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(continued after index)

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3.2 Continuity and Convergence of Nets

One can, of course, define convergence for sequences in topological spaces as for metric spaces.

Definition 3.2.1. *Let $(X, \mathcal{T})$ be a topological space. A sequence $(x_n)_{n=1}^{\infty}$ in X is said to* converge *to $x \in X$ if, for each $N \in \mathcal{N}_x$, there is $n_N \in \mathbb{N}$ such that $x_n \in N$ for all $n \geq n_N$.*

This definition is perfectly fine, but if one attempts to prove analogues of results for convergent sequences in metric spaces, problems show up. Proposition 2.3.4, for example, is no longer true in general topological spaces.

Example 3.2.2. Let X be an uncountable set equipped with the topology of Example 3.1.2(e); that is, the open sets are $\varnothing$ and those with a countable complement. Fix a point $x_0 \in X$. Then $X \setminus \{x_0\}$ is not closed, so that $\overline{X \setminus \{x_0\}} = X$ must hold. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $X \setminus \{x_0\}$, and let $U := X \setminus \{x_1, x_2, \ldots\}$. Due to the nature of our topology, U is open and thus is a neighborhood of x_0. However, $x_n \notin U$ for all $n \in \mathbb{N}$ by definition, so that $(x_n)_{n=1}^{\infty}$ cannot converge to x_0.

A less contrived example for the failure of Proposition 2.3.4 in general topological spaces is given in Exercise 11 below.

So, how are we going to define continuity on arbitrary topological spaces? Of course, we could try it via sequences as for metric spaces, but in view of Example 3.2.2, we are likely to run into unexpected difficulties. Of the four equivalent conditions of Theorem 2.3.7, the fourth one doesn't make any explicit reference to a metric. We thus use it as the definition of continuity.

Definition 3.2.3. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then $f : X \to Y$ is said to be* continuous at $x_0 \in X$ *if $f^{-1}(N) \in \mathcal{N}_{x_0}$ for each $N \in \mathcal{N}_{f(x_0)}$. If f is continuous at every point of X, we simply call f* continuous.

As for metric spaces (Corollary 2.3.10), we have (with an identical proof):

Proposition 3.2.4. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then the following are equivalent for $f : X \to Y$.*

(i) *f is continuous.*
(ii) *$f^{-1}(U)$ is open in X for each open subset U of Y.*
(iii) *$f^{-1}(F)$ is closed in X for each closed subset F of Y.*

Examples 3.2.5. (a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces such that X is discrete. Then every function $f : X \to Y$ is continuous.

(b) Let $(X, \mathcal{T}_X)$ be a chaotic topological space (i.e., $\mathcal{T}_X = \{\varnothing, X\}$), let $(Y, \mathcal{T}_Y)$ be a Hausdorff space, and let $f : X \to Y$ be continuous. We claim that f has to be constant. Otherwise, there are $x, y \in X$ such that $f(x) \neq f(y)$. Choose $U, V \in \mathcal{T}_Y$ such that $f(x) \in U$, $f(y) \in V$, and $U \cap V = \varnothing$. Since f is continuous, $f^{-1}(U)$ is open in X and nonempty because it contains x. Due to the definition of a chaotic topology, $f^{-1}(U)$ must equal X. This, in turn, implies that $y \in f^{-1}(U)$ and therefore $f(y) \in U$, which is a contradiction. (The demand that $(Y, \mathcal{T}_Y)$ be Hausdorff cannot be dropped; if Y is also chaotic, every map from X to Y is continuous.)

These two examples show drastically, how the continuous functions between topological spaces depend on the topologies with which those spaces are equipped. Sometimes, we want particular maps between sets to be continuous and will adjust the topologies accordingly.

Definition 3.2.6. *Let X be a nonempty set, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. We say that $\mathcal{T}_1$ and $\mathcal{T}_2$ are* comparable *if $\mathcal{T}_1 \subset \mathcal{T}_2$ or $\mathcal{T}_2 \subset \mathcal{T}_1$. If $\mathcal{T}_1 \subset \mathcal{T}_2$, we say that $\mathcal{T}_1$ is* coarser *than $\mathcal{T}_2$ or, equivalently, that $\mathcal{T}_2$ is* finer *than $\mathcal{T}_1$.*

Clearly, $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if and only if $\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous.

Proposition 3.2.7. *Let X be a set, let $((Y_i, \mathcal{T}_i))_{i \in \mathbb{I}}$ be a family of topological spaces, and let $f_i : X \to Y_i$ be a function for each $i \in \mathbb{I}$. Then there is a coarsest topology on X such that each of the maps f_i is continuous. The collection $\{f_i^{-1}(U) : i \in \mathbb{I}, U \in \mathcal{T}_i\}$ is a subbase for this topology.*

Proof. Let $\mathcal{S} := \{f_i^{-1}(U) : i \in \mathbb{I}, U \in \mathcal{T}_i\}$, and let $\mathcal{T}$ be the collection of all unions of finite intersections of sets from $\mathcal{S}$. Then $\mathcal{T}$ is a topology on X having $\mathcal{S}$ as a subbase and turning each $f_i : (X, \mathcal{T}) \to (Y_i, \mathcal{T}_i)$ into a continuous function by Proposition 3.2.4.

Let $\mathcal{T}'$ be any topology on X such that $f_i : (X, \mathcal{T}') \to (Y_i, \mathcal{T}_i)$ is continuous for each $i \in \mathbb{I}$. By Proposition 3.2.4, it is clear that $\mathcal{S} \subset \mathcal{T}'$ and thus $\mathcal{T} \subset \mathcal{T}'$; that is. $\mathcal{T}$ is coarser than $\mathcal{T}'$. □

The relevance of Proposition 3.2.7 becomes clear in the next section.

As we have seen at the beginning of this section, sequences are too limited an instrument in the study of topological spaces (and the continuous functions on them). Nevertheless, arguments involving sequences were often very convenient when we studied metric spaces in the previous chapter. Isn't there somehow a way to "rescue" sequences for the use in general topological spaces? There is, but it comes at a price: instead of just $\mathbb{N}$, we have to admit more general index sets.

Definition 3.2.8. *An ordered set $\mathbb{A}$ is called* directed *if, for any $\alpha, \beta \in \mathbb{A}$, there is $\gamma \in \mathbb{A}$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.*

Examples 3.2.9. (a) Every totally ordered set is directed. In particular, $\mathbb{N}$ is directed.

(b) Let S be any set. Then $\mathfrak{P}(S)$, ordered by set inclusion, is directed.

Definition 3.2.10. *A* net *or a* generalized sequence *in a set S is a function from a directed set into S.*

If $\mathbb{A}$ is a directed set serving as the domain of some net, we often use the notation $(x_\alpha)_{\alpha\in\mathbb{A}}$; if no ambiguity can arise about $\mathbb{A}$, we sometimes simply write $(x_\alpha)_\alpha$. It is clear that a sequence is just a particular case of a net.

Example 3.2.11. Let $a < b$ be real numbers. A *partition* $\mathcal{P}$ of $[a,b]$ consists of finitely many numbers $t_0, t_1, \ldots, t_n$ such that $a = t_0 < t_1 < \cdots < t_n = b$. We write

$$\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}. \qquad (*)$$

The collection $\mathbb{P}$ of all partitions of $[a,b]$ is naturally ordered; given $\mathcal{P}$ as in $(*)$ and

$$\mathcal{Q} = \{a = s_0 < s_1 < \cdots < s_m = b\},$$

we define

$$\mathcal{P} \preceq \mathcal{Q} \quad :\Longleftrightarrow \quad \{t_0, t_1, \ldots, t_n\} \subset \{s_0, s_1, \ldots, s_m\}.$$

Clearly, this turns $\mathbb{P}$ into a directed set. For $\mathcal{P} \in \mathbb{P}$ as in $(*)$, a *tag* associated with $\mathcal{P}$ is an n-tuple $\xi = (\xi_1, \ldots, \xi_n)$ with $\xi_j \in [t_{j-1}, t_j]$ for $j = 1, \ldots, n$. Given $\mathcal{P} \in \mathbb{P}$, an associated tag ξ, and a function $f : [a,b] \to \mathbb{R}$, the corresponding *Riemann sum* is defined as

$$R(f; \mathcal{P}, \xi) := \sum_{j=1}^{n} f(\xi_j)(t_j - t_{j-1}).$$

If, for each $\mathcal{P} \in \mathbb{P}$, we fix a tag $\xi_\mathcal{P}$, the Riemann sums $(R(f; \mathcal{P}, \xi_\mathcal{P}))_{\mathcal{P}\in\mathbb{P}}$ form a net in $\mathbb{R}$.

It is clear how Definition 3.2.1 has to be extended to general nets.

Definition 3.2.12. *Let $(X, \mathcal{T})$ be a topological space. A net $(x_\alpha)_{\alpha\in\mathbb{A}}$ in X is said to* converge *to $x \in X$ if, for each $N \in \mathcal{N}_x$, there is $\alpha_N \in \mathbb{A}$ such that $x_\alpha \in N$ for all $\alpha \in \mathbb{A}$ such that $\alpha_N \preceq \alpha$. We then say that x is a* limit *of $(x_\alpha)_{\alpha\in\mathbb{A}}$ and write $x = \lim_\alpha x_\alpha$ or $x_\alpha \to x$.*

The notation $\lim_\alpha x_\alpha = x$ has to be handled with caution: a limit of a net in a general topological space need not be unique. An easy, albeit extreme, example is a chaotic topological space with more than one point: every net converges to every point. Writing $\lim_\alpha x_\alpha = x$ therefore does *not* mean that x is *the* limit of the net $(x_\alpha)_\alpha$, but rather that x is one (of possibly many) limits of that net.

We now give a few examples of convergent nets that may not be all that unfamiliar.

Examples 3.2.13. (a) In the situation of Example 3.2.11, suppose that f is Riemann integrable on $[a,b]$. Then $(R(f; \mathcal{P}, \xi_\mathcal{P}))_{\mathcal{P}\in\mathbb{P}}$ converges, namely

$$\lim_{\mathcal{P}} R(f; \mathcal{P}, \xi_\mathcal{P}) = \int_a^b f(t)\,dt.$$

Conversely, if $(R(f; \mathcal{P}, \xi_\mathcal{P}))_{\mathcal{P}\in\mathbb{P}}$ converges for each choice $(\xi_\mathcal{P})_{\mathcal{P}\in\mathbb{P}}$ of tags, and if this limit is independent of $(\xi_\mathcal{P})_{\mathcal{P}\in\mathbb{P}}$, then f is Riemann integrable.

(b) Let $S \neq \varnothing$ be a set, and let (Y, d) be a metric space. We say that a net $(f_\alpha)_{\alpha\in\mathbb{A}}$ in $F(S,Y)$ converges to $f \in F(S,Y)$ *pointwise* on S if $\lim_\alpha f_\alpha(x) = f(x)$, for each $x \in S$; that is, if for each $x \in S$ and for each $\epsilon > 0$, there is $\alpha_{x,\epsilon} \in \mathbb{A}$ with

$$d(f_\alpha(x), f(x)) < \epsilon \qquad (\alpha \in \mathbb{A},\ \alpha_{x,\epsilon} \preceq \alpha);$$

and we say that $(f_\alpha)_{\alpha\in\mathbb{A}}$ converges to f *uniformly* on S if, for each $\epsilon > 0$, there is $\alpha_\epsilon \in \mathbb{A}$ with

$$d(f_\alpha(x), f(x)) < \epsilon \qquad (x \in S,\ \alpha \in \mathbb{A},\ \alpha_\epsilon \preceq \alpha).$$

(In the situation where $S \subset \mathbb{R}^n$, $Y = \mathbb{R}$, and with $(f_\alpha)_{\alpha\in\mathbb{A}}$ a sequence, these definitions are just the familiar ones.)
We now show how to express these notions of convergence in terms of the topologies $\mathcal{T}_\mathcal{C}$ (discussed in Example 3.1.11), where $\varnothing \neq \mathcal{C} \subset \mathfrak{P}(X)$ is stable under finite unions. Let $(f_\alpha)_{\alpha\in\mathbb{A}}$ be a net in $F(S,Y)$, and let $f \in F(S,Y)$. Since the collection of sets $\{N_{f,C,\epsilon} : C \in \mathcal{C},\ \epsilon > 0\}$ forms a base for $\mathcal{N}_f$, it is clear that $\lim_\alpha f_\alpha = f$ with respect to $\mathcal{T}_\mathcal{C}$ if and only if, for each $C \in \mathcal{C}$ and for each $\epsilon > 0$, there is $\alpha_{C,\epsilon} \in \mathbb{A}$ such that

$$\sup_{x\in C} d(f_\alpha(x), f(x)) < \epsilon \qquad (\alpha \in \mathbb{A},\ \alpha_{C,\epsilon} \preceq \alpha).$$

Let $\mathcal{F}$ be the collection of all finite subsets of S, and suppose that $(f_\alpha)_{\alpha\in\mathbb{A}}$ in $F(S,Y)$ converges pointwise to $f \in F(S,Y)$. Let $F \in \mathcal{F}$, say $F = \{x_1, \ldots, x_n\}$, and let $\epsilon > 0$. Then, for $j = 1, \ldots, n$, there is $\alpha_{j,\epsilon} \in \mathbb{A}$ such that

$$d(f_\alpha(x_j), f(x_j)) < \epsilon \qquad (\alpha \in \mathbb{A},\ \alpha_{j,\epsilon} \preceq \alpha).$$

Since $\mathbb{A}$ is directed, there is $\alpha_{F,\epsilon} \in \mathbb{A}$ with $\alpha_{j,\epsilon} \preceq \alpha_{F,\epsilon}$ for $j = 1, \ldots, n$. It follows that

$$\max_{j=1,\ldots,n} d(f_\alpha(x_j), f(x_j)) < \epsilon \qquad (\alpha \in \mathbb{A},\ \alpha_{F,\epsilon} \preceq \alpha),$$

so that we have convergence with respect to $\mathcal{T}_\mathcal{F}$. Conversely (and straightforward to verify), we have pointwise convergence whenever we have convergence with respect to $\mathcal{T}_\mathcal{F}$. We therefore sometimes call $\mathcal{T}_\mathcal{F}$ *the topology of pointwise convergence* (on S).
Similarly, uniform convergence is nothing but convergence with respect to $\mathcal{T}_{\{S\}}$, so that we sometimes refer to this topology as to *the topology of uniform convergence*.

As we saw at the beginning of this section, sequences are an inadequate instrument in the study of general topological spaces because, among other things, Proposition 2.3.4 is no longer true. As it turns out, this changes if we only replace sequences by nets.

Proposition 3.2.14. *Let $(X, \mathcal{T})$ be a topological space, and let $S \subset X$. Then $\overline{S}$ consists of those points in X that are a limit of a net in S.*

Proof. Let $x \in \overline{S}$. Turn $\mathcal{N}_x$ into a directed set via reversed set inclusion; that is, by letting

$$M \preceq N \quad :\Longleftrightarrow \quad N \subset M \qquad (N, M \in \mathcal{N}_x).$$

By Proposition 3.1.18, there is, for each $N \in \mathcal{N}_x$, an element $x_N \in N \cap S$. Then $(x_N)_{N \in \mathcal{N}_x}$ is a net in S such that $x = \lim_N x_N$.

Let $(x_\alpha)_{\alpha \in \mathbb{A}}$ be a net in S such that $x = \lim_\alpha x_\alpha$, and assume that $x \in U := X \setminus \overline{S}$. Then there is $\alpha_U \in \mathbb{A}$ such that $x_\alpha \in U \subset X \setminus S$ for all $\alpha \in \mathbb{A}$ such that $\alpha_U \preceq \alpha$, which is impossible. □

Corollary 3.2.15. *Let $(X, \mathcal{T})$ be a topological space. Then $F \subset X$ is closed if and only if every net in F that converges in X has its limits in F.*

Proposition 3.2.14 yields further examples for the nonuniqueness of limits in general topological spaces (and more natural ones than chaotic spaces).

Example 3.2.16. Let R be a commutative ring with identity that is an integral domain. The singleton subset $\{(0)\}$ is then dense in all of $\operatorname{Spec}(R)$ (Example 3.1.22). By Proposition 3.2.14, the constant net $((0))_\alpha$, no matter what the index set is, therefore converges to every point in $\operatorname{Spec}(R)$. If R is not a field (e.g., $R = \mathbb{Z}$), then $\operatorname{Spec}(R)$ has other elements besides (0). In this case, the net $((0))_\alpha$ has several limits, namely each point in $\operatorname{Spec}(R)$.

Our next proposition shows that Definition 3.1.3 is crucial when it comes to uniqueness of the limit in topological spaces.

Proposition 3.2.17. *The following are equivalent for a topological space $(X, \mathcal{T})$.*

(i) *X is Hausdorff.*
(ii) *Every convergent net in X has a unique limit.*

Proof. (i) $\Longrightarrow$ (ii): Let $(x_\alpha)_{\alpha \in \mathbb{A}}$ be a net in X such that there are $x, x' \in X$ such that $x_\alpha \to x$ and $x_\alpha \to x'$, but $x \neq x'$. By Definition 3.1.3, there are $N \in \mathcal{N}_x$ and $M \in \mathcal{N}_{x'}$ such that $N \cap M = \varnothing$. By the definition of convergence, there are $\alpha_N, \alpha_M \in \mathbb{A}$ such that $x_\alpha \in N$ for all $\alpha \in \mathbb{A}$ with $\alpha_N \preceq \alpha$ and $x_\alpha \in M$ for all $\alpha \in \mathbb{A}$ with $\alpha_M \preceq \alpha$. Since $\mathbb{A}$ is directed, we can find $\alpha \in \mathbb{A}$ with both $\alpha_N \preceq \alpha$ and $\alpha_M \preceq \alpha$. Consequently, $x_\alpha \in N \cap M$ must hold for such α, which is impossible because $N \cap M = \varnothing$.

(ii) $\Longrightarrow$ (i): Assume that X is not Hausdorff. Then there are $x, y \in X$ with $x \neq y$ such that $N \cap M \neq \varnothing$ for all $N \in \mathcal{N}_x$ and $M \in \mathcal{N}_y$. Turn $\mathcal{N}_x \times \mathcal{N}_y$ into a directed set by letting, for $(N_1, M_1), (N_2, M_2) \in \mathcal{N}_x \times \mathcal{N}_y$,

$$(N_1, M_1) \preceq (N_2, M_2) \quad :\Longleftrightarrow \quad N_2 \subset N_1 \text{ and } M_2 \subset M_1.$$

For any $(N, M) \in \mathcal{N}_x \times \mathcal{N}_y$ pick $x_{(N,M)} \in N \cap M$. It is routinely checked that $\big(x_{(N,M)}\big)_{(N,M) \in \mathcal{N}_x \times \mathcal{N}_y}$ is a net in X that converges to both x and y. □

As it turns out, with nets instead of sequences, we have an analogue of Theorem 2.3.7.

Theorem 3.2.18. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $x_0 \in X$. Then the following are equivalent for a function $f\colon X \to Y$.*

(i) *f is continuous at x_0.*
(ii) *For each net $(x_\alpha)_\alpha$ in X with $\lim_\alpha x_\alpha = x_0$, we have $\lim_\alpha f(x_\alpha) = f(x_0)$.*

Proof. (i) $\Longrightarrow$ (ii): Let $(x_\alpha)_\alpha$ be a net in X with $\lim_\alpha x_\alpha = x_0$, and let $N \in \mathcal{N}_{f(x_0)}$. By the definition of continuity at a point, we have $f^{-1}(N) \in \mathcal{N}_{x_0}$. Consequently, there is an index $\alpha_{f^{-1}(N)}$ such that $x_\alpha \in f^{-1}(N)$ for all α such that $\alpha_{f^{-1}(N)} \preceq \alpha$. But this means that $f(x_\alpha) \in N$ for those α. Since $N \in \mathcal{N}_{f(x_0)}$ was arbitrary, this means that $\lim_\alpha f(x_\alpha) = f(x_0)$.

(ii) $\Longrightarrow$ (i): Let $N \in \mathcal{N}_{f(x_0)}$, and assume towards a contradiction that $f^{-1}(N) \notin \mathcal{N}_{x_0}$. It follows that $U \not\subset f^{-1}(N)$ for each open subset U of X containing x_0. Let $\mathcal{U}_{x_0}$ denote the collection of all open subsets of X containing x_0. Then $\mathcal{U}_{x_0}$ is a directed subset of $\mathcal{N}_{x_0}$ (with respect to reversed set inclusion). By our assumption, we can choose $x_U \in U \setminus f^{-1}(N)$ for each $U \in \mathcal{U}_{x_0}$. It is clear that $\lim_U x_U = x_0$. However, since $f(x_U) \notin N$ for all $U \in \mathcal{U}_{x_0}$, it follows that $f(x_U) \not\to f(x_0)$. □

We can use Theorem 3.2.18 to give an alternative description of the topology introduced in Proposition 3.2.7.

Proposition 3.2.19. *Let X be a set, let $((Y_i, \mathcal{T}_i))_{i \in \mathbb{I}}$ be a family of topological spaces, let $f_i\colon X \to Y_i$ be a function for each $i \in \mathbb{I}$, and let $\mathcal{T}$ be the coarsest topology on X such that each of the maps f_i is continuous. Then a net $(x_\alpha)_{\alpha \in \mathbb{A}}$ converges in $(X, \mathcal{T})$ to $x \in X$ if and only if $(f_i(x_\alpha))_{\alpha \in \mathbb{A}}$ converges to $f_i(x)$ in $(Y_i, \mathcal{T}_i)$ for each $i \in \mathbb{I}$.*

Proof. Suppose that $x = \lim_\alpha x_\alpha$ in $(X, \mathcal{T})$. Since each $f_i\colon X \to Y_i$ is continuous, it follows that $\lim_\alpha f_i(x_\alpha) = f_i(x)$ in $(Y_i, \mathcal{T}_i)$ for each $i \in \mathbb{I}$.

Conversely, suppose that $\lim_\alpha f_i(x_\alpha) = f_i(x)$ in $(Y_i, \mathcal{T}_i)$ for each $i \in \mathbb{I}$. Let N be a neighborhood of x in $(X, \mathcal{T})$. By the definition of a neighborhood, there is an open subset U of $(X, \mathcal{T})$ with $x \in U \subset N$. By the description of a subbase for $\mathcal{T}$ given in Proposition 3.2.7, there are $i_1, \ldots, i_n \in \mathbb{I}$ along with sets $U_j \in \mathcal{T}_{i_j}$ for $j = 1, \ldots, n$ such that

$$x \in f_{i_1}^{-1}(U_1) \cap \cdots \cap f_{i_n}^{-1}(U_n) \subset U.$$

Since $\lim_\alpha f_{i_j}(x_\alpha) = f_{i_j}(x)$ for $j = 1, \ldots, n$, there are $\alpha_1, \ldots, \alpha_n \in \mathbb{A}$ such that

$$f_{i_j}(x_\alpha) \in U_j \qquad (j = 1, \ldots, n,\ \alpha \in \mathbb{A},\ \alpha_j \preceq \alpha).$$

Since $\mathbb{A}$ is directed, there is $\alpha_N \in \mathbb{A}$ such that $\alpha_j \preceq \alpha_N$ for $j = 1, \ldots, n$. Consequently, we have

$$x_\alpha \in f_{i_1}^{-1}(U_1) \cap \cdots \cap f_{i_n}^{-1}(U_n) \subset U \subset N \qquad (\alpha \in \mathbb{A},\ \alpha_N \preceq \alpha),$$

so that $x = \lim_\alpha x_\alpha$ holds in $(X, \mathcal{T})$. □

Exercises

1. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, and $(Z, \mathcal{T}_Z)$ be topological spaces, let $g : X \to Y$ be continuous at $x_0 \in X$, and let $f : Y \to Z$ be continuous at $g(x_0) \in Y$. Show that $f \circ g$ is continuous at x_0.
2. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $\mathcal{B}_Y$ and $\mathcal{S}_Y$ be a base and a subbase, respectively, for $\mathcal{T}_Y$. Show that $f : X \to Y$ is continuous if and only if $f^{-1}(B) \in \mathcal{T}_X$ for each $B \in \mathcal{B}_Y$ and if and only if $f^{-1}(S) \in \mathcal{T}_X$ for each $S \in \mathcal{S}_Y$.
3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let X_1 and X_2 be two subspaces of X such that $X_1 \cup X_2 = X$ which are both open or both closed. Show that $f : X \to Y$ is continuous if and only if $f|_{X_j}$ is continuous for $j = 1, 2$.
4. Let $(X, \mathcal{T})$ be a first countable topological space. Show that, for any $S \subset X$, the closure $\overline{S}$ of S consists of those points $x \in X$ such that there is a *sequence* in S converging to x.
5. Let R and S be commutative rings with identity, and let $\phi : R \to S$ be a unital ring homomorphism, that is, an additive and multiplicative map that maps the identity of R to the identity of S. Show that:
   (a) For any ideal I of S, its inverse image $\phi^{-1}(I)$ is an ideal of R; if I is proper or prime, respectively, the same is true for $\phi^{-1}(I)$;
   (b) The map
   $$\phi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R), \quad \mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$$
   is continuous if both $\mathrm{Spec}(S)$ and $\mathrm{Spec}(R)$ are equipped with their respective Zariski topology.
6. Let $(X, \mathcal{T})$ be a topological space, and let $\approx$ be an equivalence relation on X. Show that the quotient topology on $X/\!\approx$ (see Exercise 3.1.8) is the finest topology on $X/\!\approx$ making the *quotient map*
   $$X \to X/\!\approx, \quad x \mapsto [x]$$
   continuous.
7. Let $(X, \mathcal{T})$ be a topological space, and let $f, g : X \to \mathbb{F}$ be continuous. Show that $f + g$, fg, and (provided that $f(x) \neq 0$ for $x \in X$) $\frac{1}{f}$ are also continuous.
8. Let $(X, \mathcal{T})$ be a topological space, and let $f, g : X \to \mathbb{R}$ be continuous. Show that $\max\{f, g\}$ and $\min\{f, g\}$ are continuous.
9. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, let $S \subset X$, and let $f : X \to Y$ be continuous. Show that $f(\overline{S}) \subset \overline{f(S)}$.
10. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces such that Y is Hausdorff, let D be dense in X, and let $f, g : X \to Y$ be continuous functions such that $f|_D = g|_D$. Show that $f = g$. What happens if we drop the demand that Y be Hausdorff?
11. Let $\mathcal{F}$ denote the collection of all finite subsets of $[0, 1]$, and let $F([0, 1], \mathbb{F})$ be equipped with the topology $\mathcal{T}_{\mathcal{F}}$. Show that the continuous functions from $[0, 1]$ to $\mathbb{F}$ are dense in $(F([0, 1], \mathbb{F}), \mathcal{T}_{\mathcal{F}})$, but that there is no *sequence* of continuous functions from $[0, 1]$ to $\mathbb{F}$ converging to $f : [0, 1] \to \mathbb{F}$ given by
    $$f(t) := \begin{cases} 1, & t \notin [0, 1] \cap \mathbb{Q}, \\ 0, & t \in [0, 1] \cap \mathbb{Q}. \end{cases}$$
    (*Hint*: Recall that convergence in $\mathcal{T}_{\mathcal{F}}$ is pointwise convergence by Example 3.2.13(b), and use Exercise 2.4.7 to show that f cannot be the limit of a *sequence* of continuous functions.)

12. Let (X, d) be a metric space. A net $(x_\alpha)_{\alpha \in \mathbb{A}}$ in X is called a *Cauchy net* if, for each $\epsilon > 0$, there is $\alpha_\epsilon \in \mathbb{A}$ such that $d(x_\alpha, x_\beta) < \epsilon$ for all $\alpha, \beta \in \mathbb{A}$ such that $\alpha_\epsilon \preceq \alpha, \beta$.
    (a) Show that every convergent net in X is a Cauchy net.
    (b) Show that X is complete if and only if each Cauchy net in X converges.

3.3 Compactness

Compactness for topological spaces is defined as in the metric situation.

Definition 3.3.1. *Let $(X, \mathcal{T})$ be a topological space, and let $S \subset X$. An* open cover *for S is a collection $\mathcal{U}$ of open subsets of X such that $S \subset \bigcup\{U : U \in \mathcal{U}\}$.*

Definition 3.3.2. *A subset K of a topological space $(X, \mathcal{T})$ is called* compact *if, for each open cover $\mathcal{U}$ of K, there are $U_1, \ldots, U_n \in \mathcal{U}$ such that $K \subset U_1 \cup \cdots \cup U_n$.*

Before we flesh out this definition with examples (nonmetrizable ones), we introduce yet another definition.

Definition 3.3.3. *A topological space $(X, \mathcal{T})$ has the* finite intersection property *if, for any collection $\mathcal{F}$ of closed subsets of X such that $\bigcap\{F : F \in \mathcal{F}\} = \varnothing$, there are $F_1, \ldots, F_n \in \mathcal{F}$ such that $F_1 \cap \cdots \cap F_n = \varnothing$.*

The following is straightforward (just pass to complements).

Proposition 3.3.4. *Let $(X, \mathcal{T})$ be a topological space. Then the following are equivalent.*

(i) *X is compact.*
(ii) *X has the finite intersection property.*

The reason why we introduced the finite intersection property at all is that it is sometimes easier to verify than compactness.

Example 3.3.5. Let R be a commutative ring with identity. We claim that $\mathrm{Spec}(R)$ has the finite intersection property (and thus is compact). Let $\mathcal{I}$ be a family of ideals of R such that
$$\bigcap\{V(I) : I \in \mathcal{I}\} = V\left(\sum\{I : I \in \mathcal{I}\}\right) = \varnothing.$$
This implies that $\sum\{I : I \in \mathcal{I}\} = R$: otherwise, Exercise 1.3.4 would yield a maximal ideal containing $\sum\{I : I \in \mathcal{I}\}$, and since maximal ideals are prime, this would be a contradiction. Since $1 \in \sum\{I : I \in \mathcal{I}\}$, there are $I_1, \ldots, I_n \in \mathcal{I}$ and $a_j \in I_j$ for $j = 1, \ldots, n$ such that $1 = \sum_{j=1}^n a_j$. It follows that $1 \in I_1 + \cdots + I_n$. Since the only ideal of R containing 1 is R itself, this means that $I_1 + \cdots + I_n = R$, so that
$$\varnothing = V(R) = V(I_1 + \cdots + I_n) = V(I_1) \cap \cdots \cap V(I_n).$$
Consequently, $\mathrm{Spec}(R)$ has the finite intersection property.

As can be expected, many of the properties of compact metric spaces do, in fact, hold for compact topological spaces.

Proposition 3.3.6. *Let $(X, \mathcal{T})$ be a topological space, and let Y be a subspace of X.*

(i) *If X is compact and Y is closed in X, then Y is compact.*
(ii) *If X is Hausdorff and Y is compact, then Y is closed in X.*

Proof. (i) is proved as for metric spaces (Proposition 2.5.4(i)).

For (ii), let $x \in X \setminus Y$. For each $y \in Y$, there are open subsets U_y, V_y of X such that $x \in U_y$, $y \in V_y$, and $U_y \cap V_y = \varnothing$. Since $\{V_y : y \in Y\}$ is an open cover for Y, there are $y_1, \ldots, y_n \in Y$ such that

$$Y \subset V_{y_1} \cup \cdots \cup V_{y_n}.$$

Letting $U := U_{y_1} \cap \cdots \cap U_{y_n}$, we obtain that

$$U \cap Y \subset U \cap (V_{y_1} \cup \cdots \cup V_{y_n}) = \varnothing,$$

so that $X \setminus Y$ is a neighborhood of x. Since $x \in X \setminus Y$ was arbitrary, this shows that $X \setminus Y$ is open. □

The requirement in Proposition 3.3.6(ii) that X be Hausdorff cannot be dropped.

Example 3.3.7. Let R be a commutative ring with identity that is an integral domain, but not a field; for example, $R = \mathbb{Z}$. Then the singleton subset $\{(0)\}$ of $\mathrm{Spec}(R)$ is dense in R, but not closed (otherwise (0) would be the only prime ideal of R making it a field). Nevertheless, as a singleton subset, $\{(0)\}$ is trivially compact.

The following also holds (with an identical proof) as in the metric case.

Proposition 3.3.8. *Let $(K, \mathcal{T}_K)$ be a compact topological space, let $(Y, \mathcal{T}_Y)$ be any topological space, and let $f : K \to Y$ be continuous. Then $f(K)$ is compact.*

Corollary 3.3.9. *Let $(K, \mathcal{T}_K)$ be a nonempty, compact topological space, and let $f : K \to \mathbb{R}$ be continuous. Then f attains both a minimum and a maximum on K.*

We now use Proposition 3.3.8 to prove one of the most useful results on compact topological spaces. We first state another definition.

Definition 3.3.10. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A* homeomorphism *between X and Y is a bijective map $f : X \to Y$ such that both f and f^{-1} are continuous. If there is a homeomorphism between X and Y, the two spaces are called* homeomorphic.

To put it in a nutshell: two homeomorphic topological spaces cannot be told apart as topological spaces; that is., whatever topological property one space has is also enjoyed by the other.

Theorem 3.3.11. *Let $(K, \mathcal{T}_K)$ and $(Y, \mathcal{T}_Y)$ be topological spaces such that K is compact and Y is Hausdorff, and let $f : K \to Y$ be bijective and continuous. Then f is a homeomorphism.*

Proof. We verify Proposition 3.2.4(iii) for f^{-1}. Let $F \subset K$ be closed. Since f is bijective, it is clear that the inverse image of F under f^{-1} is just $f(F)$.

Since K is compact, so is F by Proposition 3.3.6(i). Consequently, $f(F)$ is compact in Y by Proposition 3.3.8 and thus closed in Y (because Y is Hausdorff) by Proposition 3.3.6(ii). This proves Proposition 3.2.4(iii) for f^{-1} and thus the continuity of f^{-1}. □

Corollary 3.3.12. *Let X be a set, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be comparable topologies on X each turning it into a compact Hausdorff space. Then $\mathcal{T}_1 = \mathcal{T}_2$ holds.*

A metric space is compact if and only if each sequence in the space has a convergent subsequence. As we saw in the previous section, sequences are inadequate when it comes to dealing with general topological spaces. We now characterize the compact topological spaces in a similar way involving nets. We first need to define what we mean by a subnet.

Definition 3.3.13. *Let $\mathbb{A}$ and $\mathbb{B}$ be directed sets. A map $\phi : \mathbb{B} \to \mathbb{A}$ is called* cofinal *if, for each $\alpha \in \mathbb{A}$, there is $\beta_\alpha \in \mathbb{B}$ such that $\alpha \preceq \phi(\beta)$ for all $\beta \in \mathbb{B}$ such that $\beta_\alpha \preceq \beta$.*

Definition 3.3.14. *Let S be a nonempty set, and let $(x_\alpha)_{\alpha \in \mathbb{A}}$ and $(y_\beta)_{\beta \in \mathbb{B}}$ be nets in S. Then $(y_\beta)_{\beta \in \mathbb{B}}$ is a* subnet *of $(x_\alpha)_{\alpha \in \mathbb{A}}$ if $y_\beta = x_{\phi(\beta)}$ for a cofinal map $\phi : \mathbb{B} \to \mathbb{A}$.*

If $(y_\beta)_\beta$ is a subnet of $(x_\alpha)_\alpha$, we sometimes write $(x_{\alpha_\beta})_\beta$ instead of $(y_\beta)_\beta$ or $\left(x_{\phi(\beta)}\right)_\beta$.

Of course, a subsequence of a sequence is a subnet. We want to stress, however, that a subnet of a sequence need not be a subsequence anymore: in Example 3.3.22 below, we encounter a sequence that has a convergent subnet, but no convergent subsequence.

Nevertheless, as for sequences, we have the following.

Proposition 3.3.15. *Let $(X, \mathcal{T})$ be a topological space, let $(x_\alpha)_{\alpha \in \mathbb{A}}$ be a net in X, and let $x \in X$ be a limit of $(x_\alpha)_{\alpha \in \mathbb{A}}$. Then each subnet of $(x_\alpha)_{\alpha \in \mathbb{A}}$ converges to x as well.*

Proof. Let $(y_\beta)_{\beta \in \mathbb{B}}$ be a subnet of $(x_\alpha)_\alpha$ with corresponding cofinal map $\phi : \mathbb{B} \to \mathbb{A}$.

Let $N \in \mathcal{N}_x$. Since $x = \lim_\alpha x_\alpha$, there is $\alpha_N \in \mathbb{A}$ such that $x_\alpha \in N$ for $\alpha \in \mathbb{A}$ with $\alpha_N \preceq \alpha$. Since ϕ is cofinal, there is $\beta_N \in \mathbb{B}$ such that $\alpha_N \preceq \phi(\beta)$ for all $\beta \in \mathbb{B}$ with $\beta_N \preceq \beta$. It follows that $y_\beta = x_{\phi(\beta)} \in N$ for all $\beta \in \mathbb{B}$ with $\beta_N \preceq \beta$. □

The following definition and proposition are also helpful.

Definition 3.3.16. *Let $(X, \mathcal{T})$ be a topological space, and let $(x_\alpha)_{\alpha\in\mathbb{A}}$ be a net in X. A point $x \in X$ is an* accumulation point *for $(x_\alpha)_{\alpha\in\mathbb{A}}$ if, for each $\alpha \in \mathbb{A}$ and for each $N \in \mathcal{N}_x$, there is $\beta \in \mathbb{A}$ with $\alpha \preceq \beta$ such that $x_\beta \in N$.*

Proposition 3.3.17. *Let $(X, \mathcal{T})$ be a topological space, and let $(x_\alpha)_{\alpha\in\mathbb{A}}$ be a net in X. Then the following are equivalent for $x \in X$.*

(i) *x is an accumulation point of $(x_\alpha)_{\alpha\in\mathbb{A}}$.*
(ii) *There is a subnet of $(x_\alpha)_{\alpha\in\mathbb{A}}$ converging to x.*

Proof. (i) $\Longrightarrow$ (ii): Let $\mathbb{B} := \mathbb{A} \times \mathcal{N}_x$. For $(\alpha_1, N_1), (\alpha_2, N_2) \in \mathbb{B}$ define:

$$(\alpha_1, N_1) \preceq (\alpha_2, N_2) \quad :\Longleftrightarrow \quad \alpha_1 \preceq \alpha_2 \text{ and } N_2 \subset N_1.$$

This turns $\mathbb{B}$ into a directed set. Let $(\alpha, N) \in \mathbb{B}$. By the definition of an accumulation point, there is $\phi(\alpha, N) \in \mathbb{A}$ with $\alpha \preceq \phi(\alpha, N)$ such that $x_{\phi(\alpha,N)} \in N$. The map $\phi\colon \mathbb{B} \to \mathbb{A}$ is cofinal, and the net $\left(x_{\phi(\alpha,N)}\right)_{(\alpha,N)\in\mathbb{B}}$ converges to x.

(ii) $\Longrightarrow$ (i): Let $\mathbb{B}$ be a directed set, and let $\phi\colon \mathbb{B} \to \mathbb{A}$ be cofinal, such that $\left(x_{\phi(\beta)}\right)_{\beta\in\mathbb{B}}$ converges to x. Let $N \in \mathcal{N}_x$ and let $\alpha \in \mathbb{A}$. Since ϕ is cofinal, there is $\beta_\alpha \in \mathbb{B}$ such that $\alpha \preceq \phi(\beta)$ for all $\beta \in \mathbb{B}$ with $\beta_\alpha \preceq \beta$. Since $x = \lim_\beta x_{\phi(\beta)}$, there is $\beta_N \in \mathbb{B}$ such that $x_{\phi(\beta)} \in N$ for all $\beta \in \mathbb{B}$ with $\beta_N \preceq \beta$. Since $\mathbb{B}$ is directed, there is $\beta \in \mathbb{B}$ such that $\beta_\alpha \preceq \beta$ and $\beta_N \preceq \beta$. It follows that $\alpha \preceq \phi(\beta)$ and $x_{\phi(\beta)} \in N$. □

We can now prove an analogue of Theorem 2.5.10 for general topological spaces.

Theorem 3.3.18. *For a topological space $(X, \mathcal{T})$ the following are equivalent:*

(i) *X is compact.*
(ii) *Each net in X has a convergent subnet.*

Proof. (i) $\Longrightarrow$ (ii): Let $(x_\alpha)_{\alpha\in\mathbb{A}}$ be a net in X. By Proposition 3.3.17, it is sufficient to show that $(x_\alpha)_{\alpha\in\mathbb{A}}$ has an accumulation point. Assume towards a contradiction that $(x_\alpha)_{\alpha\in\mathbb{A}}$ has no accumulation point. Then, for each $x \in X$, there is a neighborhood U_x of x (which we can choose to be open by making it smaller if necessary) and an index $\alpha_x \in \mathbb{A}$ such that $x_\alpha \notin U_x$ for all $\alpha \in \mathbb{A}$ such that $\alpha_x \preceq \alpha$. The collection $\{U_x : x \in X\}$ is an open cover of X. Since X is compact, there are $x_1, \ldots, x_n \in X$ such that

$$X = U_{x_1} \cup \cdots \cup U_{x_n}.$$

Choose $\alpha \in \mathbb{A}$ such that $\alpha_{x_j} \preceq \alpha$ for $j = 1, \ldots, n$, so that

$$x_\alpha \notin U_{x_1} \cup \cdots \cup U_{x_n} = X,$$

which is absurd.

(ii) $\Longrightarrow$ (i): Assume that X is not compact. Then there is an open cover $\mathcal{U}$ of X that has no finite subcover. Let $\mathbb{U}$ be the collection of all finite subsets of $\mathcal{U}$ ordered by set inclusion. For each $\upsilon \in \mathbb{U}$, there is

$$x_\upsilon \in X \setminus \bigcup\{U : U \in \upsilon\} = \bigcap\{X \setminus U : U \in \upsilon\}$$

(otherwise, $\mathcal{U}$ would have a finite subcover). By hypothesis, the net $(x_\upsilon)_{\upsilon\in\mathbb{U}}$ has an accumulation point $x \in X$.

Fix $U \in \mathcal{U}$. By definition, $\{U\} \in \mathbb{U}$ holds. Therefore, by the definition of an accumulation point, there is, for any neighborhood N of x, an element $\upsilon_U \in \mathbb{U}$ with $\{U\} \preceq \upsilon_U$ (meaning $U \in \upsilon_U$) such that $x_{\upsilon_U} \in N$. Since also

$$x_{\upsilon_U} \in \bigcap\{X \setminus V : V \in \upsilon_U\} \subset X \setminus U,$$

it follows that $N \cap (X \setminus U) \neq \varnothing$. Assume that $x \in U$. Since U is then an open neighborhood of x, the preceding argument implies that $U \cap (X \setminus U) \neq \varnothing$, which is absurd. It follows that $x \in X \setminus U$.

Since $U \in \mathcal{U}$ is arbitrary, we eventually obtain

$$x \in \bigcap\{X \setminus U : U \in \mathcal{U}\} = X \setminus \bigcup\{U : U \in \mathcal{U}\} = \varnothing,$$

which is again absurd. □

We shall see in Example 3.3.22 below that the corresponding statement about convergent sub*sequences* of *sequences* in compact spaces becomes false in the general topological setting.

We now prepare the ground for Tychonoff's theorem, one of the most fundamental results in set-theoretic topology.

Definition 3.3.19. *Let $((X_i, \mathcal{T}_i))_{i\in\mathbb{I}}$ be a family of topological spaces, and let $X := \prod_{i\in\mathbb{I}} X_i$. Then the* product topology *on X is the coarsest topology $\mathcal{T}$ on X making the* coordinate projections

$$\pi_i\colon X \to X_i, \quad f \mapsto f(i) \qquad (i \in \mathbb{I})$$

continuous. We call $(X, \mathcal{T})$ the topological product *of $((X_i, \mathcal{T}_i))_{i\in\mathbb{I}}$.*

By Proposition 3.2.7, the product topology does exist, and its open sets are the unions of sets of the form

$$\pi_{i_1}^{-1}(U_1) \cap \cdots \cap \pi_{i_n}^{-1}(U_n),$$

where $i_1, \ldots, i_n \in \mathbb{I}$ and $U_j \in \mathcal{T}_{i_j}$ for $j = 1, \ldots, n$. From Proposition 3.2.19 we know that a net $(f_\alpha)_\alpha$ converges to f in $(X, \mathcal{T})$ if and only if $(f_\alpha(i))_\alpha = (\pi_i(f_\alpha))_\alpha$ converges to $f(i) = \pi_i(f)$ in $(X_i, \mathcal{T}_i)$ for each $i \in \mathbb{I}$: the product topology is the *topology of coordinatewise convergence*. This second fact shows that, implicitly, we have already encountered the product topology in two special instances.

Examples 3.3.20. (a) Let $S \neq \varnothing$ be a set, and let (Y, d) be a metric space. Then $F(S, Y)$ is just another symbol for Y^S, and the product topology on Y^S is the topology of pointwise convergence, that is, $\mathcal{T}_{\mathcal{F}}$, where $\mathcal{F}$ is the collection of finite subsets of S.

(b) Let $((X_n, d_n))_{n=1}^{\infty}$ be a sequence of metric spaces. Then the product topology on $X := \prod_{n=1}^{\infty} X_n$ is metrizable via the metric

$$d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d(x_n, y_n)}$$

for $(x_1, x_2, \ldots), (y_1, y_2, \ldots) \in X$; this follows from Exercise 2.3.1.

Theorem 3.3.21 (Tychonoff's theorem). *Let $((K_i, \mathcal{T}_i))_{i \in \mathbb{I}}$ be a nonempty family of compact topological spaces. Then their topological product is also compact.*

Proof. Let $(f_\alpha)_\alpha$ be a net in $K := \prod_{i \in \mathbb{I}} K_i$. Let $\mathbb{J} \subset \mathbb{I}$ be nonempty and let $f \in K$. We call $(\mathbb{J}, f)$ a *partial accumulation point* of $(f_\alpha)_\alpha$ if $f|_{\mathbb{J}}$ is an accumulation point of $(f_\alpha|_{\mathbb{J}})_\alpha$ in $\prod_{j \in \mathbb{J}} K_j$. Obviously, a partial accumulation point $(\mathbb{J}, f)$ is an accumulation point of $(f_\alpha)_\alpha$ if and only if $\mathbb{J}$ is all of $\mathbb{I}$.

Let $\mathcal{P}$ be the set of all partial accumulation points of $(f_\alpha)_\alpha$. For any two $(\mathbb{J}_f, f), (\mathbb{J}_g, g) \in \mathcal{P}$, define

$$(\mathbb{J}_f, f) \preceq (\mathbb{J}_g, g) \quad :\Longleftrightarrow \quad \mathbb{J}_f \subset \mathbb{J}_g \text{ and } g|_{\mathbb{J}_f} = f.$$

Since K_i is compact for each $i \in \mathbb{I}$, the net $(f_\alpha)_\alpha$ has partial accumulation points $(\{i\}, f_i)$ for each $i \in \mathbb{I}$ by Theorem 3.3.18; in particular, $\mathcal{P}$ is not empty.

Let $\mathcal{Q}$ be a totally ordered subset of $\mathcal{P}$. Let $\mathbb{J}_g := \bigcup\{\mathbb{J}_f : (\mathbb{J}_f, f) \in \mathcal{Q}\}$. Define $g \in K$ by letting $g(j) := f(j)$ for $j \in \mathbb{J}_f$ with $(\mathbb{J}_f, f) \in \mathcal{Q}$ (and arbitrarily on $\mathbb{I} \setminus \mathbb{J}_g$). Since $\mathcal{Q}$ is totally ordered, g is well defined. We claim that $(\mathbb{J}_g, g)$ is a partial accumulation point of $(f_\alpha)_\alpha$. Let $N \subset \prod_{j \in \mathbb{J}_g} K_j$ be a neighborhood of $g|_{\mathbb{J}_g}$. By Proposition 3.2.7, we may suppose that

$$N = \pi_{j_1}^{-1}(U_{j_1}) \cap \cdots \cap \pi_{j_n}^{-1}(U_{j_n}),$$

where $j_1, \ldots, j_n \in \mathbb{J}_g$, and $U_{j_1} \subset K_{j_1}, \ldots, U_{j_n} \subset K_{j_n}$ are open. Let $(\mathbb{J}_h, h) \in \mathcal{Q}$ be such that $\{j_1, \ldots, j_n\} \subset \mathbb{J}_h$ (this is possible because $\mathcal{Q}$ is totally ordered). Since $(\mathbb{J}_h, h)$ is a partial accumulation point of $(f_\alpha)_\alpha$, it follows that there is, for each index α, an index β with $\alpha \preceq \beta$ and

$$f_\beta(j_k) = \pi_{j_k}(f_\beta) \in U_{j_k} \qquad (k = 1, \ldots, n),$$

so that $f_\beta \in N$. Hence, $(\mathbb{J}_g, g)$ is indeed a partial accumulation point of $(f_\alpha)_\alpha$ and thus lies in $\mathcal{P}$.

By Zorn's lemma, $\mathcal{P}$ has a maximal element $(\mathbb{J}_{\max}, f_{\max})$. Assume that $\mathbb{J}_{\max} \subsetneq \mathbb{I}$; that is, there is $i_0 \in \mathbb{I} \setminus \mathbb{J}_{\max}$. Since $(\mathbb{J}_{\max}, f_{\max})$ is a partial accumulation point of $(f_\alpha)_\alpha$, there is a subnet $(f_{\alpha_\beta})_\beta$ of $(f_\alpha)_\alpha$ such that $\pi_j(f_{\alpha_\beta}) \to \pi_j(f_{\max})$ for each $j \in \mathbb{J}_{\max}$. Since K_{i_0} is compact, we may find a subnet $\left(f_{\alpha_{\beta_\gamma}}\right)_\gamma$ of $(f_{\alpha_\beta})_\beta$ such that $\pi_{i_0}(f_{\alpha_{\beta_\gamma}})_\gamma$ converges to some x_{i_0} in K_{i_0}. Define $\tilde{f} \in K$ by letting $\tilde{f}|_{\mathbb{J}_{\max}} = f_{\max}$ and $\tilde{f}(i_0) = x_{i_0}$. It follows that $\left(\mathbb{J}_{\max} \cup \{i_0\}, \tilde{f}\right)$ is a partial accumulation point of $(f_\alpha)_\alpha$, which contradicts the maximality of $(\mathbb{J}_{\max}, f_{\max})$. □

We now use Tychonoff's theorem to exhibit a compact topological space and a sequence in that space without a convergent subsequence:

Example 3.3.22. Let $\mathbb{I}$ be the set of all strictly increasing sequences in $\mathbb{N}$, that is, sequences $(n_k)_{k=1}^{\infty}$ such that $n_1 < n_2 < n_3 < \cdots$. It is not difficult to see that $\mathbb{I}$ has cardinality $\mathfrak{c}$; we won't need this, however. For each $i = (n_k)_{k=1}^{\infty}$ in $\mathbb{I}$, define a sequence $(f_n(i))_{n=1}^{\infty}$ as follows,

$$f_n(i) := \begin{cases} (-1)^k, & \text{if } n = n_k \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(f_n(i))_{n=1}^{\infty}$ is a sequence in $[-1, 1]$ such that the subsequence $(f_{n_k}(i))_{k=1}^{\infty}$ diverges. The topological product $[-1, 1]^{\mathbb{I}}$ is compact by Tychonoff's theorem. Assume that the sequence $(f_n)_{n=1}^{\infty}$ has a convergent subsequence, say $(f_{n_k})_{k=1}^{\infty}$. By the definition of the product topology, this means that $(f_{n_k}(i))_{k=1}^{\infty}$ converges for each $i \in \mathbb{I}$. For $i = (n_k)_{k=1}^{\infty}$, this is impossible according to our construction. Of course, the sequence $(f_n)_{n=1}^{\infty}$ has a convergent subnet.

Another application of Tychonoff's theorem, namely a proof of the Arzelà–Ascoli theorem, is given in Appendix C.

Concluding this section, we deal with a class of possibly noncompact topological spaces which, nevertheless, are reasonably close to being compact.

Definition 3.3.23. *Let $(X, \mathcal{T})$ be a topological space. Then X is said to be* locally compact *if $\mathcal{N}_x$ contains a compact subset of X for each $x \in X$.*

Examples 3.3.24. (a) Compact spaces are (obviously and trivially) locally compact.

(b) $\mathbb{R}^n$ is locally compact, but fails to be compact.

(c) Every discrete topological space is locally compact.

(d) Let E be the linear space $C([0, 1], \mathbb{F})$ equipped with the norm $\|\cdot\|_\infty$, and assume that it is locally compact. Then 0 has a compact neighborhood, say K. By the definition of a neighborhood, there is $\epsilon > 0$ such that $B_\epsilon(0) \subset K$, and since K is closed in E, it follows that $B_\epsilon[0] = \overline{B_\epsilon(0)} \subset K$ as well. Consequently, $B_\epsilon[0]$ must be compact. Since

$$B_\epsilon[0] \to B_1[0], \quad x \mapsto \frac{1}{\epsilon} x$$

is a homeomorphism, this implies that $B_1[0]$ is compact, too. As we have seen in Example 2.5.13, this is not true. (Invoking Theorem B.5 instead

of Example 2.5.13, one can see that no infinite-dimensional, normed space is locally compact.)

Nevertheless, even though $\mathbb{R}^n$ is not compact, it is not far away from a certain compact space. We illustrate this in the case where $n = 2$.

Example 3.3.25. We may identify $\mathbb{C} \cong \mathbb{R}^2$ with the xy-plane in $\mathbb{R}^3$. Let $\mathbb{C}_\infty$ be the unit sphere in $\mathbb{R}^3$ (i.e., those vectors with Euclidean length one): this space is compact. Let $p = (0, 0, 1)$. Then, for each $z = x + iy \in \mathbb{C}$, there is a unique line in $\mathbb{R}^3$ connecting $(x, y, 0)$ with p. Let $\iota(z)$ be the intersection point of this line with $\mathbb{C}_\infty$. Then $\iota : \mathbb{C} \to \mathbb{C}_\infty$ is a continuous injective map whose range is $\mathbb{C}_\infty \setminus \{p\}$ and that is a homeomorphism onto its image. (All this needs to be checked, of course.) The sphere $\mathbb{C}_\infty$ is sometimes referred to as the *Riemann sphere*.

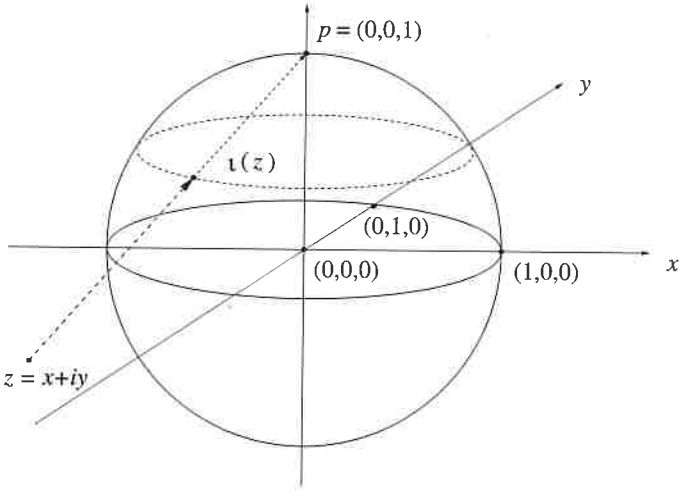

Fig. 3.2: Riemann sphere

Interestingly, a similar construction works for arbitrary locally compact Hausdorff spaces.

Theorem 3.3.26. *Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. Then there is a compact Hausdorff space $(X_\infty, \mathcal{T}_\infty)$, the* one-point compactification *of X, along with a map $\iota : X \to X_\infty$ such that*

(i) *ι is a homeomorphism onto its image, and*
(ii) *$X_\infty \setminus \iota(X)$ consists of just one point.*

Moreover, $(X_\infty, \mathcal{T}_\infty)$ is unique up to homeomorphism.

Proof. Let ∞ be any point not contained in X, and let $X_\infty := X \cup \{\infty\}$. Define

$$\mathcal{T}_\infty := \mathcal{T} \cup \{X_\infty \setminus K : K \subset X \text{ is compact}\}.$$

From this definition, it is clear that $\mathcal{T} \subset \mathcal{T}_\infty$ and that $X \cap U \in \mathcal{T}$ for each $U \in \mathcal{T}_\infty$ (since X is Hausdorff, all its compact subsets are closed).

We claim that $\mathcal{T}_\infty$ is a topology on X_∞. It is clear that $\varnothing, X_\infty \in \mathcal{T}_\infty$.

Let $\mathcal{U}$ be an arbitrary family of sets in $\mathcal{T}_\infty$. If $\mathcal{U} \subset \mathcal{T}$, nothing has to be shown. Hence, we may suppose that there is a compact subset K of X such that $X_\infty \setminus K \in \mathcal{U}$. Since $K \cap U \in \mathcal{T}|_K$ for each $U \in \mathcal{U}$, there is $V \in \mathcal{T}$ such that

$$\bigcup\{K \cap U : U \in \mathcal{U}\} = K \cap V.$$

It follows that

$$\begin{aligned}\bigcup\{U : U \in \mathcal{U}\} &= \bigcup\{K \cap U : U \in \mathcal{U}\} \cup (X_\infty \setminus K)\\ &= (K \cap V) \cup (X_\infty \setminus K)\\ &= V \cup (X_\infty \setminus K)\\ &= X_\infty \setminus ((X \setminus V) \cap K).\end{aligned}$$

Since K is compact and $X \setminus V$ is closed, the subset $(X \setminus V) \cap K$ of X is closed in K and thus compact. It follows that $\bigcup\{U : U \in \mathcal{U}\} \in \mathcal{T}_\infty$.

Let $U_1, U_2 \in \mathcal{T}_\infty$. If $U_1, U_2 \subset X$, nothing has to be shown. Hence, suppose that $U_2 = X_\infty \setminus K_2$ for some compact subset K_2 of X. If $U_1 \subset X$, it follows that

$$U_1 \cap U_2 = U_1 \cap (X_\infty \setminus K_2) = U_1 \cap (X \setminus K_2),$$

which belongs to $\mathcal{T}$ because K_2 is closed. If $U_1 \not\subset X$, there is a compact set $K_1 \subset X$ with $U_1 = X_\infty \setminus K_1$. Since $K_1 \cup K_2$ is again compact (see Exercise 1 below), it follows that

$$U_1 \cap U_2 = X_\infty \setminus (K_1 \cup K_2) \in \mathcal{T}_\infty.$$

All in all, $\mathcal{T}_\infty$ is a topology on X_∞. By the remarks at the beginning of this proof, $\mathcal{T}_\infty|_X$ is just $\mathcal{T}$, so that the canonical embedding $\iota : X \to X_\infty$ is indeed a homeomorphism onto its image.

We claim that $(X_\infty, \mathcal{T}_\infty)$ is Hausdorff. To see this let $x, y \in X_\infty$ be such that $x \neq y$. Without loss of generality, suppose that $y = \infty$. Let $K \subset X$ be a compact neighborhood of x. Then there is $U \in \mathcal{T}$ with $x \in U \subset K$. Let $V := X_\infty \setminus K$. Then $V \in \mathcal{T}_\infty$ with $\infty \in V$, and $U \cap V = \varnothing$ holds.

Let $\mathcal{U}$ be an open cover for X_∞. Then there is $U_0 \in \mathcal{U}$ such that $\infty \in U_0$. Consequently, there is a compact subset K of X with $U_0 = X_\infty \setminus K$. Since $\{X \cap U : U \in \mathcal{U}\}$ is an open cover for K, there are $U_1, \ldots, U_n \in \mathcal{U}$ with $K \subset U_1 \cup \cdots \cup U_n$. It follows that

$$X_\infty = U_0 \cup K = U_0 \cup U_1 \cup \cdots \cup U_n.$$

Consequently, X_∞ is compact.

Let $(X'_\infty, \mathcal{T}'_\infty)$ be any other compact Hausdorff space containing a homeomorphic copy of X such that $X'_\infty \setminus X$ consists of only one element, say ∞'. Define $f: X_\infty \to X'_\infty$ as the identity on X and such that $f(\infty) := \infty'$, and let $U \in \mathcal{T}'_\infty$. Then there are two possibilities: $U \in \mathcal{T}$ or $\infty' \in U$. If $U \in \mathcal{T}$, then $f^{-1}(U) = U \in \mathcal{T} \subset \mathcal{T}_\infty$ holds. Suppose that $\infty' \in U$. Then $K := X_\infty \setminus U$ is a subset of X that is closed in X'_∞ and thus compact. It follows that $f^{-1}(U) = X_\infty \setminus K \in \mathcal{T}_\infty$. Hence, in either case, $f^{-1}(U)$ belongs to $\mathcal{T}_\infty$. Since $U \in \mathcal{T}'_\infty$ was arbitrary, this means that f is continuous. Reversing the rôles of X_∞ and X'_∞ yields the continuity of f^{-1}, so that f is a homeomorphism. This proves the uniqueness part of the theorem. □

Note that this construction also works when X is already compact.

Exercises

1. Let $(X, \mathcal{T})$ be a topological space, and let $K_1, \dots, K_n$ be compact subsets of X. Show that $K_1 \cup \cdots \cup K_n$ is compact.
2. Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a base for $\mathcal{T}$. Show that X is compact if and only if each open cover $\mathcal{U} \subset \mathcal{B}$ of X has a finite subcover.
3. *Dini's lemma.* Let $(K, \mathcal{T})$ be a compact topological space, let $f: K \to \mathbb{R}$ be continuous, and let $(f_\alpha)_{\alpha \in \mathbb{A}}$ be a net of continuous functions from K to $\mathbb{R}$ such that $f(x) = \lim_\alpha f_\alpha(x)$ for all $x \in K$ and $f_\beta(x) \le f_\alpha(x)$ for all $x \in K$ and $\alpha, \beta \in \mathbb{A}$ with $\alpha \preceq \beta$. Show that $(f_\alpha)_\alpha$ converges to f uniformly on K. (*Hint*: Fix $\epsilon > 0$ and consider, for each $\alpha \in \mathbb{A}$, the set $U_\alpha := \{x \in K : 0 \le f_\alpha(x) - f(x) < \epsilon\}$.)
4. Let $(X, \mathcal{T}_X)$ be a topological space, let $((Y_i, \mathcal{T}_i))_{i \in \mathbb{I}}$ be a family of topological spaces, and let $(Y, \mathcal{T}_Y)$ denote its topological product. Show that $f: X \to Y$ is continuous if and only if $\pi_i \circ f: X \to Y_i$ is continuous for each $i \in \mathbb{I}$, where $\pi_i: Y \to Y_i$ is the coordinate projection.
5. Let $(X_1, \mathcal{T}_1), \dots, (X_n, \mathcal{T}_n)$ be topological spaces, and, for $j = 1, \dots, n$, let $\mathcal{B}_j$ be a base for $\mathcal{T}_j$. Show that the subsets of $X_1 \times \cdots \times X_n$ of the form $B_1 \times \cdots \times B_n$ with $B_j \in \mathcal{B}_j$ for $j = 1, \dots, n$ form a base for the product topology on $X_1 \times \cdots \times X_n$.
6. Let $((X_i, \mathcal{T}_i))_{i \in \mathbb{I}}$ be a family of topological spaces. The sets of the form $\prod_{i \in \mathbb{I}} U_i$, where $U_i \in \mathcal{T}_i$ for $i \in \mathbb{I}$, form a base for a topology on $\prod_{i \in \mathbb{I}} X_i$, the *box topology*. Show that the box topology is finer than the product topology and that the two topologies coincide if and only if $(X_i, \mathcal{T}_i)$ is chaotic for all but finitely many $i \in \mathbb{I}$.
7. Let $((X_i, \mathcal{T}_i))_{i \in \mathbb{I}}$ be a family of topological spaces, and let $(X, \mathcal{T})$ be their topological product. Show that $(X, \mathcal{T})$ is Hausdorff if and only if $(X_i, \mathcal{T}_i)$ is Hausdorff for each $i \in \mathbb{I}$.
8. Let $(X, \mathcal{T})$ be a topological space, and let $X \times X$ be equipped with the product topology. Show that $\{(x, x) : x \in X\}$ is closed in $X \times X$ if and only if X is Hausdorff.
9. Let E be a normed space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. A *linear functional* on E is a map $\phi: E \to \mathbb{F}$ such that $\phi(\lambda x + \mu y) = \lambda \phi(x) + \mu \phi(y)$ for all $\lambda, \mu \in \mathbb{F}$ and $x, y \in E$. Let $\mathcal{F}$ denote the collection of all finite subsets of E, and let

$$K := \left\{ \phi \in \mathbb{F}^E : \phi \text{ is a linear functional on } E \text{ with } |\phi(x)| \le \|x\| \text{ for } x \in E \right\}.$$

Prove the *Alaoglu–Bourbaki theorem*: $(K, \mathcal{T}_\mathcal{F}|_K)$ is compact.

10. Show that $\mathbb{Q}$ with the topology inherited from $\mathbb{R}$ is not locally compact.

3.4 Connectedness

Intuitively, one would call a topological space connected if one can "walk" from any point of the space to any other point of the space without having to "jump" over a "gap". How can this be made precise?

Definition 3.4.1. *Let $(X, \mathcal{T})$ be a topological space. A* path *in X is a continuous map from $[0,1]$ to X.*

Of course, any nondegenerate, closed, and bounded interval in $\mathbb{R}$ may serve as a domain for a path.

Definition 3.4.2. *Let $(X, \mathcal{T})$ be a topological space, and let $x_0, x_1 \in X$. Then x_0 and x_1 can be* connected by a path *in X if there is a path $\gamma: [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. We say that γ* connects *x_0 with x_1.*

Definition 3.4.3. *A topological space $(X, \mathcal{T})$ is called* path connected *if any two of its points can be connected by a path in X.*

Example 3.4.4. Let E be a normed space. Recall that a subset C of E is called *convex* if, for any $x, y \in C$, the line segment $\{x + t(y - x) : t \in [0,1]\}$ also lies in C. Trivially, the whole space is convex as are all its singleton subsets. The same is true for all open balls. Let $x_0 \in E$, let $r > 0$, and let $x, y \in B_r(x_0)$; then we have

$$\begin{aligned} \|x + t(y - x) - x_0\| &= \|(1 - t)x + ty - (1 - t)x_0 - tx_0\| \\ &\le (1 - t)\|x - x_0\| + t\|y - x_0\| \\ &< r \qquad (t \in [0,1]). \end{aligned}$$

Similarly, one sees that all closed balls are convex. Every convex subset C of E is path connected: for $x_0, x_1 \in C$, the function

$$\gamma: [0,1] \to E, \quad t \mapsto x_0 + t(x_1 - x_0)$$

is a path in C connecting x_0 with x_1. In particular, all intervals in $\mathbb{R}$—possibly unbounded or degenerate—are path connected.

Here comes an easily proven hereditary property for path connectedness.

Proposition 3.4.5. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces such that X is path connected, and let $f: X \to Y$ be continuous. Then $f(X)$ is path connected.*

Proof. Let $y_0, y_1 \in f(X)$, and let $x_0, x_1 \in X$ be such that $y_j = f(x_j)$ for $j = 0, 1$. Let $\gamma: [0,1] \to X$ be a path connecting x_0 with x_1. Then $f \circ \gamma$ is a path in $f(X)$ that connects y_0 with y_1. □