Proseminar Functionalanalysis
Problems 8 18.5.2005

51. Let M be a subset of a Hilbert space H, and let v, $w \in H$. Suppose that $(v, x) = (w, x)$ for all $x \in M$ implies $v = w$. If this holds for all v, $w \in H$, show that M is total (complete) in H, i.e $\overline{\text{span}} M = H$.

52. Let $x_n(t) = t^n$, $n \in \mathbb{N}_0$. Show that $\{x_n : n \in \mathbb{N}_0\}$ is complete in $L^2(-1, 1)$.
Hint: Assume $f \in L^2(-1, 1)$ satisfies $f \perp x_n$, $n \in \mathbb{N}_0$. Show $F \perp x_n$, $n \in \mathbb{N}_0$, where $F(t) = \int_{-1}^{t} f(s)\, ds$. Approximate F by a polynomial Q using the theorem of Weierstraß and use

$$\int_{-1}^{1} F(t)^2\, dt = \int_{-1}^{1} F(t)(F(t) - Q(t))\, dt$$

(justify) to argue $F = 0$, which implies $f = 0$.

53. On $\mathbb{C}^2$ let the operator $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $Tx = (\xi_1 + i\xi_2, \xi_1 - i\xi_2)$, where $x = (\xi_1, \xi_2)$. Find T'. Show that we have $T'T = TT' = 2\text{id}$.

54. Let $\{y_k : 1 \le k \le n\}$ be linearly independent elements of a Hilbert space H and define $\mathcal{K} = \{x \in H : (x, y_k) = c_k, 1 \le k \le n\}$, where $c_k \in \mathbb{R}$ are fixed constants. Show that there exists a unique element $x_0 \in \mathcal{K}$ with minimum norm. Furthermore show that x_0 is given by

$$x_0 = \sum_{i=1}^{n} \beta_i y_i$$

where the coefficients β_i are given by the solution of the system

$$\begin{aligned}
(y_1, y_1)\beta_1 + (y_2, y_1)\beta_2 + \cdots + (y_n, y_1)\beta_n &= c_1 \\
(y_1, y_2)\beta_1 + (y_2, y_2)\beta_2 + \cdots + (y_n, y_2)\beta_n &= c_2 \\
&\vdots \\
(y_1, y_n)\beta_1 + (y_2, y_n)\beta_2 + \cdots + (y_n, y_n)\beta_n &= c_n.
\end{aligned}$$

55. The shaft angular velocity of an d–c motor driven from a variable current source u is governed by the following first order differential equation

$$\dot{\omega}(t) + \omega(t) = u(t).$$

The angular position θ of the motor shaft is the time integral of ω. Assume that the motor is initially at rest $\theta(0) = \omega(0) = 0$. Determine the field current u of minimum energy which rotates the shaft within one second to the new rest position $\theta = 1$, $\omega = 0$. The energy is measured by $\int_0^1 u^2(t)\, dt$.
Hint: Use problem 54