Proseminar Functionalanalysis
Problems 8 2.6.2005

56. Let K be a compact subset of a Hilbert space H. Prove that every sequence in K that converges weakly also converges strongly.

57. Let (x_n) be a weakly convergent sequence in a Hilbert space and let x be its weak limit. Prove that x lies in the closed convex hull of the set $\{x_n : n \in \mathbb{N}\}$.

58. Let $T \in \mathcal{L}(X,Y)$, X, Y Banach spaces. Establish the identity

$$K^{**} i_X = i_Y K$$

where i_X and i_Y stand for the canonical injections of X into X^{**}, respectively of Y into Y^{**}.

59. A selfadjoint operator T on a Hilbert space H is called positive if $(Tx,x) \geq 0$ for all $x \in H$. Show

$$|(Tx,y)|^2 \leq (Tx,x)(Ty,y)$$

for all x, $y \in H$. (Hint: Prove that $(x,y) \to (Tx,y)$ behaves like an inner product and therefore satisfies the Cauchy-Schwarz inequality.

60. Show that $\mathcal{K}(X,Y)$ is a subspace of $\mathcal{L}(X,Y)$.

61. Let X be an infinite dimensional Banach space and Y be a normed space. Let $T \in \mathcal{L}(X,Y)$ be an operator for which there is $\alpha > 0$ such that $||Tx|| \geq \alpha ||x||$ holds for all $x \in X$. Show that T is not compact.

62. Let (x_n) be a sequence of pairwise orthogonal elements in a Hilbert space H. Show the equivalence of the following statements:

    (a) $\sum_{i=1}^{\infty} x_i$ converges.

    (b) $\sum_{i=1}^{\infty} ||x_i||^2$ converges.

    (c) $\sum_{i=1}^{\infty} x_i$ converges weakly.