglecting the movement of the fluid in the tank, the equation of motion, without friction, is

$$\rho(\ell w)'(t) = p(t) - p^e(t) \tag{1}$$

where w is the velocity of the fluid in the direction from the piston to the tank, and p^e is the pressure at the end of the pipe at the entrance to the tank. Throughout we use the symbol $'$ for the derivative with respect to t. The level height h varies according to

$$h'(t) = cw(t), \tag{2}$$

where $c = A_p/A_t$ and $\ell(t)$ is coupled to $h(t)$ by $A_p\ell(t) + A_t h(t) \equiv V_0$; V_0 is the given constant total volume of fluid. In the configurations that we have in mind, the cross sections of the tanks are much larger than the cross sections of the pipes, and thus c is much smaller than 1.

Modelling the fluid in the tank to be at rest, one version of the difference of the hydrostatic pressure at the bottom of the tank and the pressure p^e in the flow at the entrance of the pipe is

$$\rho g h(t) - p^e(t) = \zeta^e \frac{\rho}{2} w(t)^2. \tag{3}$$

Here $\zeta^e \geq 1$ is a friction coefficient depending on the particular geometry and smoothness of the junction of the tank and the pipe. For an ideally smooth junction $\zeta^e = 1$, and (3) becomes Bernoulli's equation for perfect acceleration and deceleration of the fluid, cf. eg. [14], (2.33).

Suppose that $p(t)$ is periodic and that w, h is a periodic solution of (1) and (2) of the same period $T > 0$. The existence of such solutions will be shown in the next section. Integrating (1) from 0 to T and dividing by T, one obtains mean values on the right-hand side, and the left-hand side vanishes, because $\ell(t)w(t)$ is periodic with period T,

$$0 = \overline{p} - \rho g \overline{h} + \zeta^e \frac{\rho}{2} \overline{w^2}. \tag{4}$$

Here and throughout $^-$ denotes mean values of the periodic functions, e.g. $\overline{w^2} = \frac{1}{T}\int_0^T w(t)^2 \mathrm{d}t > 0$. Thus, the mean level height in the tank of the periodically forced system,

$$\overline{h} = \frac{1}{\rho g}\left(\overline{p} + \zeta^e \frac{\rho}{2}\overline{w^2}\right),$$

is larger than in the system with constant $p \equiv \overline{p}$. In other words, there is a pumping effect by periodic excitation. It is produced by the average pressure loss at the junction of the pipe and the tank that is modelled by (3).

Another common model, used for example in [7–9], for the pressure loss at the pipe–tank junction is

$$\rho g h(t) - p^e(t) = \zeta^e \frac{1 - \mathrm{sgn} w(t)}{2} \frac{\rho}{2} w(t)^2 \tag{5}$$

which only accounts for acceleration for flow from the tank into the pipe. One obtains

$$\overline{h} = \frac{1}{\rho g}\left(\overline{p} + \zeta^e \frac{\rho}{4}(\overline{w^2} - \widehat{w^2})\right)$$

where

$$\widehat{w^2} = \frac{1}{T}\int_0^T \mathrm{sgn} w(t) w(t)^2 \mathrm{d}t.$$

$\widehat{w^2}$ might be zero, positive or negative. For example for $w(t) = \sin t$, $\widehat{w^2}$ is zero, for $w(t) = \sin t^2$ (periodically extended outside $[0, \sqrt{2\pi}]$) it is positive. The difference $\overline{w^2} - \widehat{w^2}$ is always nonnegative. Again, the pumping effect corresponds to the mean of the pressure loss at the pipe–tank junction.

Next, consider two tanks that are connected by a pipe of fixed length ℓ, as in Fig. 1(b), p being the pressure above the fluid in the left tank with cross section A_0. The equation of motion for this configuration is

$$\rho \ell w'(t) = p^b(t) - p^e(t), \tag{6}$$

where the pressure p^e at the end of the pipe is given in (3) or (5), and

$$p^b(t) = p(t) + \rho g h_0(t) - \zeta^b \frac{\rho}{2} w(t)^2, \tag{7}$$

or

$$p^b(t) = p(t) + \rho g h_0(t) - \zeta^b \frac{1 + \mathrm{sgn} w(t)}{2} \frac{\rho}{2} w(t)^2, \tag{8}$$

are analogous models for the pressure at the beginning of the pipe. The evolution of h is coupled to w by (2) and $h_0(t)$ is given by $A_0 h_0(t) + A_t h(t) \equiv V_0$, where V_0 now is the total volume of fluid outside the always filled pipe.

The system is in equilibrium iff $p + \rho g h_0 \equiv \rho g h$. Suppose the forcing p is periodic and w, h is a periodic solution of the same period $T > 0$. The existence of such solutions is demonstrated in the next section. For simplicity of exposition let $\overline{p} = 0$. Then for the model (3) and (7), taking the mean in (6) yields

$$\overline{h} - \overline{h_0} = \frac{1}{2g}(\zeta^e - \zeta^b)\overline{w^2}. \tag{9}$$

If $\zeta^e \neq \zeta^b$, the pumping goes in the direction of lower pressure due to the larger pressure loss at one of the junctions. However, in the case $\zeta^e = \zeta^b$ there is no pumping effect.

In the model (5), (8) the situation is different because of the $\pm\mathrm{sgn} w$ terms. Taking the mean yields

$$\overline{h} - \overline{h_0} = \frac{1}{4g}(\zeta^e - \zeta^b)\overline{w^2} - \frac{1}{4g}(\zeta^e + \zeta^b)\widehat{w^2}. \tag{10}$$

Thus, in the case $\zeta^e = \zeta^b$, whether or not there is a pumping effect depends on $\widehat{w^2} \neq 0$, which in turn depends on the shape of the periodic forcing. The effect might be negative ($\overline{h} < \overline{h_0}$) or positive.

In [8] a combination of (5) and (7) was used. Then, even in the case $\zeta^b = \zeta^e$, there is a pumping effect of height

$$\overline{h} - \overline{h_0} = \frac{1}{4g}(\zeta^e - 2\zeta^b)\overline{w^2} - \frac{1}{4g}\zeta^e \widehat{w^2}. \tag{11}$$

To see the impact of friction, a term $p^f(t)$ has to be subtracted on the right-hand side of (6); Poiseuille's law for laminar flow is

$$p^f = \ell r_0 w \tag{12}$$

and Blasius' law for turbulent flow reads

$$p^f = \ell r_1 \mathrm{sgn} w |w|^{7/4} \tag{13}$$

with constants $r_0, r_1 > 0$. For a periodic w with mean 0 (otherwise h would not be periodic), the presence of (12) does not alter the pumping effect. For (13), the term $\ell r_1 \widetilde{w}/\rho g$ has to be subtracted from the right-hand side in (9)–(11) with

$$\widetilde{w} = \frac{1}{T}\int_0^T \mathrm{sgn} w |w|^{7/4} \mathrm{d}t.$$

$\widetilde{w}$ might be zero, positive or negative, depending on the shape of w.

In summary, the 2-tank configuration does not exhibit pumping effects in the case of the symmetric model (3) and (7) with $\zeta^b = \zeta^e$. For the sgn-model (5) and (8) with $\zeta^b = \zeta^e$, pumping takes place only if $\widehat{w^2} \neq 0$. Linear friction (12) does not alter the pumping effect; nonlinear friction (13) only does so when $\widetilde{w} \neq 0$. The conclusion is that pumping effects are introduced by unsymmetric modelling of the pressure losses at the junctions ($\zeta^b - \zeta^e \neq 0$ or $\mathrm{sgn} w$ terms) which on average act like a valve that favours the flow in the direction of the larger pressure loss. For example the model (3), (7) with $\zeta^b > \zeta^e$ exhibits a negative pumping effect in the sense that the average level height is higher in the tank in which the periodic pressure is applied, as indicated in Fig. 1(b).

In Fig. 1(c), a configuration with three tanks that are connected by two pipes of length $\ell_1 < \ell_2$ is shown. The pressure p above the fluid level in the middle tank is a given forcing function, the level heights in the outer tanks are h_1, h_2 and the flow velocities in the outward direction are denoted by w_1, w_2. The model for this configuration is a system of four differential equations,

$$\begin{aligned} h_i' &= c w_i, \\ \rho \ell_i w_i' &= \rho g (h_0 - h_i) + (\zeta_i^e - \zeta_i^b)\frac{\rho}{2} w_i^2 - r_0 \ell_i w_i + p, \\ i &= 1, 2, \end{aligned} \tag{14}$$

where h_0 is given by the constant volume of fluid outside the always filled pipes, $h_0 = V_0/A_t - h_1 - h_2$. For simplicity we let the cross section of all tanks equal A_t, the cross sections of both pipes equal A_p and we employed Poiseuille's law with uniform friction coefficient r_0. Moreover, we used (3) (without sgn terms) for the pressure losses at all four pipe–tank junctions. For constant $p \equiv \overline{p}$, the system is in equilibrium iff $h_1 \equiv h_2$ and $\overline{p} + \rho g h_0 \equiv \rho g h_1$.

In 3-tank models, as in [8–10], pumping is defined by the difference of the average level heights in the two outer tanks. The 3-tank pumping effect can be understood as a combination of two 2-tank systems as follows: let

$$\zeta_i = \zeta_i^e - \zeta_i^b$$

be the net pressure loss in pipe i. From the 2-tank analysis the difference of the average level height in the middle and the outer tank i should be $\overline{h_i} - \overline{h_0} = \zeta_i \overline{w_i^2}/2g$, as in (9), and we expect a 3-tank pumping $\overline{h_2} - \overline{h_1} = (\overline{h_2} - \overline{h_0}) - (\overline{h_1} - \overline{h_0})$.

Indeed, let p be periodic and w_1, w_2, h_1, h_2 be a solution of (14) of the same period $T > 0$. The existence and stability of such solutions will be demonstrated in the next section. Then, because of the h_i-equations in (14), $\overline{w_i} = 0$. Taking the mean in the w_i-equations of (14) and subtracting the two equations, we get

$$\overline{h_2} - \overline{h_1} = \frac{1}{2g}(\zeta_2\overline{w_2^2} - \zeta_1\overline{w_1^2}). \tag{15}$$

In particular in the case $\zeta_i^e = \zeta_i^b, i = 1, 2$, the pumping effect is zero because of $\zeta_1 = \zeta_2 = 0$. This result is obvious by the above 2-tank analysis but it was not included or discussed in the papers on the subject [8–10,12]. In these references the pumping effect was attributed to the difference of the lengths $\ell_1 \neq \ell_2$ and not explained any further.

In order to understand the role of the difference of the pipe lengths, we assume $\zeta_1 = \zeta_2 =: \zeta \neq 0$. The average level difference then is $\zeta(\overline{w_2^2} - \overline{w_1^2})/2g$. In the next section it will be proven that $\operatorname{sgn}(\overline{w_2^2} - \overline{w_1^2}) = \operatorname{sgn}(\ell_1 - \ell_2)$ for periodic solutions of (14) and sufficiently small ζ. The physical interpretation is that the amplitude of the velocity in the longer pipe is smaller, because there is more mass and inertia than in the shorter pipe, and the periodically moving system allocates its total momentum among the two pipes according to the proportion of ℓ_1 and ℓ_2. In the limit $\zeta, c \to 0$ the proportion is $\ell_1|w_1| = \ell_2|w_2|$; see (26).

To illustrate the analysis by examples, again let $\zeta_1 = \zeta_2 =: \zeta$. In the case $\zeta = 0$, the pumping effect is 0. In the case $\zeta > 0$, the pumping goes from the middle tank to the outer ones, because the pressure loss at the outer junctions is larger. If $\ell_1 < \ell_2$, then the amplitude of the velocity in pipe 1 is larger and therefore the positive difference $\overline{h_1} - \overline{h_0}$ is larger than $\overline{h_2} - \overline{h_0}$, which implies a negative 3-tank pumping effect from tank 2 into tank 1. In the case $\zeta < 0$ and still $\ell_1 < \ell_2$, the middle tank sucks more volume out of tank 1 than out of tank 2 and the pumping effect $\overline{h_2} - \overline{h_1}$ is positive, as indicated in Fig. 1(c).

In the case of the sgn-versions (8) and (5) for (some of) the junctions at the middle tank and the outer tanks, respectively, various combinations of (10) and/or (11) are possible. For example, in the case $\zeta_i^b = \zeta_i^e =: \zeta_0, i = 1, 2$, and therefore $\zeta_i = 0$ in (15), we get

$$\overline{h_2} - \overline{h_1} = \frac{\zeta_0}{2g}(\widehat{w_1^2} - \widehat{w_2^2}),$$

when all pressure losses include sgn-terms. We do not further pursue these models.

Finally, consider a configuration with three tanks where the level height $h_0(t)$ is enforced by a piston, as in Fig. 1(d). The momentum equations for the flow in the pipes are

$$\rho\ell_i w_i' = p^m - \zeta_i^b\frac{\rho}{2}w_i^2 - p_i^e - r_0\ell_i w_i, \quad i = 1, 2, \tag{16}$$

where p_i^e is as in (3) and p^m is the pressure at the bottom of the middle tank. In the piston enforcement configuration, p^m is not modelled explicitly. Rather, we introduce

$$w_- = \ell_1 w_1 - \ell_2 w_2, \quad w_+ = w_1 + w_2$$

and take the difference of the two equations in (16), so that p^m cancels. We get

$$w_-' = g(h_2 - h_1) + \frac{\zeta_1}{2}w_1^2 - \frac{\zeta_2}{2}w_2^2 - rw_-$$

where we used the abbreviations $\zeta_i = \zeta_i^e - \zeta_i^b$ and $r = r_0/\rho$. $h_i' = cw_i$ as in (14). Because

$$cw_+ = -h_0' =: v,$$

w_+ is proportional to the given velocity $v(t)$ of the piston. In order to set up a system for the two state variables h_1, w_-, we need to express w_i in terms of w_-, w_+, and h_2 in terms of h_0, h_1, namely

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{1}{\ell_1 + \ell_2}\begin{pmatrix} \ell_2 w_+ + w_- \\ \ell_1 w_+ - w_- \end{pmatrix}, \quad h_0 + h_1 + h_2 = V_0/A_t.$$

In the particular case $\zeta_1 = \zeta_2 =: \zeta$, the term $\zeta(w_1^2 - w_2^2)$ can be rewritten as a product of w_+ and w_-; thereby the w_-^2 terms cancel and we arrive at

$$\begin{aligned} h_1' &= \frac{1}{\ell_1 + \ell_2}(cw_- + \ell_2 v), \\ w_-' &= -2gh_1 - \left(r - \frac{\zeta}{c}\frac{v}{\ell_1 + \ell_2}\right)w_- + g\left(\frac{V_0}{A_t} - h_0\right) \\ &\quad + \frac{\zeta}{2c^2}\frac{\ell_2 - \ell_1}{\ell_1 + \ell_2}v^2. \end{aligned} \tag{17}$$

This is a *linear* model; the forcing and the time dependence of the w_--coefficient in the w_- equation is determined by the given functions h_0 and $v = -h_0'$.

The unforced system ($h_0 \equiv \overline{h_0}, v \equiv 0$) is in equilibrium iff $w_- \equiv 0$ and $h_2 - h_1 \equiv 0$. Suppose h_0 (and thus v) is periodic and h_1, w_- is a solution of (17) with the same period $T > 0$. The existence of such solutions is demonstrated in Section 3. Since $\overline{v} = -\overline{h_0'} = 0$, taking the mean in the h_1-equation implies $\overline{w_-} = 0$, and the w_--equation, using $V_0/A_t - h_0 = h_1 + h_2$, yields

$$\overline{h_2} - \overline{h_1} = \frac{\zeta}{2g}(\overline{w_2^2} - \overline{w_1^2}), \tag{18}$$

as in (15).

A 2-tank configuration with piston enforcement cannot exhibit pumping, since the level heights are tied to each other by the constant total volume. However, in the 3-tank model (17) the average level heights have to be adjusted according to (16),

$$0 = \overline{p^m} + (\zeta_i^e - \zeta_i^b)\frac{\rho}{2}\overline{w_i^2} - \rho g\overline{h_i}.$$

Thus, as in the pressure enforcement model, the pumping is due to the difference of the pressure losses at the pipe–tank junctions.

Let us compare the piston enforcement model (16) to configurations in the literature. In [9], in place of the middle tank a tee-junction with cross section A_p of all three of its arms

was used and the flow characteristics and pressure losses in the tee were taken from independent laboratory experiments. The model in [9] is not linear because of the nonlinearities in the pressure loss coefficients at the tee-junction. In [9,10] formulas of the type (18) are derived by making a sinusoidal substitution for approximate solutions and neglecting the pressure loss coefficients in the tee. That a pumping effect can occur in a linear model with periodic coefficients was already shown in [6] and demonstrated in [8] by simulations with an infinite dimensional model for elastic tubes. The observation that a rigid pipe model can be formulated in a linear variant with time dependent coefficients was brought to our attention by the developments in [12].

3. Existence of periodic solutions

This section is concerned with the existence of periodic solutions to four of the models in Section 2. Differentiating the product in (1) and using $\ell' = -w$ yields

$$w' = \frac{p/\rho - gh + (1+\zeta/2)w^2}{V_0/A_p - h/c}$$

along with (2). The state component h appears in the denominator of the w-equation, and therefore the 1-tank model is very different from the other models in Section 2. For the 1-tank model we do not have existence results for general periodic forcing functions p. To guarantee the existence of some periodic solutions, we choose a particular solution (w, h) and construct a forcing p such that (1), (2) is satisfied: let, for example,

$$p(t) = \rho(V_0/A_p - gc + \cos t)\cos t - \rho(1+\zeta^e/2)\sin^2 t. \quad (19)$$

Then $w(t) = \sin t$ and $h(t) = -c\cos t$ is a solution of (1), (2). This solution is 2π-periodic, as is the forcing function p.

As to the 2-tank model (2), (6) with pressures (3), (7), differentiation of the w-equation gives

$$w'' + f(w)w' + \omega_0^2 w = e(t) \quad (20)$$

with $f(w) = (\zeta^b - \zeta^e)w/\ell$, $\omega_0^2 = 2gc/\ell$ (let $A_0 = A_t$, $V_0 = 0$ for simplicity) and $e(t) = p'(t)/\rho\ell$. When friction (12) is included, the constant r is added to $f(w)$. It was established in [15,16], that the Liénard equation (20) with T-periodic forcing function e has a solution of period T, provided that f and e are continuous, $\omega_0 \neq 0$, and $0 < T < 2\pi/\omega_0$. An outline of a proof is given in the Appendix.

Thus, if p is continuously differentiable, T-periodic and $0 < T/2\pi < \sqrt{\ell/2gc}$, then the 2-tank model (2), (6), (3), (7) with or without friction has a solution of period T. For the sgn-versions (5) and/or (8) the difficulty of non-differentiability arises when trying to set up the second order formulation; we omit these cases.

Next, consider the 3-tank model with pressure forcing p in the middle tank and assume that there is some friction involved, i.e. $r_0 > 0$. Without loss of generality we can set $V_0 = 0$: if $h_i, w_i, i = 1, 2$, is a solution for some $V_0 \neq 0$, then $h_i - V_0/3, w_i$ is a solution of (14) with $V_0 = 0$, and vice versa. In order to make the model accessible to the theory of perturbations of linear systems, we define the column vector $x = \mathrm{col}(h_1, h_2, w_1, w_2)$ and write (14) in the form

$$x' = Ax + b(t) + \zeta f(x), \quad (21)$$

where, for simplicity of notation, we take the same $\zeta = \zeta_i^e - \zeta_i^b$ for both i, and

$$A = \begin{pmatrix} 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \\ -2g/\ell_1 & -g/\ell_1 & -r & 0 \\ -g/\ell_2 & -2g/\ell_2 & 0 & -r \end{pmatrix},$$

with $r = r_0/\rho$ and

$$b(t) = \begin{pmatrix} 0 \\ 0 \\ p(t)/\rho\ell_1 \\ p(t)/\rho\ell_2 \end{pmatrix}, \quad f\begin{pmatrix} h_1 \\ h_2 \\ w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ w_1^2/2\ell_1 \\ w_2^2/2\ell_2 \end{pmatrix}.$$

Suppose p is periodic with some arbitrary period $T > 0$. Eq. (21) is a special case of [11], chapter IV, equation (2.4) and so the theory in [11] guarantees the existence and uniqueness of a T-periodic solution of (21), provided $|\zeta|$ is small enough.

More specifically, the equation $x' = Ax$ is noncritical with respect to periodic functions of any period $T > 0$, because all eigenvalues of A have non-zero real parts. Indeed, since $r > 0$, they are negative. [11] IV Theorem 1.1, implies that the linear equation $x' = Ax + b(t)$ has a unique T-periodic solution that is denoted by $\mathcal{K}b$. The nonlinearity f is locally Lipschitz on $\mathbb{R}^4$. Therefore, the application of [11] IV Theorem 2.1, as in the discussion of [11] IV (2.4), implies the existence of positive constants ε_1, ρ_1 with the following property: For all ζ with $0 \le |\zeta| \le \varepsilon_1$ there exists a T-periodic solution $x^*(\cdot, \zeta, b)$ of (21), $x^*(\cdot, 0, b) = \mathcal{K}b$, and $x^*(\cdot, \zeta, b)$ is the only T-periodic solution of (21) that is contained in the ρ_1-neighbourhood of $\mathcal{K}b$. Furthermore, $x^*(\cdot, \zeta, b)$ depends continuously on ζ. See the Appendix for an existence proof.

As to the asymptotic stability of $x^*(\cdot, \zeta, b)$, [11] IV Theorem 3.1 can be applied (for our problem, the number k of eigenvalues with positive real part in the interpretation of the Theorem is $k = 0$). The stability of the zero solution of IV (3.3) can be traced back to the solution $x^*(\cdot, \zeta, b)$ of (21), and so there is an $\varepsilon_2 > 0$ such that for all $0 \le |\zeta| \le \varepsilon_2$, $x^*(\cdot, \zeta, b)$ is uniformly asymptotically stable.

Finally, the piston enforcement model (17) for $x = \mathrm{col}(h_1, w_-)$ is of the form

$$x' = Ax + b(t) + \zeta f(t, x), \quad (22)$$

with

$$A = \begin{pmatrix} 0 & c/(\ell_1+\ell_2) \\ -2g & -r \end{pmatrix}, \quad b(t) = \begin{pmatrix} \ell_2 v(t)/(\ell_1+\ell_2) \\ g(V_0/A_t - h_0(t)) \end{pmatrix},$$

$$f\left(t, \begin{pmatrix} h_1 \\ w_- \end{pmatrix}\right) = \frac{1}{c(\ell_1+\ell_2)}\begin{pmatrix} 0 \\ v(t)w_- + (\ell_2-\ell_1)v(t)^2/2c \end{pmatrix}.$$

Let $r > 0$. Suppose h_0 is differentiable and, along with its derivative v, is periodic with some arbitrary period $T > 0$. Since f is Lipschitz in x uniformly with respect to t in

$(-\infty, \infty)$, the theory of [11] can be applied to (22) as described above for (21). Provided that $|\zeta|$ is small enough, there exists a unique asymptotically stable periodic solution of (22) in the neighbourhood of the solution $\mathcal{K}b$ of $x' = Ax + b(t)$.

4. Definition and existence of pumps

In this section it is shown that the models of Section 2 are pumps according to

Definition 1. Let $(X, |\cdot|)$ and U be Banach spaces, F a map from $X \times U$ to X, u a map from $(-\infty, \infty)$ to U and $x_0 \in X$. The system

$$x'(t) = F(x(t), u(t)), \quad x(0) = x_0 \tag{23}$$

is called a *periodically forced pump* if it has the following properties:

(i) u is a nonconstant periodic function with mean $\bar{u}$.
(ii) There exists an asymptotically periodic solution $x : [0, \infty) \to X$, that is, x solves (23) and for a periodic function $y : (-\infty, \infty) \to X$, $\lim_{t\to\infty} |x(t) - y(t)| = 0$.
(iii) The mean $\bar{y}$ of y is not an equilibrium of the $\bar{u}$-forced system, i.e. $F(\bar{y}, \bar{u}) \neq 0$.

A periodically forced pump is called an *externally forced periodic pump* if it is of the form

$$x'(t) = F(x(t)) + B(u(t)), \quad x(0) = x_0$$

with maps $F : X \to X$, $B : U \to X$. Otherwise it is called an *internally forced periodic pump*.

The definition does not require that the forcing function u and the limit cycle y have the same period; thus, subharmonic limit cycles (cf. e.g. [17]) are admissible. We have formulated the definition with asymptotically periodic solutions x, because starting the trajectory x in an equilibrium x_0 of the unforced system most evidently demonstrates the pumping; cf. the simulations in [8]. On the other hand, if there exists a periodic solution y with $F(\bar{y}, \bar{u}) \neq 0$, then system (23) is a periodically forced pump (set $x = y$ and $x_0 = y(0)$).

In the case of the 3-tank models we restrict the proof of pumping to forcing functions p, that are complex valued simple harmonic. Furthermore, we require that the forcing frequency is larger than half the natural frequencies of the decoupled 2-tank subsystems. Let $C \in \mathbb{C}$, $\omega \in \mathbb{R}$ and

$$p'(t) = \rho C \exp(-\mathrm{i}\omega t) \quad \text{with } \omega^2 > gc/\min(\ell_1, \ell_2). \tag{24}$$

Note that c is small, so the requirement on ω^2 is physically reasonable.

Theorem 1. (i) *The model (1)–(3) with p in (19) and $\zeta^e \neq 0$ is an externally forced periodic pump.*
(ii) *Provided that the period T of p satisfies $0 < T < 2\pi\sqrt{\ell/2gc}$, the model (2), (6), (3), (7) with or without friction (12) and $\zeta^e \neq \zeta^b$ is an externally forced periodic pump.*
(iii) *The model (14) with p as in (24), $r_0 > 0$, $\ell_1 \neq \ell_2$ and sufficiently small $|\zeta_i^e - \zeta_i^b| = |\zeta| \neq 0$ is an externally forced periodic pump.*
(iv) *The model (17) with p as in (24), $r > 0$, $\ell_1 \neq \ell_2$ and sufficiently small $|\zeta| \neq 0$ is a periodically forced pump.*

Proof. The existence of periodic solutions under the assumptions on the parameters was demonstrated in Section 3. It remains to show $F(\overline{y}, \overline{u}) \neq 0$ for periodic solutions y.
(i) For the 1-tank model, (4) implies $\overline{p} \neq \rho g \overline{h}$, and so $\overline{h}, \overline{w}$ is not an equilibrium of the $\overline{p}$-forced 1-tank configuration.
(ii) For the 2-tank model, $\overline{h} \neq \overline{h_0}$ follows from (9).
(iii) In the 3-tank models, the existence of a pumping effect $\overline{h_1} \neq \overline{h_2}$ is equivalent to $\overline{|w_1^2|} \neq \overline{|w_2^2|}$. To show that this is the case when $\ell_1 \neq \ell_2$, we use the perturbation statements in Section 3. For the moment, let $\zeta_i^e - \zeta_i^b = \zeta = 0$ and consider the then linear system (14). Using $h_0 + h_1 + h_2 = V_0/A_t = 0$ it can be written as a system of coupled oscillators

$$\begin{aligned} w_1'' + rw_1' + \frac{gc}{\ell_1}(2w_1 + w_2) &= \frac{1}{\rho\ell_1}p', \\ w_2'' + rw_2' + \frac{gc}{\ell_2}(w_1 + 2w_2) &= \frac{1}{\rho\ell_2}p', \end{aligned} \tag{25}$$

for the flow velocities w_1, w_2. In order to apply the theory of coupled linear oscillators in [18], we need the characteristic polynomial $\Delta(-\mathrm{i}\omega)$. $\Delta(-\mathrm{i}\omega) \neq 0$, because all eigenvalues of A have negative real part. [18], formula (3.2.14) yields

$$\begin{aligned} w_1 &= \left(\frac{1}{\ell_1}\left(-\omega^2 - \mathrm{i}\omega r + 2\frac{gc}{\ell_2}\right) - \frac{gc}{\ell_1\ell_2}\right)\frac{C\exp(-\mathrm{i}\omega t)}{\Delta(-\mathrm{i}\omega)}, \\ w_2 &= \left(\frac{1}{\ell_2}\left(-\omega^2 - \mathrm{i}\omega r + 2\frac{gc}{\ell_1}\right) - \frac{gc}{\ell_1\ell_2}\right)\frac{C\exp(-\mathrm{i}\omega t)}{\Delta(-\mathrm{i}\omega)}, \end{aligned}$$

for periodic solutions of (25) with p' as in (24). Therefore,

$$\frac{|w_1|^2}{|w_2|^2} = \frac{\ell_2^2}{\ell_1^2}\frac{\omega^2 r^2 + (\omega^2 - gc/\ell_2)^2}{\omega^2 r^2 + (\omega^2 - gc/\ell_1)^2}. \tag{26}$$

This implies $\ell_1|w_1| > \ell_2|w_2|$ if $\ell_1 < \ell_2$ and $\ell_1|w_1| < \ell_2|w_2|$, if $\ell_1 > \ell_2$. Thus, although there is no pumping in the linear system (25), the amplitudes of the velocities are different if $\ell_1 \neq \ell_2$. Now, the solution $x^*(\cdot, \zeta, b)$ of (21) is in a neighbourhood of the solution $\mathcal{K}b$ to the linear system with $\zeta = 0$, and since $x^*(\cdot, \zeta, b)$ depends continuously on ζ, the inequality $|w_1| \neq |w_2|$ for the linear system implies $\overline{|w_1^2|} \neq \overline{|w_2^2|}$ in the nonlinear system if $|\zeta| \neq 0$ is sufficiently small.
(iv) Suppose $\zeta_i^e - \zeta_i^b = \zeta = 0$, so (16) becomes

$$\rho\ell_i w_i' = p^m - r_0\ell_i w_i - \rho g h_i, \quad i = 1, 2.$$

Again we consider the simple harmonic case in complex notation. Then the unknown pressure p^m is also simple harmonic, $p^{m\prime}(t) = \rho P \exp(-\mathrm{i}\omega t)$ say. We treat $p^m/\rho\ell_i$ as a forcing function in the linear oscillator

$$w_i'' + rw_i' + \frac{gc}{\ell_i}w_i = \frac{1}{\ell_i}P\exp(-\mathrm{i}\omega t). \tag{27}$$

According to [18] (2.3.2), the $2\pi/\omega$-periodic solution to (27) is

$$w_i(t) = \frac{1}{\ell_i}(-\omega^2 - \mathrm{i}\omega r + gc/\ell_i)^{-1}P\exp(-\mathrm{i}\omega t).$$

Therefore the ratio $|w_1|^2/|w_2|^2$ is exactly the same as in (26). As in (iii) the perturbation statements of Section 3 and the continuous dependence of the periodic solution on ζ imply

$\overline{|w_1|^2} \neq \overline{|w_2|^2}$ for the nonlinear system with sufficiently small $|\zeta| \neq 0$. □

Note that the system (17) is not an *externally* forced periodic pump, because it is of the form

$$x' = A(t)x + b(t),$$

with

$$A(t) = \begin{pmatrix} 0 & c/(\ell_1+\ell_2) \\ -2g & -r + \zeta v(t)/c(\ell_1+\ell_2) \end{pmatrix},$$

$$b(t) = \begin{pmatrix} \ell_2 v(t)/(\ell_1+\ell_2) \\ g(V_0/A_t - h_0(t)) + \zeta(\ell_2-\ell_1)v(t)^2/2c^2(\ell_1+\ell_2) \end{pmatrix}.$$

But it is a *linear* periodically forced pump with time dependent periodic coefficient matrix. However, there is no linear externally forced periodic pump. This is the assertion of

Theorem 2. *Let X and U be Banach spaces, A an X-valued closed linear operator with domain in X, B an X-valued closed linear operator with domain in U, u a U-valued integrable nonconstant periodic function and $x_0 \in X$. Then the system*

$$x'(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{28}$$

is not an externally forced periodic pump.

Proof. Let $T > 0$ be the minimal period of u and let P be the minimal period of the limit cycle y of an asymptotically periodic solution x of (28). Then the time derivative of x is asymptotically periodic with P-periodic limit cycle. Therefore, $Ax+Bu$ is asymptotically periodic with P-periodic limit cycle. It follows that P is a multiple of T, $P = nT$ for some $n \in \mathbb{N}$. For $k \in \mathbb{N}$ define

$$x_k = \frac{1}{P}\int_k^{k+P} x(t)\mathrm{d}t,$$

which are elements of the domain of A (cf. [19], proof of (2.14)). Because of the asymptotic behaviour of x, $\lim_{k\to\infty} x_k = \bar{y}$. Furthermore, as A is closed and linear, it interchanges with integration, and so

$$\begin{aligned}\lim_{k\to\infty} Ax_k &= \frac{1}{P}\lim_{k\to\infty} A\int_k^{k+P} x(t)\mathrm{d}t \\ &= \frac{1}{P}\lim_{k\to\infty}\int_k^{k+P} Ax(t)\mathrm{d}t \\ &= \frac{1}{P}\lim_{k\to\infty}\left(\int_k^{k+P}\frac{\mathrm{d}}{\mathrm{d}t}x(t)\mathrm{d}t - \int_k^{k+P} Bu(t)\mathrm{d}t\right) \\ &= \frac{1}{P}\lim_{k\to\infty}(x(k+P)-x(k)) - B\bar{u} = -B\bar{u}.\end{aligned}$$

Therefore $\bar{y}$ is in the domain of A and $A\bar{y} = -B\bar{u}$. Thus, no solution of (28) with the properties required in the definition of an externally forced periodic pump exists. □

The fact that A and B are closed but may be unbounded makes the theorem applicable to infinite-dimensional systems such as partial differential equations with point or boundary forcing.

5. Conclusions

Systems of ordinary differential equations were studied that are models of flow in rigid pipe configurations. It was shown that pumping effects due to periodic excitation are caused by unsymmetric modelling of pipe–tank pressure losses. The unsymmetry may be due to different pipe–tank parameters or to terms involving the signum of the flow velocity. This led to the insight that symmetric pressure loss models do not exhibit pumping effects in 3-tank configurations, even in the case of different lengths of the two pipes. A definition of periodically forced pumps was stated. It was proved that the rigid pipe models are pumps, provided that the model parameters satisfy physically reasonable relations. One of the models was shown to be a linear pump with time dependent coefficient matrix. Finally, in a formulation that includes infinite-dimensional systems, it was demonstrated that there is no linear periodically forced pump with constant coefficients.

Acknowledgements

This work was substantially helped on by Valery Romanovski, Center for Applied Mathematics and Theoretical Physics, University of Maribor, Slovenia.

Appendix

For the convenience of the reader we outline proofs of the existence of periodic solutions, and add a remark on the reversal of the pumping direction.

The 2-tank model is a first order system $u'(t) = F(t, u(t))$ with $u = \binom{w}{v}$ and

$$F\left(t, \begin{pmatrix} w \\ v \end{pmatrix}(t)\right) = \begin{pmatrix} v \\ -f(w)v - \omega_0^2 w + e \end{pmatrix}(t),$$

where $\omega_0^2 \neq 0$, $f : \mathbb{R} \to \mathbb{R}$ is continuous and e is continuous and T-periodic with mean $\bar{e} = 0$. The existence of T-periodic solutions is formulated as a fixed point problem in the Banach space $E = \{u \in C([0,T];\mathbb{R}^2) : u(T) = u(0)\}$,

$$u(t) = \int_0^t F(s, u(s))\mathrm{d}s =: (Su)(t).$$

$S : E \to C([0,T];\mathbb{R}^2)$ is a compact operator because $F : [0,T]\times\mathbb{R}^2 \to \mathbb{R}^2$ is continuous. For $\mu \in [0,1]$ let $S_\mu = \mu S$. S_0 has the fixed point 0 and, as shown below, there is an open bounded set $D \subset E$ with $0 \in D$ and $S_\mu u \neq u$ for all $u \in \partial D$ and all $\mu \in [0,1]$. Therefore, by the Leray–Schauder Theorem, S_1 has at least one fixed point in D. Indeed, for sufficiently large K, the set D can be chosen to be $D = \{(w,v) \in E : \|w\|_\infty, \|v\|_\infty < K\}$. This is so, because there is an a priori bound for any possible solution of $S_\mu u = u \in E$.

To see this, let $u = (w,v) \in E$ be such a solution. Then $\mu v = w'$ and w satisfies

$$w'' + \mu f(w)w' + \mu^2\omega_0^2 w = \mu^2 e(t). \tag{29}$$

Establishing a priori bounds on w, w' can be done in the following way, cf. [20]. Multiplication of (29) by w and integration gives

$$-\|w'\|_2^2 + \mu^2\omega_0^2\|w\|_2^2 = \mu^2\langle w, e\rangle_2 \tag{30}$$

where $\langle\cdot,\cdot\rangle_2$ and $\|\cdot\|_2$ are the inner product and norm in $L^2(0,T)$. The Fourier expansion of w ($\bar{w} = 0$ by integration of (29)) and Parseval's identity show

$$\|w\|_2^2 \le \frac{T^2}{4\pi^2}\|w'\|_2^2.$$

Using this, $\mu^2 \le 1$ and $|\langle w, e\rangle_2| \le \|w\|_2\|e\|_2$ in (30) yields

$$\frac{4\pi^2 - \omega_0^2T^2}{4\pi^2}\|w'\|_2^2 \le \frac{T}{2\pi}\|w'\|_2\|e\|_2.$$

By assumption $T < 2\pi/\omega_0$, so

$$\|w'\|_2 \le \frac{2\pi T}{4\pi^2 - \omega_0^2T^2}\|e\|_2.$$

Since $w(t_0) = 0$ for some $t_0 \in [0,T]$, $\|w\|_\infty = \max|w(t)| \le \sqrt{T}\|w'\|_2$. Thus, $M := 2\pi T^{3/2}(4\pi^2 - \omega_0^2T^2)^{-1}\|e\|_2$ is a bound on $\|w\|_\infty$, independent of $\mu \in [0,1]$. As $w \in C^2([0,T])$, $\|w'\|_\infty^2 \le 4\|w\|_\infty\|w''\|_\infty$ and (29) yields

$$\|w'\|_\infty^2 \le 4M(q\|w'\|_\infty + \omega_0^2M + \|e\|_\infty)$$

for all $\mu \in [0,1]$, where $q = \max\{|f(w)| : |w| \le M\}$. Therefore, there is a μ-independent bound on $\|w'\|_\infty$.

The 3-tank model (21), for $\zeta = 0$, has a unique periodic solution denoted by $\mathcal{K}b$. For $\zeta \ne 0$, consider

$$y' = Ay + \zeta g(y) \tag{31}$$

where $g(y) = f(y + \mathcal{K}b)$. If y is a T-periodic solution of (31), then $x = y + \mathcal{K}b$ is a T-periodic solution of (21). The existence of a T-periodic solution of (31) is argued as follows, cf. [20]. The solution of (31) with $y(0) = y_0$ is

$$y(t) = \mathrm{e}^{tA}y_0 + \zeta\int_0^t \mathrm{e}^{(t-s)A}g(y(s))\mathrm{d}s.$$

Thus, T-periodicity is enforced by

$$y_0 = (I - \mathrm{e}^{TA})^{-1}\zeta\int_0^T \mathrm{e}^{(T-s)A}g(y(s))\mathrm{d}s.$$

$I - e^{TA}$ is invertible as all eigenvalues of A have negative real part. Therefore, the existence of a T-periodic solution of (31) is equivalent to the existence of a fixed point of S_ζ, where S_ζ is given by

$$(S_\zeta y)(t) := \zeta\mathrm{e}^{tA}\left((I - e^{TA})^{-1}\int_0^T \mathrm{e}^{(T-s)A}g(y(s))\mathrm{d}s + \int_0^t \mathrm{e}^{-sA}g(y(s))\mathrm{d}s\right).$$

S_ζ is a compact operator in the Banach space $E = \{y \in C([0,T];\mathbb{R}^4) : y(T) = y(0)\}$. For arbitrary $M > 0$, let $q_M = \max\{|g(y)| : |y| \le M\}$. Then, for $y \in D_M := \{y \in E : \|y\|_\infty \le M\}$, $\|S_\zeta y\|_\infty \le 2|\zeta|KTq_M$ with

$$K = \max\left(\max_{s,t\in[0,T]}|\mathrm{e}^{tA}(I - e^{TA})^{-1}||\mathrm{e}^{(T-s)A}|, \max_{s\in[0,T]}|\mathrm{e}^{-sA}|\right),$$

wherein $|\cdot|$ denotes an appropriate matrix norm. If $|\zeta| < (2KTq_M)^{-1}M$, then $S_\zeta D_M \subset D_M$. Therefore, by the Schauder Fixed Point Theorem, for such ζ, (21) has a T-periodic solution.

The reversal of the pumping direction is an interesting resonance feature of the 3-tank models. If the tube friction is sufficiently small, there is a smooth inversion of the direction of the pumping effect in the 3-tank models (pressure or piston forcing in the middle tank) when the forcing frequency is reduced below a certain value.

Consider the complex simple harmonic forcing in (24) and omit the assumption $\omega^2 > gc/\min(\ell_1,\ell_2)$ which was only made to simplify the use of (26). In view of (15) (with $\zeta_1 = \zeta_2 = \zeta$) and (18), the question is whether $|w_1|/|w_2|$ is larger or smaller than 1. This determines the direction of the pumping effect. As the solutions depend continuously on ζ, $|w_1|/|w_2| > 1$ (<1 resp.) in the linearized ($\zeta = 0$) system carries over to the nonlinear system as long as $|\zeta|$ is sufficiently small. Suppose $\ell_2 > \ell_1$.

For $\omega^2 \ge 0$ let $f(\omega^2)$ denote the right-hand side of (26),

$$f(\omega^2) = \frac{\ell_2^2}{\ell_1^2}\frac{\omega^2r^2 + (\omega^2 - gc/\ell_2)^2}{\omega^2r^2 + (\omega^2 - gc/\ell_1)^2}.$$

We have $f(0) = 1$ and $\lim_{\omega^2\to\infty} f(\omega^2) = \ell_2^2/\ell_1^2 > 1$. Analysis of the derivative of f leads to the definition of the frequencies

$$(\omega^2)_{1,2} = \frac{gc}{2\ell_1\ell_2}\left(\ell_1 + \ell_2 \pm\sqrt{(\ell_1 - \ell_2)^2 + 2r^2\ell_1\ell_2(\ell_1 + \ell_2)/gc}\right)$$

where the $+$stands for $(\omega^2)_1$ and the $-$for $(\omega^2)_2$.

In the case $2gc < r^2(\ell_1 + \ell_2)$, i.e. when the friction coefficient r is large, $f(\omega^2)$ grows on $(0,(\omega^2)_1)$ and falls on $((\omega^2)_1,\infty)$. Therefore $|w_1|/|w_2| > 1$ for all ω in this case, and the direction of average pumping, for small $\zeta > 0$, always goes from tank 2 to tank 1 (cf. (18)).

If $2gc \ge r^2(\ell_1 + \ell_2)$, $f(\omega^2)$ falls on $(0,(\omega^2)_2) \cup ((\omega^2)_1,\infty)$ and grows on $((\omega^2)_2,(\omega^2)_1)$. Therefore $|w_1|/|w_2|$ smoothly crosses 1 on the middle interval, thereby, for small $\zeta > 0$, changing the direction of pumping from $1 \to 2$ to $2 \to 1$. This phenomenon was physically and numerically observed by Takagi and Takahashi [10].

References

[1] G. Liebau, Über ein ventilloses Pumpprinzip, Naturwissenschaften 41 (1954) 327–328.

[2] M. Moser, J.W. Huang, G.S. Schwarz, T. Kenner, A. Noordergraaf, Impedance defined flow generalisation of William Harvey's concept of the circulation — 370 years later, Int. J. Cardiovasc. Med. Sci. 1 (1998) 205–211.

[3] H. Andersson, W. van der Wijngaart, P. Nilsson, P. Enoksson, G. Stemme, A valve-less diffuser micropump for microfluidic analytical systems, in: The MicroTAS 2000 Symposium, Enschede.
[4] E. Jung, C.S. Peskin, Two-dimensional simulations of valveless pumping using the immersed boundary method, SIAM J. Sci. Comput. 23 (2001) 19–45.
[5] J.T. Ottesen, Valveless pumping in a fluid-filled closed elastic tube-system: one-dimensional theory with experimental validation, J. Math. Biol. 46 (2003) 309–332.
[6] H.J. von Bredow, Untersuchung eines ventillosen Pumpprinzips, Fortschr.-Ber. VDI-Z. Reihe 7 Nr. 9 (1968) 1–89.
[7] H.J. Rath, I. Teipel, Der Fördereffekt in ventillosen, elastischen Leitungen, Z. Angew. Math. Phys. 29 (1978) 123–133.
[8] A. Borzì, G. Propst, Numerical investigation of the Liebau phenomenon, Z. Angew. Math. Phys. 54 (2003) 1050–1072.
[9] S. Takagi, T. Saijo, Study of a Piston Pump without Valves (1st Report, On a Pipe-capacity-system with a T-junction), Bull. JSME 26 (1983) 1366–1372.
[10] S. Takagi, K. Takahashi, Study of a piston pump without valves, Bull. JSME 28 (1985) 831–836.
[11] J.K. Hale, Ordinary Differential Equations, Wiley-Interscience, New York, 1969.
[12] D. Auerbach, W. Moehring, M. Moser, An analytic approach to the Liebau phenomenon of valveless pumping, Cardiovasc. Eng. 4 (2004) 201–208.
[13] G. Propst, A. Borzì, Numerical investigation of periodically excited valveless pumping, in: W. Klonowski (Ed.), Proceedings of Euroattractor 2002, Pabst Science Publishers, Lengerich, 2004.
[14] H. Ockendon, A.B. Tayler, Inviscid Fluid Flows, Springer-Verlag, New York, 1983.
[15] R. Reissig, Über einen allgemeinen Typ erzwungener nichtlinearer Schwingungen zweiter Ordnung, Rend. Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Natur. 56 (1974) 297–302.
[16] J. Mawhin, J.R. Ward, Nonuniform nonresonance conditions at the two first eigenvalues for periodic solutions of forced Lienard and Duffing equations, Rocky Mountain J. Math. 12 (1982) 643–654.
[17] J.M.T. Thompson, H.B. Stewart, Nonlinear Dynamics and Chaos, John Wiley & Sons, Chichester, 1987.
[18] P.M. Morse, K.U. Ingard, Theoretical Acoustics, McGraw-Hill, New York, 1968.
[19] K. Ito, F. Kappel, Evolution Equations and Approximations, World Scientific, New Jersey, 2002.
[20] K. Schmitt, R.C. Thompson, Nonlinear Analysis and Differential Equations — An Introduction, University of Utah and Utah State University, 2000.